

A Simple Model of Monetary Exchange Based on Nonconvexities and Sunspots*

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Preliminary and incomplete.

Abstract

We construct a simple model of monetary exchange where, as in Lagos and Wright (2003), trades take place in both centralized and decentralized markets. However, in contrast to Lagos and Wright, we allow for general preferences and introduce nonconvexities (indivisibilities) and sunspots. In the centralized market, agents can trade state-contingent commodities. We show that nonconvexities and sunspots make the model tractable by reducing the heterogeneity generated by the randomness of the trading process in the decentralized market. We show that the allocations in the centralized and decentralized markets are determined independently. In particular, the unemployment rate in the centralized market is independent of inflation. While the allocation in the centralized market is efficient, the allocation in the decentralized market is in general inefficient.

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1 Introduction

Everyone should agree that it is desirable to construct models in macro and monetary economics with better microfoundations. Cass and Shell (1980) advocate this position with respect to the overlapping generations model of money described in Samuelson (1958), Shell (1971) and Wallace (1980), for example. Cass and Shell argue "this basic structure has two general features which we believe are indispensable to the development of macroeconomics as an intellectually convincing discipline ... First it is genuinely dynamic ... Second it is fundamentally disaggregative." They also "firmly believe that a satisfactory general theory must, at a minimum, encompass some diversity among households as well as some variety among commodities." We think these words still ring true today, although in the generation since they were written, a new microfounded monetary model based on search theory has been developed that fits this description as well or better than overlapping generations models. A drawback of early search models is that for tractability reasons they had to make extreme assumptions about how much money agents could carry, since otherwise the endogenous distribution of money across agents became difficult to handle. These restrictions not only seem inelegant, they also greatly hindered the use of the models for monetary policy as it is usually formulated.

The subsequent literature developed in several directions. Some people tried to relax these restrictions and see just how far one could get analytically, while others proceeded with numerical methods (e.g. Green and Zhou (1998), Zhou (1999), Camera and Corbae (1999), and Zhu (2003) proceed analytically, while Molico (1999, 2004) uses computational methods).

Still others tried to find assumptions to make the model less complicated, so that analytic results would be relatively easy to derive, and hence the model could be put to theoretical and practical use. There are two main approaches. Shi (1997) assumes the fundamental decision-making units in the model are not individuals, but large households consisting of agents who share their money after each round of decentralized trade, and he appeals to the law of large numbers to conclude that all households start the next round of trade with the same amount of cash. Lagos and Wright (2002) alternatively assume that there are centralized markets in cash and other goods that open after each round of decentralized trade, and show that as long as preferences are quasi-linear, all agents will leave these markets with the same money holdings because, heuristically speaking, quasi-linearity implies no wealth effect in the demand for money (assuming interior solutions).

While these approaches are quite useful, each comes with some baggage. Several conceptual and technical issues with household models are discussed in Lagos and Wright (2002); a clear problem with the alternative centralized market approach is that quasi-linear preferences are obviously special and entail several other implications that may be undesirable for some purposes. In this paper we propose an alternative structure that also generates a degenerate distribution of money holdings in search-based models: we adopt the centralized market structure in Lagos and Wright (2002), but rather than quasi-linearity, we allow general utility functions and assume that some good is indivisible. As is well known, in models with indivisibilities or some other non-convexities, efficient allocations may require randomization – i.e. they may involve lotteries (see e.g. Rogerson 1988, or the recent special issue of JET). Intuitively, the idea is this: since expected utility will be linear in the

probabilities associated with these lotteries, one might conjecture that the agents in our model may behave like the quasi-linear agents in Lagos and Wright. While it is going to take a little work to make this precise, this intuition turns out to be correct.

The way we proceed is to look for competitive equilibria that generate randomized allocations via sunspots. Following Shell and Wright (1992) our equilibrium concept is exactly that in Debreu (1959, chapter 7), where state-contingent commodities are traded in centralized markets, and where it just so happens that the state is a sunspot variable – i.e. it constitutes extrinsic uncertainty in the sense that it has no impact on preferences, endowments or technology. In equilibrium, we prove that agents who enter the market with different amounts of money will generally buy or sell the indivisible good with different probabilities, but they will all choose the same bundle of divisible goods. In particular this implies that they all take the same amount of money out of the centralized market and hence into next period's decentralized market. Hence we get a degenerate distribution of money in the decentralized market (just like Lagos-Wright). A caveat is that this works here (just like Lagos-Wright) only if all agents choose interior solutions for the set of states where they buy or sell the indivisible good. We provide conditions on fundamentals under which this is true.

The bottom line in terms of monetary theory is that we provide a new environment where money is valued due to explicit descriptions in the model of frictions in the trading process, and yet things are analytically very tractable because the distribution of money is degenerate, and hence as in other simple models of money, we can reduce the equilibrium conditions to a single difference equation. At the same time, our results provide another example

of the role of sunspots in economics, although far from being detrimental to economic activity, as in much of the early work following Cass and Shell (1980), here sunspots are a good thing. As emphasized in Shell and Wright (1992), in nonconvex economies sunspots can help us achieve an efficient randomized allocation as a standard competitive equilibrium in the sense of Debreu (1959, chapter 7). One thing not noticed in Shell and Wright (1992) however is that in this equilibrium, assuming an interior solution, agents will act as if they have quasi-linear preferences. That is, wealth effects vanish from the demand for divisible goods, including the demand for money. From the perspective of trying to build tractable models of money, this is very nice.

The rest of the paper is organized as follows...

2 A simple model

We consider a model with indivisible labor and sunspots based on Shell and Wright (1993). The economy is composed of a measure one of consumers indexed by $i \in I$, K firms indexed by $k \in \{1, \dots, K\}$, J consumption goods indexed by $j \in \{1, \dots, J\}$, and a good in fixed supply called land.¹ In the following section, we will reinterpret land as money by embedding this framework into a model of monetary exchange. Consumption goods are produced using labor. Agent i is endowed with one indivisible unit of labor, some consumption goods c_0^i , and some land m_0^i . We denote $M = \int m_0^i di$ the aggregate stock of land and $C = \int c_0^i di$ the aggregate endowment in consumption goods.

¹One can view the model as dynamic by reinterpreting the vector c as having commodities indexed by date.

Consumers value consumption, leisure and land. The utility function is $U(c, h, m)$ where $c \in \mathbb{R}_+^J$ is consumption, $m \in \mathbb{R}_+$ is land at the end of the trading period and $h \in \{0, 1\}$ is labor. The utility function is twice continuously differentiable, strictly increasing in each of its argument, and strictly concave.

Output is produced by firms according to a strictly concave production function $f(n) \in \mathbb{R}_+^J$ where $n \in \mathbb{R}_+^J$ is a vector of labor services used in the production of the J consumption goods. Each consumer receives an equal share in the profits of the firms.

The economy is subject to extrinsic uncertainty. There is a continuum of states $s \in S = [0, 1]$. The realization of a state does not affect the fundamentals of the economy (such as preferences, technology or endowments), and it is a random draw from a uniform distribution. Markets are complete and agents can trade state-contingent commodities. We consider equilibria where prices are constant across states. Denote $p \in \mathbb{R}_+^J$ the price vector of consumption goods, $w \in \mathbb{R}_+$ the price of labor, and the price of land is normalized to one.

The problem of firm k is

$$\max_{n^k(s)} \int_S \pi [n^k(s)] ds, \quad (1)$$

where $\pi [n^k(s)] = pf[n^k(s)] - w \sum n_j^k(s)$ and $n_j^k(s)$ is labor used by firm k to produce good j in state s . According to (1), firm k chooses labor input in order to maximize its expected profits taking the price of consumption goods and the wage as given.

Lemma 1 *For all $s \in S$, $n^k(s) = n^k$.*

Proof. Direct from the strict concavity of $\pi(n)$ and the assumption that prices are constant across states. ■

The previous lemma simply indicates that firms make the same labor choice in all states. Let $\pi(s) = \sum_k \pi^k(s)$ denote the aggregate profits in state s .

Each consumer chooses consumption, labor and land in each state in order to maximize his expected utility. Agent i 's problem is

$$W(m_0^i, c_0^i) = \max_{h^i(s), c^i(s), m^i(s)} \int_S \{U[c^i(s), h^i(s), m^i(s)]\} ds \quad (2)$$

$$\text{s.t.} \quad \int_S \{pc^i(s) + m^i(s)\} ds = \int_S \{pc_0^i + wh^i(s) + m_0^i + \pi(s)\} ds. \quad (3)$$

Equation (3) is a standard budget constraint where goods are state-contingent.

Denote $S_e^i = \{s \in S : h^i(s) = 1\}$ the set of states where agent i supplies his indivisible unit of labor, and $S_u^i = S \setminus S_e^i$.

Lemma 2 For all $s \in S_e^i$, $c^i(s) = c_e^i$ and $m^i(s) = m_e^i$. For all $s \in S_u^i$, $c^i(s) = c_u^i$ and $m^i(s) = m_u^i$.

Proof. Direct from the strict concavity of $U(c, h, m)$. ■

The previous Lemma states that agents make the same choices in terms of consumption and land in all states where they are employed. Note also that agents only care about the measure of states where they are employed, $\ell^i = \int_{S_e^i} ds$, and not about which particular state is in S_e^i . Agent i 's problem (2)-(3) can be reexpressed as

$$W(m_0^i, c_0^i) = \max_{\ell^i, c_e^i, c_u^i, m_e^i, m_u^i} \{\ell^i U(c_e^i, 1, m_e^i) + (1 - \ell^i)U(c_u^i, 0, m_u^i)\} \quad (4)$$

$$\text{s.t.} \quad \ell^i(pc_e^i + m_e^i) + (1 - \ell^i)(pc_u^i + m_u^i) = pc_0^i + w\ell^i + m_0^i + \pi. \quad (5)$$

Denote $I_e(s) = \{i \in I : h^i(s) = 1\}$ the set of agents who work in state s and $I_u(s) = I \setminus I_e(s)$.² Labor market clearing requires $\sum_{j,k} n_j^k(s) = \int_{I_e(s)} di$. Integrating with respect to the state, and using the fact that n is constant across states,

$$\sum_{j,k} n_j^k = \int_I \ell^i di. \quad (6)$$

The clearing of the land and goods markets implies

$$M = \int_I m^i(s) di = \int_{I_e(s)} m_e^i di + \int_{I_u(s)} m_u^i di, \quad \forall s \in S. \quad (7)$$

$$\sum_k f(n^k) + C = \int_I c^i(s) di = \int_{I_e(s)} c_e^i di + \int_{I_u(s)} c_u^i di, \quad \forall s \in S. \quad (8)$$

Definition 3 A competitive equilibrium is an allocation $\{(c_e^i, c_u^i, \ell^i, m_e^i, m_u^i), I_e(s), (n^k)\}$ and a list of prices (w, p) such that:

- (i) Given (w, p) , $(c_e^i, c_u^i, \ell^i, m_e^i, m_u^i)$ solves (4)-(5).
- (ii) Given (w, p) , n^k solves (1).
- (iii) (w, p) satisfies the market clearing conditions (6) and (7).
- (iv) $\ell^i = \int \mathbf{1}_{\{i \in I_e(s)\}} ds$.

Competitive equilibria where the choice for ℓ^i is interior for all consumers have interesting properties. Denote λ^i the Lagrange multiplier associated with (5). Assuming $\ell^i \in (0, 1)$, the first-order conditions for $(\ell^i, c_e^i, c_u^i, m_e^i, m_u^i)$ are

$$U(c_e^i, 1, m_e^i) - U(c_u^i, 0, m_u^i) + \lambda^i (w - pc_e^i - m_e^i + pc_u^i + m_u^i) = 0. \quad (9)$$

²The sets $I_e(s)$ and S_e^i are related as follows: $I_e(s) = \{i : s \in S_e^i\}$.

$$U_j(c_e^i, 1, m_e^i) = U_j(c_u^i, 0, m_u^i) = \lambda^i p_j, \quad \forall j = 1, \dots, J, \quad (10)$$

$$U_m(c_e^i, 1, m_e^i) = U_m(c_u^i, 0, m_u^i) = \lambda^i, \quad (11)$$

where U_j is the partial derivative of U with respect to c_j and U_m is the partial derivative of U with respect to m . According to (10), marginal utility of consumption is equal across states. Similarly, according to (11), marginal utility of land is equal across states.

Lemma 4 *In any competitive equilibrium, $(c_e^i, c_u^i, m_e^i, m_u^i) = (c_e, c_u, m_e, m_u)$ for all i such that $\ell^i \in (0, 1)$.*

Proof. Equations (9)-(11) determine $(c_e^i, c_u^i, m_e^i, m_u^i)$ independently of agents' initial endowments. ■

The previous lemma states that consumers make the same choices in terms of consumption and land whenever their choice of labor is interior even though they may have different initial endowments.

In the following we show how the equilibrium allocation when the initial distribution of endowments is non-degenerate relates to the allocation when it is degenerate. Denote $(\hat{c}_e, \hat{c}_u, \hat{\ell}, \hat{m}_e, \hat{m}_u)$ the equilibrium allocation, (\hat{p}, \hat{w}) the prices and $\hat{\lambda}$ the Lagrange multiplier associated with (5) when the initial distribution of endowments is degenerate, i.e., $m_0^i = M$ and $c_0^i = C$ for all $i \in I$. We assume that fundamentals (preferences, technology...) are such that $\hat{\ell} \in (0, 1)$.³ Consider then an economy where the distribution of initial endowments is nondegenerate and let F_0 be the distribution of agents' wealth

³For prices to be constant across states, the measure of the set $I_e(s)$ must be constant across states. To guarantee that this is satisfied, we construct the set of states where agent i supplies his indivisible unit of labor as $S_e^i = [i, i + \hat{n}]$ modulo 1.

where the endowments in terms of consumption goods are valued according to the price vector \hat{p} .

Proposition 5 *There exists $\underline{\omega}$ and $\bar{\omega}$ such that if $\text{supp}(F_0) \subseteq [\underline{\omega}, \bar{\omega}]$ then $(c_e^i, c_u^i, m_e^i, m_u^i) = (\hat{c}_e, \hat{c}_u, \hat{m}_e, \hat{m}_u)$ for all $i \in I$ and $\int_I \ell^i di = \hat{\ell}$. Furthermore, $W(m, c) = \hat{\lambda}(m + \hat{p}c) + \bar{W}$ where $\bar{W} \in \mathbb{R}$.*

Proof. From Lemma 4, if ℓ^i is interior and $(p, w) = (\hat{p}, \hat{w})$ then $(c_e^i, c_u^i, m_e^i, m_u^i) = (\hat{c}_e, \hat{c}_u, \hat{m}_e, \hat{m}_u)$. From (5), $\int_I \ell^i di = \hat{\ell}$. It follows that (\hat{p}, \hat{w}) are market-clearing prices. The interval $[\underline{\omega}, \bar{\omega}]$ corresponds to the range of values for the endowment $\omega^i = m_0^i + pc_0^i$ such that $\ell^i \in (0, 1)$. From (5) and (9), and after some simplification,

$$\ell^i = \frac{\hat{\lambda}(M + \hat{p}C - \omega^i)}{U(c_u, 0, m_u) - U(c_e, 1, m_e)} + \hat{\ell}, \quad (12)$$

If $U(c_u, 0, m_u) - U(c_e, 1, m_e) > 0$ then, $\ell^i \in (0, 1)$ requires $\omega^i \in (\underline{\omega}, \bar{\omega})$ where

$$\bar{\omega} = M + [U(c_u, 0, m_u) - U(c_e, 1, m_e)] \hat{\ell} / \hat{\lambda}, \quad (13)$$

$$\underline{\omega} = M - [U(c_u, 0, m_u) - U(c_e, 1, m_e)](1 - \hat{\ell}) / \hat{\lambda}. \quad (14)$$

If $U(c_u, 0, m_u) - U(c_e, 1, m_e) < 0$ then $\underline{\omega}$ and $\bar{\omega}$ are given by

$$\bar{\omega} = M - [U(c_u, 0, m_u) - U(c_e, 1, m_e)](1 - \hat{\ell}) / \hat{\lambda}, \quad (15)$$

$$\underline{\omega} = M + [U(c_u, 0, m_u) - U(c_e, 1, m_e)] \hat{\ell} / \hat{\lambda}. \quad (16)$$

From (4)-(5), the value function W satisfies

$$\begin{aligned} W(m_0^i, c_0^i) &= \ell^i U(\hat{c}_e, 1, \hat{m}_e) + (1 - \ell^i) U(\hat{c}_u, 0, \hat{m}_u) \\ &+ \hat{\lambda} \{ \hat{p}c_0^i + \hat{w}\ell^i + m_0^i + \pi - \ell^i(\hat{p}\hat{c}_e + \hat{m}_e) - (1 - \ell^i)(\hat{p}\hat{c}_u + \hat{m}_u) \} \end{aligned}$$

Using (9) the previous equation can be simplified to

$$W(m_0^i, c_0^i) = \hat{\lambda} (\hat{p}c_0^i + m_0^i) + U(\hat{c}_u, 0, \hat{m}_u) + \hat{\lambda} (\pi - \hat{p}\hat{c}_u - \hat{m}_u).$$

■

Proposition 5 shows that economies with different distributions of endowments have the same allocations for (c_e, c_u, m_e, m_u) whenever the distribution of endowments is not too disperse. In particular, they are characterized by the same ex-post distribution of land with at most two mass points. Also, it shows that the value function W is linear in the agent's wealth.

Corollary 6 *If $\partial^2 U / \partial c_j \partial m = \partial^2 U / \partial h \partial m = 0$ then $m_e = m_u = M$.*

Proof. Immediate from (11). ■

If the utility function is additively separable in m then the ex-post distribution land is degenerate. This result will be crucial to make the model of monetary exchange in the next section tractable.

In the following, we consider two examples to illustrate Proposition 5. In these examples, $c_0^i = 0$ and land is additively separable in the utility function.

Example 7 *Assume $U(c, h, m) = A \log c + L(h) + V(m)$ where $L'(\cdot) < 0$ and $V(m)$ is strictly increasing and concave. Normalize $L(1) = 0$ and denote $L(0) = L$. Assume further that production is linear, $f(n) = \alpha n$. Then, $m_e = m_u = M$, $c_e = c_u = \alpha A / L$ and $n = A / L$. So, n is interior if $A < L$. From (13) and (14), $\underline{\omega}$ and $\bar{\omega}$ satisfy*

$$\begin{aligned} \bar{\omega} &= M + A / V'(M), \\ \underline{\omega} &= M - (L - A) / V'(M). \end{aligned}$$

As in the specification studied in Rogerson (1988), consumption and leisure are additively separable in the previous example so that the reduced form utility function, $A \log c - Lh$, coincides with a quasi-linear specification. We provide next an example where the reduced form utility function is not quasi-linear.

Example 8 Assume $U(c, h, m) = c^a(1 - h + b)^{1-a} + V(m)$ with $a \in (0, 1)$ and $b > 0$. Assume further that the production is linear, $f(n) = \alpha n$. Then, $c_u = a(1 + b)\alpha/(1 - a)$, $c_e = ab\alpha/(1 - a)$ and $n = a(1 + b)$. From (13) and (14), $\underline{\omega}$ and $\bar{\omega}$ satisfy

$$\begin{aligned}\bar{\omega} &= M + \left(\frac{a\alpha}{1-a}\right)^a a(1+b)/V'(M), \\ \underline{\omega} &= M - \left(\frac{a\alpha}{1-a}\right)^a (1-a-ab)/V'(M).\end{aligned}$$

This analysis shows that one round of trading in this competitive market can eliminate the heterogeneity in terms of endowments assuming agents' initial endowments are not too far away from the average. All agents have the same consumption conditional on being employed or unemployed. The only choice variable that adjusts to make up for the differences in endowments is the measure of states in which agents are employed.

3 A model of monetary exchange

The previous section has shown how in the presence of indivisible labor the ex-ante heterogeneity in terms of endowments was endogenously washed out. This formalization can provide a useful device to make some models with heterogeneity tractable. For instance, search-theoretic models of monetary exchange assume that agents are subject to idiosyncratic risks

regarding their trading opportunities so that the distribution of money balances in equilibrium is non-degenerate. This non-degenerate distribution is what makes the model very difficult to analyze. To reduce this complexity, Lagos and Wright (2003) assumed that agents access competitive markets periodically, and were endowed with a quasi-linear utility function in order to eliminate wealth effects. As a consequence, all agents start each round of decentralized meetings with the same money balances. We show here an alternative way to obtain a degenerate distribution of money balances based on the formalization described in the previous section. In contrast to Lagos and Wright's device, we do not need the quasi-linear specification for the utility function but we do need indivisible labor and extrinsic uncertainty. Also, we will use the full strength of the competitive paradigm by allowing agents to trade state-contingent commodities.

We now consider an intertemporal economy where there is an essential role for money. Time is discrete and each period of time is divided into two subperiods called day and night. During the day, there is a centralized market similar to the one described in the previous section where agents trade state-contingent commodities and indivisible labor. With no loss, we assume that there is a single consumption good and a single firm in the centralized market, and agents have no endowment in terms of this good ($c_0^i = 0$). The utility function over these goods is $U(c, h)$.

At night, there is a decentralized market where agents are matched bilaterally and at random. Agents in this market are anonymous and cannot use debt to implement trades. As is standard in search-theoretic models of monetary exchange, there is a double coincidence of wants problem in the decentralized market that generates an essential role for money. This

problem is described as follows. There are several perfectly divisible and perishable goods that are produced and consumed in the decentralized market and agents are specialized in both production and consumption. We assume that there is no double coincidence of wants meetings. The probability for an agent to meet someone who produces the good he likes is $\sigma \leq 1/2$. Symmetrically, the probability to meet someone who likes the good that one produces is also σ . Agents' utility function in the decentralized market is $u(q^b) - c(q^s)$ where q^b is amount bought and q^s the amount sold. We assume that $u(q^b) - c(q^s)$ is strictly concave. Furthermore, we denote q^* the quantity that solves $u'(q) = c'(q)$. The discount factor between the day and night is denoted β^d and the discount factor between the night and the following day is β^n . We denote $\beta = \beta^d \beta^n$.

Fiat money is perfectly divisible, durable and is intrinsically useless. The stock of money in the economy in period t is M_t . The supply of money is growing at the gross growth rate $\gamma > \beta$, i.e., $M_{t+1} = \gamma M_t$. Money is injected, or withdrawn, through lump-sum transfers in the centralized market.

3.1 Efficient allocation

We consider a social planner who maximizes the sum of the utilities of all agents subject to some feasibility constraints as well as the search frictions in the decentralized market. Denote $\tilde{I} \subseteq I \times I$ the set of bilateral matches in the decentralized market where $(i, j) \in \tilde{I}$ if i and j are matched and i is the buyer in the match. The planner's problem can be written recursively as follows

$$W^* = \max_{c^i(s), h^i(s), q^i} \int_S \int_I U[c^i(s), h^i(s)] di ds + \beta^d \left[\int_{\tilde{I}} [u(q^i) - c(q^j)] d(i, j) \right] + \beta W^* \quad (17)$$

$$\text{s.t. } \int c^i(s)di \leq f \left[\int h^i(s)di \right], \quad \forall s \quad (18)$$

$$q^i \leq q^j, \quad \forall (i, j) \in \tilde{I} \quad (19)$$

Denote $I^e(s)$ the set of agents who are working in state s and let $\ell(s)$ be the measure of this set.

Proposition 9 *The efficient allocation is such that*

$$q^i = q^j = q^*, \quad \forall (i, j) \in \tilde{I}, \quad (20)$$

$c^i(s) = c^e$ for all $i \in I^e(s)$, $c^i(s) = c^u$ for all $i \in I \setminus I^e(s)$, $\ell(s) = \ell$ for all s .

Assuming an interior solution, (c^e, c^u, ℓ) satisfies

$$U_c(c_e, 1) = U_c(c_u, 0), \quad (21)$$

$$U(c_e, 1) - U(c_u, 0) = U_c(c_u, 0) [c_e - c_u - f'(\ell)], \quad (22)$$

$$\ell c^e + (1 - \ell)c^u = f(\ell). \quad (23)$$

Proof. From (17) and (19), $q^i = q^j = \arg \max [u(q) - c(q)]$ for all $(i, j) \in \tilde{I}$. From the strict concavity of $U(c, h)$, $c^i(s) = c^e(s)$ for all $i \in I^e(s)$ and $c^i(s) = c^u(s)$ for all $i \in I \setminus I^e(s)$. It can also be noticed that the maximization problem is separable across states. We can therefore rewrite the problem as

$$W^* = \max_{c^e, c^u, \ell} \ell U(c^e, 1) + (1 - \ell)U(c^u, 0) + \beta^d \sigma [u(q^*) - c(q^*)] + \beta W^* \quad (24)$$

$$\text{s.t. } \ell c^e + (1 - \ell)c^u \leq f(\ell). \quad (25)$$

The first-order conditions to this problem are (21) and (22). ■

3.2 Equilibrium

The problem of agent i in the centralized market is

$$W(m_0^i) = \max_{h^i(s), c^i(s), m^i(s)} \int_S \left\{ U[c^i(s), h^i(s)] + \beta^d V[m^i(s)] \right\} ds, \quad (26)$$

$$\ell^i(pc_e^i + m_e^i) + (1 - \ell^i)(pc_u^i + m_u^i) = w\ell^i + m_0^i + \pi + T.$$

where $V(m)$ is the value function of an agent with m units of money in the centralized sector and where T is the lump-sum transfer (or tax). From Proposition 5, $m^e = m^u = M$ assuming m is in some relevant range. So, the distribution of money balances at the beginning of the decentralized market is degenerate. Furthermore, the value function $W(m)$ is linear in m with $W'(m) = \lambda$.

Let n be employment in the centralized market, i.e., $n = \int \ell^i di$. From (9), (10) and (8), the allocation (c_e, c_u, n) is determined as follows.

$$U(c_e, 1) - U(c_u, 0) + U_c(c_u, 0) [f'(n) - c_e + c_u] = 0, \quad (27)$$

$$U_c(c_e, 1) = U_c(c_u, 0), \quad (28)$$

$$f(n) = nc_e + (1 - n)c_u. \quad (29)$$

From (10), the choice of money balances satisfies

$$\beta^d V'(m) = \lambda. \quad (30)$$

In a meeting, the terms of trade (q, d) , where q is the production of the seller and d the amount of money he receives from the buyer, depend on the money holdings of the buyer and the seller. The utility of agent i in the decentralized market obeys the following Bellman equation

$$\begin{aligned} V(m^i) = & \sigma \int \{u[q(m^i, m^j)] + \beta^n W_{+1}[m^i - d(m^i, m^j)]\} dF_1(m^j) \\ & + \sigma \int \{-c[q(m^j, m^i)] + \beta^n W_{+1}[m^i + d(m^j, m^i)]\} dF_1(m^j) + (1 - 2\sigma)\beta^n W_{+1}(m^i), \end{aligned} \quad (31)$$

where $F_1(m^j)$ is the distribution of money balances across agents. As we have seen above, assuming the distribution of money balances at the beginning of the period is not too disperse, $F_1(m)$ is degenerate.

One can make different assumptions regarding the formation of the terms of trade in the decentralized market. Here, we follow the literature since Shi (1995) and Trejos and Wright (1995) and assume that terms of trade are determined according to the generalized Nash bargaining solution.

$$\max_{q, d \leq m^i} [u(q) + \beta^n W_{+1}(m^i - d) - \beta^n W_{+1}(m^i)]^\theta [-c(q) + \beta^n W_{+1}(m^j + d) - \beta^n W_{+1}(m^j)]^{1-\theta}, \quad (32)$$

where $\theta \in (0, 1]$ is the buyer's bargaining power. Using the linearity of $W_{+1}(m)$, (32) can be simplified to

$$\max_{q, d \leq m^i} [u(q) - \beta^n \lambda_{+1} d]^\theta [-c(q) + \beta^n \lambda_{+1} d]^{1-\theta}. \quad (33)$$

The solution to (32) is $q = q^*$ and $d = m^* = g(q^*)/\beta\lambda_{+1}$ if $m^i \geq m^*$, and $g(q) = \beta^n \lambda_{+1} m^i$ and $d = m^i$ otherwise, where

$$g(q) = \frac{\theta u'(q)c(q) + (1-\theta)c'(q)u(q)}{\theta u'(q) + (1-\theta)c'(q)}.$$

Note that the terms of trade (q, d) do not depend directly of the money balances m^j of the seller in the match. They do depend, however, on the marginal value of money λ_{+1}^j of the seller j , but λ_{+1}^j is independent of m_{+1}^j for all m_{+1}^j in some interval (See previous section).

Using the fact that $\beta\lambda_{+1}d = g(q)$, the Bellman equation (31) can then be rewritten as

$$V(m^i) = \sigma \{u[q(m^i)] - g[q(m^i)]\} + \sigma \int \{-c[q(m^j)] + g[q(m^j)]\} dF_1(m^j) + \beta^n W_{+1}(m^i). \quad (34)$$

From (34), $V'(m^i) = \sigma [u'(q) - g'(q)] q'(m^i) + \beta^n \lambda_{+1}$ where $q'(m^i) = \beta^n \lambda_{+1} / g'(q)$ if $m^i < m^*$ and $q'(m^i) = 0$ for all $m^i > m^*$. Equation (11) yields

$$\frac{\lambda}{\beta^d} = \sigma [u'(q) - g'(q)] q'(m^i) + \beta^n \lambda_{+1}. \quad (35)$$

In the following, we focus on steady state equilibria where λM is constant. This implies $\lambda = \gamma \lambda_{+1}$. Since $q'(m^i) = 0$ for all $m^i > m^*$, it is easy to check from (35) that $m^i \leq m^*$ for all $\gamma > \beta$. Using the fact that $q'(m^i) = \beta \lambda_{+1} / g'(q)$ (35) yields

$$\frac{u'(q)}{g'(q)} = 1 + \frac{\gamma - \beta}{\beta \sigma}. \quad (36)$$

We now have to impose conditions to guarantee that $W(m)$ is linear for the values of m in the support of F_0 . In equilibrium, using the fact that agents receive a lump-sum transfer equal to γM_{-1} , the support of the distribution F_0 at time t is $\{M - M_{-1}, M, M + M_{-1}\}$. Therefore, we need to impose conditions so that $\bar{m} \geq M + M_{-1}$ and $\underline{m} \leq M - M_{-1}$. Assume $U(c_u, 0) - U(c_e, 1) > 0$ which is the common case where leisure is a normal good. Conditions (13) and (14) require

$$\lambda M_{-1} \leq [U(c_u, 0) - U(c_e, 1)] \min(1 - n, n). \quad (37)$$

Since $\beta \lambda_{+1} M = g(q)$, (37) can be rewritten as

$$g(q) \leq \beta [U(c_u, 0) - U(c_e, 1)] \min(1 - n, n). \quad (38)$$

Similarly, if $U(c_u, 0) - U(c_e, 1) < 0$ then (15) and (16) yield a condition identical to (38).

Definition 10 *A steady-state monetary equilibrium is a (q, c_e, c_u, n) that satisfies (36), (27), (28), (29).*

Proposition 11 (i) *The allocation (c_e, c_u, n) in the centralized market is efficient.* (ii) *The allocation q in the decentralized market is efficient iff $\gamma = \beta$ and $\theta = 1$. The optimal monetary policy is the Friedman rule.*

Proof. (i) Direct from the comparison of (21)-(23) and (27)-(29). (ii) Direct from (36). ■

There is a dichotomy between the centralized market and the decentralized one. The allocation in the centralized market is identical to the allocation of the nonmonetary economy. The allocation in the decentralized sector is in general inefficient because of the monetary wedge ($\gamma > \beta$) and the inefficiencies generated by Nash bargaining when $\theta < 1$.⁴

Corollary 12 *The unemployment in the centralized market is independent of the money growth rate γ .*

A change in the inflation rate does not affect aggregate production and consumption in the centralized sector. In particular, the fraction of agents who are unemployed is independent of γ . So, the model predicts a vertical long-run Phillips curve. Even though inflation does not affect aggregate employment, it does affect how employment is distributed across agents. Assuming leisure is a normal good, the fraction of rich agents who are unemployed falls and the fraction of poor agents who are unemployed increases as the inflation rate increases. In other words, the Phillips curve is downward-sloping for rich agents, upward-sloping for poor agents, and vertical for agents holding the average money holdings.

⁴These inefficiencies generated by the nonmonotonic nature of the Nash solution are not robust across bargaining solutions. See Rocheteau and Waller (2004).

4 Conclusion

What a beautiful model!

References

- [1] Camera, Gabriele and Corbae, Dean (1999). “*Money and price dispersion*”, *International Economic Review* 40, 985-1008.
- [2] Cass, D. and Shell, Karl (1980). “In Defense of the Basic Approach”, In J. H. Kareken and N. Wallace (eds.), *Models of Monetary Economies*. Minneapolis: Federal Reserve Bank of Minneapolis.
- [3] Green, E. J. and Zhou, Ruilin (1998). “*A rudimentary random-matching model with divisible money and prices*”, *Journal of Economic Theory* 81, 252-271.
- [4] Lagos, Ricardo and Wright, Randall (2003). “A unified framework for monetary theory and policy analysis”, *mimeo*, University of Pennsylvania.
- [5] Molico, Miguel, (1999) “*The distribution of money and prices in search equilibrium*”, *mimeo*.
- [6] Rogerson, Richard (1988). “Indivisible labor, lotteries and equilibrium”, *Journal of Monetary Economics* 21, 3-16.
- [7] Rogerson, Richard and Wright, Randall (1988). “Involuntary unemployment in economies with efficient risk sharing”, *Journal of Monetary Economics* 22, 501-515.
- [8] Samuelson, Paul (1958). “An exact consumption-loan model of interest with or without the social contrivance of money”, *Journal of Political Economy* 66, 467-482.

- [9] Shell, Karl (1971). “Notes on the economics of infinity”, *Journal of Political Economy*
- [10] Shell, Karl and Wright, Randall (1993). “Indivisibilities, lotteries, and sunspot equilibria”, *Economic Theory* 3, 1-17.
- [11] Shi, Shouyong (1997). “*A divisible search model of fiat money*”, *Econometrica* 65, 75-102.
- [12] Wallace, Neil (1980). “*The overlapping generations model of fiat money*”, In J. H. Kareken and N. Wallace (eds.), *Models of Monetary Economies*. Minneapolis: Federal Reserve Bank of Minneapolis 49-82.
- [13] Zhou, Ruilin (1999). “*Individual and aggregate real balances in a random-matching model*”, *International Economic Review* 40, 1009-1038.