Proof of Sufficiency:

From Principle of Maximum we know that the optimal solution must satisfy:

(1) $u'(c) \geq q$

(2) $\dot{q} = (\delta + \lambda)q - u'(c)f'(k)$

(3) $\dot{k} = f(k) - c - \lambda k$

We also know that the solution must satisfy the initial condition and the transversality condition:

(4) $k(0) = k_0$

(5) $\lim\limits_{t \to \infty} q e^{-\delta t} k(t) = 0$

We want to prove this: If u and f are concave functions, then $c(t), k(t), z(t)$ which satisfy (1)-(5) will be optimal.

Proof:

It is sufficient to prove that for any $\tilde{c}, \tilde{k}, \tilde{z}$ which are feasible and satisfy the transversality condition, we have:

$$\int [u(c) - u(\tilde{c})] e^{-\delta t} dt > 0$$

$$\int [u(c) - u(\tilde{c})] e^{-\delta t} dt$$

$$= \int \{[u(c) - u(\tilde{c})] + u'(c)[(f(k) - c - z) - (f(\tilde{k}) - \tilde{c} - \tilde{z})] + q[(z - \lambda k - \dot{k}) - (\tilde{z} - \lambda \tilde{k} - \dot{\tilde{k}})]\} e^{-\delta t} dt$$

$$= \int \{[u(c) - u(\tilde{c})] + u'(c)[(f(k) - c - z) - (f(\tilde{k}) - \tilde{c} - \tilde{z})] + q[(z - \lambda k) - (\tilde{z} - \lambda \tilde{k})]\} e^{-\delta t} dt - \int q[\dot{k} - \dot{\tilde{k}}] e^{-\delta t} dt$$

$$= \int \{[u(c) - u(\tilde{c})] + u'(c)[(f(k) - c - z) - (f(\tilde{k}) - \tilde{c} - \tilde{z})] + q[(z - \lambda k) - (\tilde{z} - \lambda \tilde{k})]\} e^{-\delta t} dt + \int [k - \tilde{k}][\dot{q} - q\delta] e^{-\delta t} dt - (k - \tilde{k})|_0^\infty$$

$$= \int \{[u(c) - u(\tilde{c}) - u'(c)(c - \tilde{c})] + [u'(c) - q][\tilde{z} - z] + u'(c)[(f(k)) - (f(\tilde{k}))] + [k - \tilde{k}][\dot{q} - q\delta - \lambda q]\} e^{-\delta t} dt$$

$$= \int \{[u(c) - u(\tilde{c}) - u'(c)(c - \tilde{c})] + [u'(c) - q][\tilde{z} - z] + u'(c)[f(k) - f(\tilde{k})] - u'(c)f'(k)(k - \tilde{k})\} e^{-\delta t} dt$$

$$= \int \{[u(c) - u(\tilde{c}) - u'(c)(c - \tilde{c})] + [u'(c) - q][\tilde{z} - z] + u'(c)[f(k) - f(\tilde{k}) - f'(k)(k - \tilde{k})]\} e^{-\delta t} dt$$

$$> 0$$