

Web Supplement to *Could Making Banks Hold Only Liquid Assets Induce Bank Runs?* by James Peck (The Ohio State University) and Karl Shell (Cornell University): Appendix

Proofs

Proof of Theorem 3.1: Given γ , the functions $\{c_I^2(\alpha_1), c_P^2(\alpha_1)\}$ that maximize W subject only to resource constraint (6) entail full consumption smoothing (8). However, the allocation defined by (6) and (8) also satisfies the incentive compatibility constraint (5), and therefore solves the more tightly constrained problem (7). Plugging (6) and (8) into the expression for W , and differentiating with respect to γ , we have

$$\left(\frac{\partial W}{\partial \gamma}\right)_{\gamma=\bar{\alpha}/y} = \int_0^{\bar{\alpha}} y(R_\ell - R_i)u'[c_P^2(\alpha) - 1 + (y - \bar{\alpha})R_i]f(\alpha)d\alpha < 0.$$

Clearly, any contract for which we have $\gamma y > \bar{\alpha}$ is inferior to the one characterized by (6) and (8), since the former provides fewer resources available in period 2, with no compensating advantage in terms of consumption smoothing in period 2 or provision of consumption in period 1. \square

Proof of Theorem 3.2: First, note that a solution to (7) must exist. Given γ , a sufficient condition for $\{c_I^2(\alpha_1), c_P^2(\alpha_1)\}$ to solve problem (7) is for (6) and (8) to hold. The specification of consumption for $\alpha_1 > \gamma y$ does not affect the objective or the incentive compatibility constraint. Plugging (6) and (8) into the expression for W , problem (7) can be transformed into an equivalent unconstrained problem of choosing γ to maximize W , which must have a solution satisfying $\gamma y < \bar{\alpha}$. Because (6) and (8) imply (5), it follows that the optimal contract is socially optimal.

Construct the contract, $(\gamma, c_I^2(\alpha_1), c_P^2(\alpha_1))$, as follows. Let γ be as in the solution to (7). Consumptions are determined by the resource equation, (6), and

$$c_I^2(\alpha_1) = c_P^2(\alpha_1) - 1 \quad \text{for all } \alpha_1,$$

It follows that patient consumers are indifferent between withdrawing in period 1 and waiting. Under the assumption that a patient consumer will choose not to run when indifferent between running and not running, the optimal contract does not have a run equilibrium. \square

Proof of Lemma 4.1: Suppose instead that $(c_P^2(\bar{\alpha}))^* > 1$ holds. Since resources remain in period 2, it follows that $\bar{\alpha} \leq \gamma^* y$ holds. Therefore, it is possible to increase welfare by reducing γ and achieving full consumption-smoothing, $(c_P^2(\alpha_1))^* = 1 + (c_I^2(\alpha_1))^*$ for all α_1 . It is easy to see that incentive compatibility and nonnegativity are satisfied, contradicting the fact that m^* solves (10).

Now suppose that $(c_P^2(\bar{\alpha}))^* = 1$ holds. Then $(c_I^2(\bar{\alpha}))^* = 0$ holds, or else (just as in the previous case) we can increase welfare by reducing γ , while maintaining full consumption-smoothing. It follows that

$$(c_P^2(\alpha_1))^* - (c_I^2(\alpha_1))^* \geq 1 \tag{4.1.1}$$

for almost all α_1 . Otherwise, for a positive-measure set of realizations of α_1 , $c_P^2(\alpha_1)$ can be increased and $c_I^2(\alpha_1)$ can be reduced to satisfy the resource and nonnegativity constraints, which increases welfare and relaxes the incentive compatibility constraint. If inequality (4.1.1) is strict for a positive-measure set of realizations of α_1 , then welfare can be feasibly increased by choosing $(c_P^2(\alpha_1))^*$ and $(c_I^2(\alpha_1))^*$ to satisfy full consumption-smoothing (where (4.1.1) holds as an equality) and the resource constraint, (6). Therefore, we have for all α_1 ,

$$(c_P^2(\alpha_1))^* - (c_I^2(\alpha_1))^* = 1. \tag{4.1.2}$$

Treating γ as a parameter, and solving the equations (6) and (4.1.2) for consumptions defines welfare $W(\gamma)$ as a function of γ . Because m^* solves (10), $W(\gamma)$ must be maximized at $\gamma = \gamma^*$. Applying the envelope theorem, one can show:

$$W'(\gamma^*) = y(R_\ell - R_i) \int_0^{\bar{\alpha}} u'[(1 - \gamma^*)yR_i + (\gamma^*y - \alpha)R_\ell + \alpha - 1]f(\alpha)d\alpha < 0.$$

It follows that reducing γ improves welfare, contradicting the fact that γ^* is part of the optimal contract, m^* . \square

Proof of Theorem 4.2: An optimal contract satisfies $c^1(z) = 1$ for all $z \leq \bar{\alpha}$. Since the possibility of bank runs is in question, a patient consumer's decision to arrive in period 1 must also take into account $c^1(z)$ for all $z > \bar{\alpha}$. Let z^* be the smallest value of z , greater than or equal to $\bar{\alpha}$, such that the following inequality holds

$$c^1(z) \leq \frac{[\gamma y - \bar{\alpha} - \int_{\bar{\alpha}}^z c^1(a) da] R_\ell}{1 - z}. \quad (4.2.1)$$

If there is no value of z satisfying (4.2.1), define z^* to equal 1. From Lemma (4.1), it follows that $(c_P^2(\bar{\alpha}))^* < 1$ holds at the optimal contract. We must also have $(c_I^2(\bar{\alpha}))^* = 0$ or else higher welfare can be achieved by transferring consumption from those who arrived in period 1 to those who did not, while continuing to satisfy the constraints.¹ It follows that, setting $z = \bar{\alpha}$, the left side of (4.2.1) is equal to unity, and the right side of (4.2.1) is equal to $(c_P^2(\bar{\alpha}))^*$. Since inequality (4.2.1) is not satisfied, we have $z^* > \bar{\alpha}$.

The claim is that there is a run equilibrium, in which all consumers arrive in period 1. Those for whom $z_j < z^*$ holds accept $c^1(z_j)$, and those for whom $z_j \geq z^*$ holds refuse $c^1(z_j)$ and do not withdraw in period 1. Without loss of generality, assume $(c_I^2(\alpha_1))^* = 0$ for $\alpha_1 > \bar{\alpha}$, because giving period-2 consumption to those who withdraw in period 1 only increases the incentive to run. Thus, for $\alpha_1 > \bar{\alpha}$, second period consumption is given by

$$(c_P^2(\alpha_1))^* = \frac{[\gamma y - \bar{\alpha} - \int_{\bar{\alpha}}^{\alpha_1} c^1(a) da] R_\ell}{1 - \alpha_1}. \quad (4.2.2)$$

Differentiating with respect to α_1 in (4.2.2) yields

$$\frac{\partial (c_P^2(\alpha_1))^*}{\partial \alpha_1} = \frac{(c_P^2(\alpha_1))^* - c^1(\alpha_1) R_\ell}{1 - \alpha_1},$$

which is negative for $\alpha_1 < z^*$, since we have $c^1(\alpha_1) R_\ell > c^1(\alpha_1) > (c_P^2(\alpha_1))^*$, with the second inequality due to inequality (4.2.1) not holding. Thus $(c_P^2(\alpha_1))^*$ is decreasing in α_1 for $\alpha_1 < z^*$.

Given the acceptance/refusal behavior specified above, everyone will refuse $c^1(z^*)$, so we have $\alpha_1 = z^*$ with probability 1. By (4.2.1) and (4.2.2), it is a best response for consumer j to refuse if $z_j = z^*$, and $z_j > z^*$ is irrelevant. If $z_j < z^*$ holds, (4.2.1) and (4.2.2) imply $c^1(z_j) > (c_P^2(z_j))^* > (c_P^2(z^*))^*$, where the last inequality follows from the fact that $(c_P^2(\alpha_1))^*$ is continuous and decreasing in α_1 for $\alpha_1 < z^*$. Therefore, it is a best response for consumer j to accept $c^1(z_j)$. \square

Proof of Theorem 4.3: From Theorem (3.1), an optimal contract in the unified system satisfies $\gamma < \bar{\alpha}/y$. In the separated system, an optimal contract must invest at least enough in technology ℓ to provide 1 unit of consumption in period 2 when everyone is patient. That is, we must have $(c_P^2(0))^* \geq 1$, or else all patient consumers will choose period 1. Thus, the optimal fraction invested in technology ℓ for the separated system, γ^* , satisfies $\gamma^* \geq 1/(R_\ell y)$. Since $R_\ell \bar{\alpha} < 1$ holds, the following obtain: First, $\gamma^* > \bar{\alpha}/y$ holds, so consumers are not rationed in period 1 unless there is a run. Second, γ^* exceeds the optimal fraction invested in technology ℓ for the unified system. \square

Computed Example:²

Let $y = 10$, $\bar{u} = 20$, $\beta = 0.7$, $u(c) = 100 \log(c)$, $R_i = 1.1$, $R_\ell = 1.08$, $\bar{\alpha} = 0.5$, and α be distributed uniformly on $[0, 0.5]$.

For the unified financial system, the optimal γ is 0.04807 so $\gamma y = 0.4807 < 0.5 = \bar{\alpha}$. Hence the probability of running out of cash is $2(0.0193) = 3.86\%$ in the unified system.

For the restricted bank, the optimal γ is 0.09445, nearly double the liquidity requirement for the unrestricted bank.

The value of $c_P^2(\bar{\alpha})$ for the restricted bank is $0.96012 < 1.0$. This implies that there is a panic-based run equilibrium to the optimal contract for the restricted bank.

For the restricted bank, the optimal γ that avoids runs is 0.09630. By comparing welfare in the best contract that avoids runs and welfare in the optimal contract (that tolerates runs), one can show that the cutoff propensity to run is

¹Remember, the optimal contract is the one that provides the highest welfare, based on the equilibrium in which the patient consumers wait until period 2.

²Computations are made using Maple version 5. The code is available from the authors for purposes of replicating the results.

0.005521. If the sunspot-induced propensity to run is less than 0.5521%, then it is better to tolerate the unlikely event of a run than to increase technology ℓ investment to prevent runs. Otherwise, the bank should set $\gamma = 0.09630 > 0.09445$.