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The economic effects of restrictions on government budget deficits: imperfect private credit markets*

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Summary. The present paper is an extension of Ghiglino and Shell [7] to the case of imperfect consumer credit markets. We show that with constraints on individual credit and only anonymous (i.e., non-personalized) lump-sum taxes, strong (or "global") irrelevance of government budget deficits is not possible, and weak (or "local") irrelevance can hold only in very special situations. This is in sharp contrast to the result for perfect credit markets. With credit constraints and anonymous consumption taxes, weak irrelevance holds if the number of tax instruments is sufficiently large and at least one consumer's credit constraint is not binding. This is an extension of the result for perfect credit markets.

Keywords and Phrases: Balanced-budget amendment, Consumption taxes, Credit constraints, Government budget deficit irrelevance, Lump-sum taxes, Overlapping generations.

JEL Classification Numbers: D50, D90, E52, E60, H62, H63.

1 Introduction and summary

In Ghiglino and Shell [7], we analyzed the economic effects of constitutional or other restrictions on the government budget deficit. We assumed that private agents have access to perfect markets for borrowing and lending. We allowed for the possibility that the full range of personalized taxes might not be available to the government. Our leading case was that of anonymous taxation, in which the government must choose taxes to be the same for every member of a given generation. The main results of Ghiglino and Shell [7] are:

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(1) When lump-sum taxes are available, deficit restrictions are (strongly or "globally") irrelevant in the sense that the set of allocations the government is able to achieve as competitive equilibria is independent of the deficit restrictions. This result holds even if taxes cannot be perfectly personalized or even if they must be anonymous [1]. The intuition behind this is that the government is able to avoid any effects of the deficit restrictions by replacing explicit borrowing with what is in effect a "social security scheme" in which individuals are taxed when young and promised offsetting transfers when old. Since taxes are lump-sum and consumer credit markets are perfect, only net lifetime taxation matters. Individuals offset their early tax bills by borrowing in the credit markets. Essentially, consumers do the explicit borrowing "on behalf" of the government.
(2) When only anonymous consumption taxes are available, the situation is more complicated. Strong (or global) irrelevance of deficit restrictions is impossible because of the constraint that before-tax and after-tax prices must be nonnegative. Nonetheless, if there is a sufficient number of tax instruments, the government is able to offset local changes in the deficit restrictions while holding constant (normalized) after-tax prices and varying before-tax prices so as to leave lifetime incomes unchanged. In particular, there is weak (or "local") irrelevance of the deficit restrictions when the number of consumers per generation does not exceed the number of goods per period [2].

The assumption of perfect private borrowing and lending markets is very strong in this context. In the real world, some consumers do face binding credit constraints or other imperfections in the borrowing market. If government borrowing is restricted but private borrowing is unconstrained, then the government can ease the effects of its own borrowing restrictions by in effect "borrowing off the books" by increasing the early-life taxes on some individuals while at least partially offsetting this by increasing late-life subsidies to the same individuals. It is natural therefore to ask how private credit constraints affect the government's ability to avoid restrictions on its deficit.

In the present paper, we assume that the government *and* individuals face credit restrictions. The restrictions on the government are from the constitution or other law, or from international borrowing agreements. The reasons for private credit constraints include imperfect collateral and other "moral hazards". The underlying sources of private credit rationing will not be analyzed here. We simply assume that there are (exogenously given) private credit constraints which possibly differ across individuals.

Following Ghiglino and Shell [7], we employ a pure-exchange overlapping-generations model with several consumers per generation and several commodities per period. We allow for lump-sum taxes and consumption taxes. We also allow for the fact that tax schedules cannot be made perfectly individual-specific. In this paper, we focus for simplicity on the perfectly anonymous case in which each consumer from the same generation faces the same tax situation. Finally, we also assume for simplicity that each commodity can be taxed at its own rate. In Ghiglino and Shell

[1] See Ghiglino and Shell [7], Proposition 5.
[2] See Ghiglino and Shell [7], Proposition 12.

[7], the situation is somewhat more general: The number of commodity tax classes is less than or equal to the number of commodities, while the number of consumer tax classes is less than or equal to the number of consumers per generation.

We use the classic economic definitions [3] of *relevance* and *irrelevance* applied in this case to government-budget-deficit restrictions. The government-budget-deficit restriction is said to be *irrelevant* if the set of achievable equilibrium allocations is unaffected by the restriction. Otherwise, the restriction is said to be *relevant*. Of course, saying that the restriction is irrelevant is not saying that the restriction does not matter. If the restriction either directly or indirectly affects expectations in such a manner that it affects the selection of the equilibrium, then the restriction does matter.

As one would expect, irrelevance of deficit restrictions is less likely with individual credit constraints than in the case with perfect consumer credit markets. We show in the present paper:

(1) If private credit is constrained but these constraints are not binding on any individual and the only tax instruments are anonymous lump-sum taxes, then there is weak (local) irrelevance of the government budget restrictions. If some credit constraints are binding, then with only anonymous lump-sum taxes even local irrelevance is unlikely or impossible.
(2) If there are private credit constraints, the case with only anonymous consumption taxes is more interesting. Surprisingly, consumption taxes, although distorting, are more likely to provide (at least some form of) irrelevance than lump-sum taxes. With credit constraints and anonymous consumption taxes, there is weak (local) irrelevance if the number of tax instruments is sufficiently large and at least one consumer's credit constraint is not binding. This generalizes the result for the case with no private credit constraints. In particular, we show that if in all periods the number of commodities is no less than the sum of the number of individuals *plus* the number of individuals for whom credit rationing is binding, then the deficit restriction is weakly (locally) irrelevant [4]. In moving from the unconstrained credit case to the constrained case, the minimal number of tools for weak irrelevance is increased by the number of constrained consumers. To undo the effects of the local change in the deficit restriction, the government has to maintain not only the lifetime incomes of all consumers but also the first period incomes of the constrained consumers.

Why is weak (i.e. local) budget-deficit irrelevance more likely with anonymous consumption taxes than with anonymous lump-sum taxes? With only anonymous lump-sum taxation, if taxes must be increased on the young in order to reduce the deficit, constrained consumers will typically have to decrease their early-life consumptions. This means that the deficit restriction is relevant. With anonymous consumptions taxes, the government will be able to accommodate local changes in the deficit restriction if there are sufficiently many types of commodities to tax. This is because, by altering tax rates, the government is able to affect early-life incomes,

[3] See Barro [6]. See also Ghiglino and Shell [7] and the references therein.

[4] If none of the private credit constraints are binding, this inequality reduces (as it should) to the one in Proposition 12 of Ghiglino and Shell [7].

late-life incomes, and the rates of commodity substitution, while generating the necessary revenue.

We choose, because we consider it most natural, to work within the monetary, overlapping-generations framework. If we had adopted real taxes, none of our analysis would have been altered. We see no reason that our basic results would be affected by going to infinite-lived agents, but since agent heterogeneity is essential to our analysis, the infinite-lifetime model would have been more clumsy to work with.

2 The model

We employ a pure-exchange overlapping-generations model in which there are n different consumers per generation and ℓ perishable commodities per period. We suppose that consumers live for two periods. The government collects taxes, distributes transfers (negative taxes), and finances government consumption. We focus on two types of government instruments: lump-sum taxes and consumption taxes. We assume that lump-sum taxes and consumption tax rates must be the same for every member of a given generation, but that consumption taxes can vary freely over the l commodities. For the general case, see Ghiglino and Shell [7], which allows for more general consumer tax classes and more general commodity tax classes. We assume that government consumption of commodities is exogenously determined. It is denoted by the sequence $g = (g^1, \dots, g^t, \dots)$ with $g^t \in \mathbb{R}^l_+$ for $t = 1, 2, \dots$. It is assumed that use of capital markets is constrained, viz. each individual faces exogenously given constraints on his borrowing.

Our set-up is based on the Samuelson [10] overlapping-generations model presented in Balasko and Shell [2,3,4], but new tax instruments and the individual credit constraints must be defined. As in Balasko and Shell [3], let $m^s_{th} \in \mathbb{R}$ be the lump-sum money transfer to consumer h of generation t in period s; if m^s_{th} is negative, then the consumer is paying a lump-sum tax. Following Ghiglino and Shell [7], we let $\tau^{si}_{th} \in \mathbb{R}$ be the present tax rate levied on consumer h of generation t on his consumption of commodity i in period s.

Let $x^s_{th} = (x^{s1}_{th}, \dots, x^{si}_{th}, \dots, x^{sl}_{th}) \in \mathbb{R}^\ell_{++}$ be the vector of consumption in period s by individual h of generation t and $\omega^s_{th} = (\omega^{s1}_{th}, \dots, \omega^{si}_{th}, \dots, w^{s\ell}_{th}) \in \mathbb{R}^\ell_{++}$ be the vector of commodity endowments in period s of individual h from generation t for $t = 0, 1, \dots, s = 1, 2, \dots,$ and $h = 1, \dots, n$. Let $m^s_t \in \mathbb{R}$ be the money transfer in period s to each consumer from generation t, and $\tau^s_t = (\tau^{s1}_t, \dots, \tau^{si}_t, \dots, \tau^{sl}_t) \in \mathbb{R}^\ell$ be the vector of anonymous consumption tax rates in period s for consumers from generation t. Consumers from generation 0 are alive in period 1, while consumers from generation t ($t = 1, 2, \dots,$) are alive in periods t and $t + 1$. Hence it is convenient to define the following vectors:

$$x_{0h} = x^1_{0h} \in \mathbb{R}^\ell_{++}, \quad x_{th} = (x^t_{th}, x^{t+1}_{th}) \in \mathbb{R}^{2\ell}_{++},$$

$$\omega_{0h} = \omega^1_{0h} \in \mathbb{R}^\ell_{++}, \quad \omega_{th} = (\omega^t_{th}, \omega^{t+1}_{th}) \in \mathbb{R}^{2\ell}_{++},$$

$$m_0 = m^1_0 \in \mathbb{R}, \qquad m_t = (m^t_t, m^{t+1}_t) \in \mathbb{R}^2,$$

and

$$\tau_0 = \tau_0^1 \in \mathbb{R}^\ell, \quad \tau_t = (\tau_t^t, \tau_t^{t+1}) \in \mathbb{R}^{2\ell}.$$

Let $p^s = (p^{s1}, \ldots, p^{si}, \ldots, p^{s\ell}) \in \mathbb{R}_{++}^\ell$ be the vector of present (before-tax) prices for commodities available in period s and let

$$q_t^s = (q_t^{s1}, \ldots, q_t^{si}, \ldots, q_t^{s\ell}) \in \mathbb{R}_{++}^\ell$$

be the present after-tax vector of commodity prices facing consumers of generation t in period s. Define the after-tax present price vectors facing consumers of generation $t = 0, 1, 2, \ldots,$ by

$$q_0 = q_0^1 = p^1 + \tau_0 \in R_{++}^\ell \text{ for } t = 0$$

and

$$q_t = (q_t^t, q_t^{t+h}) = (p^t, p^{t+1}) + (\tau_t^t, \tau_t^{t+1}) \in R_{++}^{2\ell} \text{ for } t = 1, 2, \ldots . \quad (2.1)$$

Then define the following quantity and price sequences: $x = ((x_{0h})_{h=1}^{h=n}, \ldots, (x_{th})_{h=1}^{h=n}, \ldots)$, $\omega = ((\omega_{0h})_{h=1}^{h=n}, \ldots, (\omega_{th})_{h=1}^{h=n}, \ldots)$, $p = (p^1, \ldots, p^t, \ldots)$, $m = (m_0, \ldots, m_t, \ldots)$, $\tau = (\tau_0, \ldots, \tau_t, \ldots)$ and $q = (q_0, \ldots, q_t, \ldots)$.

We assume that the preferences of consumer h from generation t can be described by the utility function u_{th} defined over the consumption set of all strictly positive x_t's (i.e. $\mathbb{R}_{++}^\ell$ or $\mathbb{R}_{++}^{2\ell}$) with the properties:

(i) u_{th} is twice differentiable with strictly positive first-order derivatives and with corresponding negative definite Hessian

and

(ii) the closure of every indifference surface of u_{th} is in the consumption set (i.e. $\mathbb{R}_{++}^\ell$ or $R_{++}^{2\ell}$).

These rather standard assumptions simplify the comparative statics [5]. Note that we have also assumed that the endowment of the consumer lies in his consumption set, i.e. we have ω_{th} is in $\mathbb{R}_{++}^\ell$ or $\mathbb{R}_{++}^{2\ell}$.

Let $b_{th}^s \in \mathbb{R}_+$ be the maximum credit in money units available in period s to consumer h from generation t. The behavior of consumer h $(h = 1, 2, \ldots, n)$ from generation t $(t = 1, 2, \ldots)$ is then described by

$$\begin{aligned}
&\text{maximize } u_{th}(x_{th}^t, x_{th}^{t+1})\\
&\text{subject to}\\
&q_{th}^t \cdot x_{th}^t + x_{th}^{tm} &&= p^t \cdot \omega_{th}^t + m_{th}^t,\\
&q_{th}^{t+1} \cdot x_{th}^{t+1} + x_{th}^{t+1,m} &&= p^{t+1} \cdot \omega_{th}^{t+1} + m_{th}^{t+1}, \qquad (2.2)\\
&x_{th}^{tm} &&\geq -b_{th}^t,\\
&\text{and}\\
&x_{th}^{tm} + x_{th}^{t+1,m} &&= 0,
\end{aligned}$$

[5] See Balasko [1]. See Balasko and Shell [2,3] for their application in overlapping generations models.

where $x_{th}^{sm} \in \mathbb{R}$ is the gross addition to money holding in period s by consumer h of generation t. The last equation in (2.2) is the requirement that the consumer's indebtedness be zero in his final period of life. The borrowing constraint is not binding on consumer th if in equilibrium $x_{th}^{tm} > -b_{th}^{t}$. The inequality in (2.2) is the credit constraint. We have implicitly assumed in writing (2.2) that the borrowing constraint of at least one consumer is not binding, so that we can use the usual no-arbitrage argument to establish that the present price of money is constant, i.e.,

$$p^{t,m} = p^{t+1,m} = p^m \in \mathbb{R}_+ \tag{2.3}$$

where $p^{s,m} \in \mathbb{R}_+$ is the present price of money in period $s = 1, 2, \cdots$. Assuming that the economy is in proper monetary equilibrium, we can set $p^m = 1$.[6]

The nominal (coupon) rate of interest on money is assumed without loss of generality to be zero.[7] Hence the only return on holding money is the capital gain relative to commodities. Condition (2.3) is thus that money appreciate in value relative to any commodity at the commodity rate of interest. For consumers for which the credit restriction is not binding, Condition (2.3) allows us to rewrite (2.2) somewhat as Balasko and Shell [3] to yield

$$\begin{aligned}
&\text{maximize } u_{th}(x_{th}^{t}, x_{th}^{t+1}) \\
&\text{subject to} \\
&q_t^t \cdot x_{th}^t + q_t^{t+1} \cdot x_{th}^{t+1} \\
&= p^t \cdot \omega_{th}^t + p^{t+1} \cdot \omega_{th}^{t+1} + m_t^t + m_t^{t+1}
\end{aligned} \tag{2.4}$$

for $h = 1, 2, \ldots, n$ and $t = 1, 2, \ldots$, where by choice of numeraire we set $q_{01}^1 = 1$. The transfers $m_t = (m_t^t, m_t^{t+1}) \in \mathbb{R}^2$ affect the behavior of the consumer only through the lifetime transfer $\mu_t = m_t^t + m_t^{t+1} \in \mathbb{R}$.

It remains to describe the behavior of the older generation $(t = 0)$ in period 1. Consumer 0h maximizes his utility subject to his one-period budget constraint:

$$\begin{aligned}
&\text{maximize } u_{0h}(x_{0h}^1) \\
&\text{subject to} \\
&q_0^1 \cdot x_{0h}^1 + x_{0h}^{1m} = p^1 \cdot \omega_{0h}^1 + m_0^1 \\
&x_{0h}^{1m} = 0, \\
&\text{and} \\
&x_{0h}^1 \in \mathbb{R}_{++}^{\ell}.
\end{aligned} \tag{2.5}$$

[6] Strictly speaking, setting $p^m = 1$ is not without loss of generality. We know, however, that we can reconstruct the full set of perfect-foresight equilibria by using the absence-of-money-illusion property.

[7] This is because the super-neutrality of money.

3 Fiscal policy

We assume in this paper that the government has at its disposal either anonymous lump-sum taxation *or* anonymous consumption taxation. Thus, the government's fiscal policy is either the sequence of anonymous lump-sum transfers m or the sequence of the consumption tax rates τ.

Let d^t be the present commodity value (and also the dollar value) of the government budget deficit incurred in period t. Hence we have for the case of lump-sum taxation

$$d^t = p^t g^t + n\left(m_{t-1}^t + m_t^t\right)$$

for $t = 1, 2, \ldots$, where n is the number of consumers per generation. For the case of consumption taxes

$$d^t = p^t g^t - \sum_{h=1}^{n} \sum_{i=1}^{l} \left(\tau_{t-1}^{ti} x_{t-1,h}^{ti} + \tau_t^{ti} x_{th}^{ti}\right)$$

for $t = 1, 2, ...$ Let d denote the sequence $(d^1, \ldots, d^t, \ldots)$. Let δ^t be the present value (and money value) of the constitutionally imposed deficit restriction (assumed for convenience in the form of an equality) in period t. Let δ denote the sequence $(\delta^1, \ldots, \delta^t, \ldots)$. The budget deficit restriction is then

$$d = \delta.$$

According to the previous definition, the deficit is denominated in money, or equivalently in Arrow-Debreu units of accounts. Our results do not depend on this convention: They still hold if the deficit restrictions are expressed in real terms.

4 Equilibrium

We maintain throughout this paper some strong assumptions. We suppose perfect-foresight on the part of consumers and the government. We also suppose that the government is able to perfectly commit to its announced fiscal policy.

Next we define equilibrium in the economy with taxes.

Definition *Given the sequence of endowments ω, the feasible fiscal policy m or τ, the exogenous consumption g, the behavior of consumers described by the systems (2.2), (2.4) and (2.5), the numeraire choice yielding $p^{11} = 1$, the (further) monetary normalization yielding $p^m = 1$ and the deficit-restriction sequence δ, a constitutional competitive equilibrium is defined by a positive price sequence p and the allocation sequence x such that markets clear, so that we have*

$$g^t + \sum_{h=1}^{h=n} (x_{t-1,h}^t + x_{t,h}^t) = \sum_{h=1}^{h=n} (\omega_{t-1,h}^t + \omega_{t,h}^t)$$

for $t = 1, 2, \ldots$, and the deficit restriction $d = \delta$ is satisfied.

From Balasko and Shell [2], one might expect that the existence of competitive equilibrium to be guaranteed in "nice" overlapping-generation models, but this does not extend to our Definition. There are three reasons that competitive equilibrium as defined above could fail to exist. The first reason is because we are seeking a *proper* monetary equilibrium, one for which the price of money is strictly positive. For a proper monetary equilibrium to exist the fiscal policy must be bonafide[8]. The second reason applies only to commodity taxation. It might not be possible to equilibrate supply and demand while maintaining the positivity of the two price sequences p and q. The third reason is that equilibrium may fail to exist because of excessive government consumption.

When the model is stationary, i.e., preferences, endowments, and government consumption are constant across generations, one is tempted to focus on equilibria in which allocations are constant across periods. We provide separate definitions of the steady state for the two tax regimes.

Definition L (Steady state with lump-sum taxes). *Let* $p = (p^1, \cdots, p^t, \cdots) \in (\mathbb{R}^l_{++})^\infty$ *be the equilibrium sequence of commodity prices when the fiscal policy is given by the sequence of lump-sum transfers* $(m_0^1, m_1^1, m_1^2, \cdots, m_t^t, m_t^{t+1}, \cdots)$. *These describe a steady-state equilibrium if there is a scalar* $\beta \in \mathbb{R}_{++}$ *such that*

$$\begin{aligned} p^t &= \beta^{t-1}\mathbf{p} \\ m_t^t &= \beta^{t-1}\mathbf{m}^1 \quad \text{and} \\ m_t^{t+1} &= \beta^t \mathbf{m}^0 \end{aligned}$$

for $t = 1, 2, \cdots$, *where* $\mathbf{p} = p^1, \mathbf{m}^1 = m_1^1$ *and* $\mathbf{m}^0 = m_0^1$.

Definition C (Steady state with consumption-taxes). *Let* $p = (p^1, \cdots, p^t, \cdots) \in (\mathbb{R}^l_{++})^\infty$ *be an equilibrium vector of before-tax commodity prices when the fiscal policy is given by the sequence of consumption taxes* $(\tau_0^1, \tau_1^1, \tau_1^2, \cdots, \tau_t^t, \tau_t^{t+1}, \cdots) \in (\mathbb{R}^l)^\infty$. *These describe a steady-state equilibrium if there is a scalar* $\beta \in \mathbb{R}_{++}$ *such that*

$$\begin{aligned} p^t &= \beta^{t-1}\mathbf{p} \\ \tau_t^t &= \beta^{t-1}\tau^1 \quad \text{and} \\ \tau_t^{t+1} &= \beta^t \tau^0 \end{aligned}$$

for $t = 1, 2, \cdots$, *where* $\mathbf{p} = p^1, \tau^1 = \tau_1^1$ *and* $\tau^0 = \tau_0^1$.

When focusing on steady states it makes sense to focus on budget deficits $d = (d^1, \cdots, d^t, \cdots)$ that are constant in current terms, so that we have for a scalar $\mathbf{d} \in \mathbb{R}$

$$d^t = \beta^{t-1}\mathbf{d}$$

for $t = 1, 2, \ldots$ From Walras's law and market clearing, we have the two steady-state relations:

$$(\beta - 1)\left[\sum_{h=1}^{n} \left(\mathbf{p}\left(x_h^1 - \omega_h^1\right)\right) - n\mathbf{m}^1\right] + \mathbf{d} = 0 \tag{SS-L}$$

[8] See Balasko and Shell [3,4,5] and Ghiglino and Shell [7].

for lump-sum taxation, and

$$(\beta - 1)\sum_{h=1}^{n}\left[\mathbf{p}\left(x_h^1 - \omega_h^1\right) + \sum_{i=1}^{l}\tau^{1i}x_h^{1i}\right] + \mathbf{d} = 0 \qquad \text{(SS-C)}$$

for consumption taxation, where $\tau^{1i} \in \mathbb{R}$ is the *i*th component of τ^1. The steady-state conditions (SS-L) and (SS-C) do not directly involve government consumption, but steady-state $\mathbf{g} = g^t$ for $t = 1, 2, \cdots$ is implied through the equilibrium allocations and prices. If $d = 0$, from (SS-L) and (SS-C), we have the familiar OG steady-state result that either the interest rate is zero ($\beta = 1$) or aggregate savings is zero. Our aim is to find conditions under which the government is able to "avoid" the restrictions on its deficit with changing neither its own consumption nor the consumption of any private consumer. When this is possible the deficit restriction is said to be $irrelevant$. We recall the formal definitions given in Ghiglino and Shell [7].

Definition (Irrelevance of the deficit restriction). *Let g be government consumption sequences and let x be an allocation that can be implemented as a competitive equilibrium with some feasible fiscal policy m (resp. τ) and with the resulting deficits given by the sequence d. The deficit restriction $\mathrm{d} = \delta$ is said to be irrelevant if for any other deficit restriction sequence δ' there exists a feasible fiscal policy m (resp. τ) that implements the allocation x as a competitive equilibrium and is compatible with g, but with the resulting deficit given by the sequence δ'.*

The above notion of irrelevance is very strong because it involves any possible deficit sequence other than the pre-reform, or baseline, deficit d. In many situations, this type of irrelevance does not obtain. Following Ghiglino and Shell [7], we employ a weaker notion of irrelevance. The first characteristic of the weaker deficit restriction is that it is based on finite but arbitrarily long time horizons. Consider the time horizon $T = 1, 2, \ldots$. Then define $\delta(T) = (\delta^1, \delta^2, \ldots, \delta^t, \ldots, \delta^T) \in \mathbb{R}^T$ as a deficit restriction of (finite) length T. For a competitive equilibrium to be weakly constitutionally feasible the deficit in period t, d^t, must be equal to δ^t if $t = 1, 2, \ldots, T$, while for $t > T$, the deficit is unrestricted. The second characteristic of the weaker deficit restriction is that only restrictions "near" the base-line deficit are considered, i.e., only period-by-period deficits that are not too different from the baseline deficits are considered. In other words, only a neighborhood (in any topology, since T is finite) of the original sequence is considered. According to the weaker notion of irrelevance only restrictions of finite length, $\delta(T)$, that belong to a first T-period neighborhood of the base deficit vector $d = (d^1, d^2, \ldots, d^t, \ldots, d^T, \ldots)$, denoted $\mathcal{D}^T(d)$, are considered.

Definition (Weak irrelevance of the deficit restriction). *Let g be the sequence of government consumptions and let x be an allocation that can be implemented as a competitive equilibrium with some feasible fiscal policy m (resp. τ) and with the resulting deficits given by the sequence d. A deficit restriction is said to be weakly irrelevant if $for\ any$ positive integer T there is a set $\mathcal{D}^T(d)$ such that for all $\delta \in \mathcal{D}^T(d)$ there is a fiscal policy $m^{'}$ (resp. $\tau^{'}$) that implements the allocations x and g, but with the resulting deficit given by the sequence δ.*

Note that the time horizon of the deficit specification is arbitrary: For every T (no matter how large) there must be a neighborhood $\mathcal{D}^T(d)$. These neighborhoods will typically depend on T. The limit of these neighborhood as T goes to infinity might be empty.

5 Relevance of government budget deficit restrictions with lump-sum taxes

The relevance of government deficit restrictions is first investigated in economies in which some consumers face credit constraints and only lump-sum taxation is available. It is shown that deficit restrictions are likely to be relevant unless the government can use non-anonymous taxes. In the leading example considered below, the government uses only anonymous lump-sum taxes and transfers and thus has no way to treat differently the consumers, so that the likely outcome is that some consumers are hurt or benefited by the change to a fiscal scheme that respects the government deficit restrictions.

For overlapping-generations economies with perfect borrowing markets and lump-sum taxes and transfers, restrictions on the government budget have no impact on the set of equilibrium allocations (see Ghiglino and Shell [7], Proposition 5). The reason for this is that in these economies only the present value of taxes and transfers, not their timing, matters to consumers. In this case, the government can "borrow off the books" from taxpayers by adjusting the timing of individual taxes and transfers.

When credit restrictions are included, weak (or local) irrelevance of the budget deficit restriction obtains if non-anonymous taxes can be personalized to some consumer whose constraint is not binding. In this case, the government respects the market constraints using personalized taxes. Being able to personalize taxes is not always possible. If this is not possible, then matters dramatically change. This is illustrated in the following example. For simplicity, a stationary equilibrium is considered. This amounts to ignoring the transition path. In other words, in this example we will assume that a suitable money transfer is made so that the economy "starts" at the steady state and only deficit specifications from $t = 2$ onward are considered.[9]

Example (Relevance of government deficit restrictions when consumer credit is constrained). Let the economy be stationary with one commodity per period and two consumers. Perfectly anonymous lump-sum taxation is available. No other tax instruments are available. The two consumers, 1 and 2, have log-linear utility functions

$$u_{th} = 1/2 \log x_{th}^t + 1/2 \log x_{th}^{t+1}$$

for $h = 1, 2$ and $t = 1, 2, \ldots$. Endowments are given by

[9] Another, equivalent, way to view steady states is to consider a model with no beginning as well as no end (see Ghiglino and Tvede [8]).

$$\omega_{t1} = (\omega_{t1}^{t}, \omega_{t1}^{t+1}) = (1, 20),$$

and

$$\omega_{t2} = (\omega_{t2}^{t}, \omega_{t2}^{t+1}) = (0.75, 1).$$

Consider generation t. Suppose that the borrowing of consumer 1 is unconstrained, $b_{t1}^{t} = \infty$, but that consumer 2 cannot borrow, $b_{t2}^{t} = 0$. At a steady state the individual demands of the consumers depend on the interest factor β. When the credit constraint is not binding, the demands must satisfy

$$x_{t1}^{t} = \frac{1+20\beta}{2},$$

$$x_{t1}^{t+1} = \frac{1+20\beta}{2\beta},$$

$$x_{t2}^{t} = \frac{0.75+\beta}{2},$$

and

$$x_{t2}^{t+1} = \frac{0.75+\beta}{2\beta}.$$

For $\beta \geq 0.75$, we have

$$\left(x_{t2}^{t}, x_{t2}^{t+1}\right) = (0.75, 1)$$

because then the credit restriction is binding.

Suppose that $g^{t} = 1$ for $t = 1, 2, \ldots$ Without credit restrictions $\beta = 0.89498$ and $\beta = 0.093112$ solve the equilibrium equations, but $\beta = 0.89498$ is not an equilibrium interest factor because the borrowing constraint for consumer 2 is violated. The steady-state equilibrium interest factor is $\beta = 0.89408$ and the corresponding equilibrium allocations are

$$x_{t1} = (x_{t1}^{t}, x_{t1}^{t+1}) = (9.4408, 10.5592)$$

and

$$x_{t2} = (x_{t2}^{t}, x_{t2}^{t+1}) = (0.75, 1).$$

Since the government is not taxing any consumer, the associated deficit is $d^{t} = p^{t} g^{t} = p^{t} = (0.89408)^{t-1}$ in present real or money units.

Suppose now that the government is required to balance its budget in every period, so that $d^{t} = \delta^{t} = 0$ for $t = 2, 3, \ldots$. We will show that the new restriction on the deficits leads to a modification of the existing allocation. First, note that in order to keep unchanged the consumption of consumer 1, β should be unchanged at

$\beta = 0.89408$. Now, the government can either tax the young or tax the old. Suppose first that consumers are taxed in their youth and receive transfers in their old age. The procedure is similar to that used in the proof of Proposition 5 in Ghiglino and Shell [7]. Since the consumers of the same generation are perfectly anonymous for tax purposes, suppose that we tax each young equally with a lump-sum tax $-m_2^2 > 0$ and no tax on the consumers born in the first period, $m_1^2 = 0$. The government budget constraint is then $\beta^{t-1}g^t + 2m_2^2 = 0$ so that $m_{21}^2 = m_{22}^2 = -\beta/2$ (or $1/2$ in current terms) in order that the deficit be zero, $d^2 = 0$. Note in the next period, these same consumers have to be compensated by a positive transfer of $\beta/2$ in present terms, or $1/(2\beta)$ in current terms. After the transfer, the endowments (in current terms) of consumer 2 are $(0.75 - 0.5, 1 + 0.5\beta)$. At $\beta = 0.89408$, consumer 2 would still like to borrow. However, due to the borrowing constraint his first period consumption is now 0.25. The equilibrium allocation has been affected by the fiscal policy. The other possibility is to subsidize the young and tax the old. A similar reasoning shows that also in this case the fiscal policy affects the equilibrium allocation. Therefore, the deficit sequence is relevant. □

The previous example suggests that when some consumers are credit-constrained, anonymous lump-sum taxes are not powerful enough to achieve irrelevance of the government budget deficit. This is generalized in the following

Proposition (Relevance of the government deficit restrictions with credit constraints). *Let the allocation x be implemented as a constitutional competitive equilibrium with a fiscal policy consisting only of lump-sum taxes and transfers compatible with the deficit restriction δ. If at least one consumer's credit constraint is binding then the deficit sequence δ is weakly (and strongly) relevant. Otherwise, it is weakly irrelevant.*

Proof. If no consumer's credit constraint is binding, then Proposition 5 in Ghiglino and Shell [7] applies. However, in general since the government is employing only anonymous taxation, any transfer changes the consumer's actual borrowings (or savings) and therefore affects his demand for commodities. Indeed, assume there are two consumers and that $h = 2$ is the consumer whose credit constraint is binding (if the credit constraint of consumer 1 is also binding then the deficit restriction is obviously relevant). Consider consumer 2 first. His demands are the solutions to the problem

$$\begin{aligned}
&\text{maximize } u_{t2}(x_{t2}^t, x_{t2}^{t+1})\\
&\text{subject to}\\
&p^t \cdot x_{t2}^t + x_{t2}^{tm} && = p^t \cdot \omega_{t2}^t + m_t^t,\\
&p^{t+1} \cdot x_{t2}^{t+1} + x_{t2}^{t+1,m} && = p^{t+1} \cdot \omega_{t2}^{t+1} + m_t^{t+1},\\
&x_{t2}^{tm} && = -b_{t2}^t,\\
&\text{and}\\
&x_{t2}^{tm} + x_{t2}^{t+1,m} && = 0.
\end{aligned} \tag{5.1}$$

The sum of the transfers made to the two consumers of type 2 in period t are

$$m^t_{t-1} + m^t_t = p^t \cdot (x^t_{t2} - \omega^t_{t2}) + p^t \cdot (x^t_{t-1,2} - \omega^t_{t-1,2}) - b^t_{t2} + b^t_{t-1,2}$$

On the other hand, because transfers are anonymous the government budget deficit is

$$d^t = 2(m^t_t + m^t_{t-1}) + p^{t,1} g^t.$$

Then we obtain

$$\delta^t = d^t = 2p^t \cdot (x^t_{t2} - \omega^t_{t2} + x^t_{t-1,2} - \omega^t_{t-1,2}) - 2b^t_{t2} + 2b^t_{t-1,2} + p^{t,1} g^t.$$

However, in order to keep consumer 1 unaffected by the fiscal policy, $p^t/p^{t-1,1}$ should be unchanged for all t. Since $p^{1,1} = 1$, the entire sequence of prices is predetermined as is the deficit sequence δ. □

Remark. If the government were able to use personalized lump-sum taxes, then the deficit sequence δ would be *weakly irrelevant*. To show this, renumber the consumers so that consumer 1 is the unconstrained consumer and reproduce the proof of Proposition 5 in Ghiglino and Shell [7].

6 Restoring irrelevance with consumption taxes

In this section we assume that only anonymous taxes on consumption are available. The question is then whether the government is able to "avoid" the deficit restriction with these instruments even though some consumers are credit constrained. As in the case with unconstrained borrowing and lending, the answer depends on the number of tax instruments compared to the number of goals (consumers) and on the duration (in periods) of the restriction. We start with an example.

Example (Irrelevance of deficit restrictions in an economy with several tax instruments). Consider a stationary, overlapping-generations economy with four commodities per period ($\ell = 4$) and two consumers per generation ($n = 2$). Assume that the second consumer faces credit restrictions while the other has free access to the credit market. The government has a constant consumption of (only) the first good, $g^t = (g^{t1}, g^{t2}, g^{t3}, g^{t4}) = (3, 0, 0, 0)$. Preferences and endowments of consumer th are given by:

$$u_{th}(x^t_{th}, x^{t+1}_{th}) = \sum_{k=1}^{4} \alpha_{kh} \log x^{tk}_{th} + \sum_{k=1}^{4} \beta_{kh} \log x^{t+1k}_{th}$$

and

$$\omega_{th} = (\omega^{0k}_h, \omega^{1k}_h)^4_{k=1},$$

where

$$(\alpha_{kh})_{h=1,2,}^{k=1,\ldots,4} = \begin{bmatrix} 1/8 & 5/8 & 1/8 & 1/8 \\ 1/8 & 4/7 & 1/7 & 9/56 \end{bmatrix}, (\beta_{kh})_{h=1,2}^{k=1,\ldots,4} = \begin{bmatrix} 1/4 & 1/4 & 1/5 & 6/20 \\ 1/5 & 1/4 & 1/4 & 6/20 \end{bmatrix}$$

and

$$\omega_1^{01} = 300, \omega_1^{13} = \omega_1^{14} = \omega_2^{13} = 200, \omega_2^{14} = 230, \omega_2^{12} = 120, \omega_2^{01} = 250,$$
$$\omega_1^{12} = 500,\ \omega_2^{02} = 1000,\ \text{and}\ \omega_i^{jk} = 100 \text{ for all other } h, j, k.$$

With anonymous consumption taxes, we have

$$\tau_{t-1,1}^{tk} = \tau_{t-1,2}^{tk} = \tau_{t-1}^{tk} \text{ and } \tau_{t1}^{tk} = \tau_{t2}^{tk} = \tau_t^{tk} \text{ for } k = 1, .., 4 \text{ and } t = 1, 2, \ldots .$$

For convenience, we look at steady state competitive equilibrium. As noted earlier, we will only consider the periods from $t = 2$ onward. First, we assume that the government finances its consumption by running a deficit, i.e. we look at a steady state with $\tau_t^{tk} = 0$ and $\tau_{t-1}^{tk} = 0$. By restricting our attention to prices of the form $p^{tk} = (\beta)^{t-1}\mathbf{p}^k, k = 1, \ldots, 4$, it can be shown that the following set of allocations and prices represents a steady state

$$\mathbf{p}^2 = 1.06975, \mathbf{p}^3 = 1.30350, \mathbf{p}^4 = 1.51932, \beta = 1.03351$$

and

$$\begin{aligned}
(x_1^{0k})_{k=1}^4 &= (120.5557, 563.4745, 92.4864, 79.3485),\\
(x_1^{1k})_{k=1}^4 &= (233.2935, 218.0816, 143.1801, 184.2615),\\
(x_2^{0k})_{k=1}^4 &= (154.2905, 659.3371, 135.2762, 130.5673),\\
(x_2^{1k})_{k=1}^4 &= (238.8603, 279.1068, 229.0573, 235.8228).
\end{aligned}$$

The associated deficit is $p^{t1} = 3(\beta)^{t-1}\mathbf{p}^1 = 3(\beta)^{t-1} = 3(1.03351)^{t-1}$ in present real (and money) terms. In current units, the savings are -275.1889 for consumer 1, 367.7109 for consumer 2 producing aggregate savings of 92.5221.

The issue is whether $(\tau_{t-1}^t, \tau_t^t)_{k=1}^4$ can be used in order to meet the deficit requirement $\delta^t = 0$ in period *t* without disturbing these allocations. Such a tax scheme must at least satisfy for each t $(t = 2, 3, \ldots)$ the following equations

$$x_{t-1,1}^{tk} = (1 - \alpha_1)\frac{\beta_{k1}W_{t-11}}{p^{tk} + \tau_{t-1}^{tk}} = x_1^{0k},$$

$$x_{t1}^{tk} = \alpha_1 \frac{\alpha_{k1}W_{t1}}{p^{tk} + \tau_t^{tk}} = x_1^{1k}, \tag{6.1}$$

$$x_{t-1,2}^{tk} = (1 - \alpha_2)\frac{\beta_{k2}W_{t-12}}{p^{tk} + \tau_{t-1}^{tk}} = x_2^{0k},$$

and

$$x_{t2}^{tk} = \alpha_2 \frac{\alpha_{k2}W_{t2}}{p^{tk} + \tau_t^{tk}} = x_2^{1k},$$

where

$$W_{th} = \sum_{k=1}^{4} p^{tk}\omega_h^{0k} + \sum_{k=1}^{4} p^{t+1k}\omega_h^{1k}$$

and

$$\sum_{k=1}^{4}(x_1^{1k} + x_2^{1k})\tau_{t-1}^{tk} + \sum_{k=1}^{4}(x_1^{0k} + x_2^{0k})\tau_t^{tk} + 3p^{t1} = 0.$$

A natural candidate for a solution to the first four equations of (6.1) is of the form $p^{tk} = (\beta)^{t-1}\mathbf{p}^k$, $\tau_t^{tk} = (\beta)^{t-1}\tau^{0k}$, and $\tau_{t-1}^{tk} = (\beta)^{t-1}\tau^{1k}$, where $\beta \in \mathbb{R}$ is the interest factor, $\tau^{0k} \in \mathbb{R}$ is the present and nominal value of the tax rate on the young and $\tau^{1k} \in \mathbb{R}$ is the present and nominal value of the tax rate on the old.

In this example, consumer 2 is assumed to face a binding credit constraint while the government cannot personalize his taxes. Then it is required that his saving or borrowing should remain exactly as it was in the untaxed situation. The budget equation for this consumer when young is

$$\sum_{k=1}^{4}(p_t^k + \tau_{t-1}^{tk})x_2^{0k} - x_{t2}^m = \sum_{k=1}^{4} p_t^k\omega_2^{0k}.$$

In the initial situation with no taxes, this yields at the steady state

$$\sum_{k=1}^{4}(\beta^{t-1}\mathbf{p}^k x_2^{0k}) - x_{t2}^m = \sum_{k=1}^{4}\beta^{t-1}\mathbf{p}^k\omega_2^{0k},$$

which can be rewritten as

$$\beta^{t-1}\left(\sum_{k=1}^{4}\mathbf{p}^k(x_2^{0k} - \omega_2^{0k})\right) = x_{t2}^m$$

or as

$$x_{t2}^m = \beta^{t-1}x_2^m.$$

We should point out that in the absence of taxes, a strictly positive government consumption is financed through a permanent deficit (see Equation (SS-C)) implying that both aggregate and individual savings are non-zero and β is different from unity.

Since we assume that consumer 2's savings are not affected by the fiscal policy, if $\widehat{\beta}$ is the value of the interest factor obtained with taxes we should have

$$\sum_{k=1}^{4}(\widehat{\beta}^{t-1}\mathbf{p}^k + \widehat{\beta}^{t-1}\tau^{0k})x_2^{0k} - \widehat{\beta}^{t-1}\mathbf{p}^k\omega_2^{0k} = \beta^{t-1}x_2^m \text{ for } t = 1, 2, \ldots .$$

These equations imply that $\widehat{\beta}$ should remain unaffected by the introduction of taxes, $\widehat{\beta} = \beta$. By replacing the demand functions, savings are unaltered for consumer 2 if

$$\frac{1}{\beta^{t-1}}\left(\alpha_2\sum_{k=1}^{4}\alpha_{k2}W_{t2}-\sum_{k=1}^{4}\beta^{t-1}\mathbf{p}^k\omega_2^{0k}\right)=\alpha_2\textstyle\sum_{k=1}^{4}(\mathbf{p}^k\omega_2^{0k}+\beta\mathbf{p}^k\omega_2^{1k}) \\ -\displaystyle\sum_{k=1}^{4}\mathbf{p}^k\omega_2^{0k}=x_2^m \qquad \text{(CS)}$$

In this example the total system is composed of 11 equations; 7 normalized prices, 2 normalized incomes, the government budget deficit equation and Equation (CS) concerning the individual borrowing/saving constraint. On the other hand, there are 3 prices (β is fixed) and 8 taxes. Hence a solution to the set of equations is likely to exist. Indeed, the following set of values represents the steady state equilibrium

$$\mathbf{p}^2=1.2399, \mathbf{p}^3=1.24845, \mathbf{p}^4=2.64413$$

and

$$(t_1^{0k})_{k=1}^4=(t_1^{1k})_{k=1}^4=(0.22449, 0.07003, 0.3477, -0.7837).$$

Finally, saving is now -367.7109 for consumer 1 and 367.7109 for consumer 2, so that aggregate savings are zero. This last result agrees with Equation (SS-C) as in the new situation a balanced budget ($\delta=0$) co-exists with β different from unity, implying that the aggregate savings are zero. □

The previous example illustrates the mechanism for irrelevance. Starting from a steady state with non-zero aggregate savings, there is a tax scheme which keeps the interest rate unchanged while achieving a zero government budget deficit. In the new situation, the government is taxing the consumers just enough to balance its budget. Moreover, this tax-transfer is such that aggregate, after tax, savings become zero. In this game, the change in the aggregate savings is completely done through the unrestricted consumer. A further important fact is that the taxes required to obtain irrelevance are age-anonymous, i.e. the tax on the young is the same as the tax on the old.

The role played by the consumer whose credit constraint is not binding is crucial. Indeed, if all consumers would be kept at their initial levels of borrowing or saving, achieving irrelevance would be impossible. This is clearly seen by considering an economy consisting of only one consumer.

The budget constraint for the generation $t-1$ consumer during his old age can be rearranged as

$$\tau_{t-1}^t x_{t-1}^t=-p_t x_{t-1}^t+p_t\omega_{t-1}^t-x_{t-1}^{tm},$$

while for the consumer in generation t, his first period constraint is

$$\tau_t^t x_t^t=-p_t x_t^t+p_t\omega_t^t-x_t^{tm}.$$

Adding these two, we obtain

$$\tau_t^t x_t^t+\tau_{t-1}^t x_{t-1}^t=p_t\left[\omega_t^t+\omega_{t-1}^t-x_t^t-x_{t-1}^t\right]-x_{t-1}^{tm}-x_t^{tm}.$$

Considering now the government budget deficit equation

$$\begin{aligned} d^t &= p^t g^t - \left[\tau_t^t x_t^t + \tau_{t-1}^t x_{t-1}^t\right] \\ &= p^t \left[g^t + x_t^t + x_{t-1}^t - \omega_t^t - \omega_{t-1}^t\right] + x_{t-1}^{tm} + x_t^{tm} \\ &= x_{t-1}^{tm} + x_t^{tm}\,, \end{aligned}$$

we see that the deficit is completely determined by the aggregate borrowing and saving decisions of the consumers. If these are unaffected by the fiscal policy, the deficit will remain unchanged.

The example above shows that when consumption tax instruments are sufficiently diversified, irrelevance of deficit restrictions may hold. However, in general as was the case with no credit restrictions (see Ghiglino and Shell [7], Example 11), having several tax instruments is not sufficient for irrelevance, which also depends on the length and the magnitude of the deficit restriction. Indeed, even when there are enough instruments, it is not assured that $q_{th}^{sk} - \tau_{th}^{sk}$ is positive, i.e. we could have for some s $(s = 1, 2, \dots)$ and some k $(k = 1, \dots, \ell)$ that $\mathbf{p}^{sk} < 0$. This would be consistent with the formal model, but is, of course, inconsistent with free disposal of endowments. As a consequence, the next proposition which generalizes the former example, gives only a *sufficient* condition for *relevance* and a *sufficient* condition for *weak irrelevance.* This proposition holds only generically–i.e. for an open and dense set of economies. In this way, degenerate cases–principally those in which individual endowments are co-linear–are excluded.

Proposition (Relevance of deficit restrictions in economies with consumption taxes and consumer credit restrictions). *Suppose that only anonymous consumption taxes are available and that the credit constraint on at least one consumer is not binding. Let x be an allocation that can be implemented as a constitutional equilibrium with a fiscal policy and deficit restriction δ and let $r_t, 0 \le r_t < n$, be the number of consumers of generation t for which the credit constraint is binding.*

Then, if $n + r_t > \ell$ for all t the deficit restriction is weakly (and strongly) relevant. On the other hand, if $n + r_t \le \ell$ for all t then the deficit restriction is weakly irrelevant.

Proof. When consumers are potentially credit constrained, demand for commodities may depend on the individual borrowings or lendings, so that these must be kept constant when the policy changes. Formally, x_{th}^{tm}, with $x_{th}^{tm} = p^t \cdot \omega_{th}^t - q_t^t \cdot x_{th}^t$, is kept constant for constrained consumers. Denote this quantity by $\overline{x}_{th}^{tm}$. Furthermore, since there is some consumer whose credit constraint is not binding, prices in successive periods are linked. Therefore, in period t the relevant system consists of $2\ell - 1$ conditions on prices and $r_t + n$ conditions on individual wealths. Let the consumers whose credit constraint is binding be denoted by $h = 1, \cdots, r_t$ while the remaining $h = r_t + 1, \cdots, n$ have non-binding credit restrictions. Taking into account the restriction on the deficit, the system of $2\ell + r_t + n$ equations can be written as

$$\widehat{p}^t + \widehat{\tau}_t^t = (p^{t1} + \tau_t^{t1})R_t^t,$$

$$p^{t+1} + \tau_t^{t+1} = (p^{t1} + \tau_t^{t1})R_t^{t+1},$$

$$p^t \cdot \omega_{th}^t - \overline{b}_{th}^t = (p^{t1} + \tau_t^{t1})W_{th}^t, \quad h = 1, \cdots, r_t$$

$$p^t \cdot \omega_{th}^t + p^{t+1} \cdot \omega_{th}^{t+1} = (p^{t1} + \tau_t^{t1})W_{th}, \quad h = 1, \cdots, n$$

and

$$\sum\nolimits_{h=1}^{h=n} \sum\nolimits_{i=1}^{i=\ell} \tau_t^{ti} f_{th}^{ti}(R_t^t, R_t^{t+1}, W_{th}) \\ + \tau_{t-1}^{ti} f_{t-1,h}^{ti}(R_{t-1}^{t-1}, R_{t-1}^t, W_{t-1,h}) = -\delta^t,$$

for $i = 1, \ldots, \ell$, where the rates of commodity substitution R, the normalized incomes W, and the government deficits δ are fixed. Suppose that $n \leq \ell$ and that $r_t = r$ is constant across time. In the Appendix, it is shown that in this case it is useful to consider as "free" variables the last $\ell - n$ prices of period t: $p^{t,n+1}, \ldots, p^{t\ell}$ and the first n prices of period $t+1$: $p^{t+1,1}, \ldots, p^{t+1,n}$. This system, which is linear in 3ℓ unknowns, has a solution if and only if $n + r \leq \ell$. The usual sign restrictions on the p's apply so this condition is not sufficient for irrelevance. The proof can easily be generalized to the case in which r_t is not constant. □

The above result does not concern paths such that there exist periods t' and t'' such that $n + r_{t'} \leq \ell$ and $n + r_{t''} > \ell$. In this case, $n + r_t$ should be compared to the number of available degrees of freedom in period t keeping in mind that this number depends also on the number of variables used in the other periods. The relevant quantity is $v_t + w_t$, where v_t and w_t are sequences defined by $v_t = \min(\ell, v_{t-1} + \ell - r_{t-1} - n)$ and $w_t = \min(\ell, w_{t+1} + \ell - r_{t+1} - n)$. The condition $n + r_t \leq \ell + v_t + w_t$ then indicates that the number of available instruments in period t is sufficient. However, in order to prove irrelevance the rank of the relevant matrices should also be checked. Unfortunately, there is no general result concerning these ranks and a case-by-case analysis is necessary.

7 Concluding remarks

We consider an OG exchange economy with anonymous taxes and transfers and constraints on individual borrowings. We ask whether or not the set of equilibrium allocations is affected by constitutional restrictions on the government's budget deficits.

Consumer credit constraints are important. With credit constraints and only anonymous lump-sum taxes, global irrelevance of government deficit restrictions is impossible and local irrelevance can obtain only in uninteresting circumstances. This contrasts with the case of unconstrained consumer credit, in which case deficit restrictions are globally (and locally) irrelevant. However, with credit constraints on individuals and only anonymous consumption taxes, global deficit irrelevance is impossible just as it is for the case without credit constraints. If there are a sufficient number of tax instruments and at least one consumer's credit constraint is not binding, then there is local irrelevance of the deficit restriction. This generalizes a similar result for the model without consumer credit constraints. More tools are needed for local irrelevance in the credit-constrained economy: For each consumer whose credit constraint is binding, there must be another tax tool. Hence the requirement for local irrelevance becomes: The number of commodities cannot be less then the number of consumers plus the number of consumers for whom the borrowing constraint is binding. (Of course, the number of consumers with binding credit restrictions might vary over time. In this case, the irrelevance result would be based on the maximum over time of the number of consumers with binding credit constraints.)

Consumption taxes are better for avoiding deficit restrictions than are lump-sum taxes in the case with constraints on consumer credit. With only anonymous lump-sum taxation, there is no way to increase the taxes on the young without reducing the early expenditures of the constrained youth. On the other hand, with a sufficient number of consumption tax rates, the government can increase locally the taxes on the young while leaving unchanged the consumptions of the credit-constrained young.

Compare our approach to taxation in the credit-constrained economy to that of Sargent and Smith [11]. Both Sargent and Smith [11] and we recognize that the tax powers of the government are limited. For us, this is captured by the assumption that all taxes must be anonymous. Sargent and Smith do allow for personalized lump-sum taxes but only as long as they do no not alleviate the credit constraint of any consumer. We make the strong assumption that the credit constraints (the b's) are unaffected by taxes and hence reducing early-life taxes can alleviate these constraints. Sargent and Smith make the strong assumption that the government cannot alleviate credit constraints by reducing early-life (lump-sum) taxes.

The implicit goal of the government in this paper (and other "irrelevance" papers) is to leave consumptions unchanged in the face of changed deficit restrictions. It is natural to ask how our results would be affected if the goal of the government were instead to maximize social welfare. To do this, one needs to characterize how taxes affect the equilibrium path of the economy. This is not so simple, at least for OG models. We leave this for further research.

A simpler question that can be addressed within our present approach is whether or not the government can free each consumer from his credit constraint. The government has the power to alleviate credit constraints by making transfers. Of course, the government might not be able to target the transfers at only credit-constrained individuals because of the anonymity requirement for taxes and transfers. Because of the need for anonymity, the government's power to alleviate binding credit constraints is limited, but this power is not zero. It turns out that the conditions for making each consumer liquid are similar to those for irrelevance of deficit restrictions. When only anonymous lump-sum transfers are used, freeing all consumer credit constraints is not possible in general, unless the deficit restriction does not include the first period of the economy. In the latter case, the government might be able to run a large first-period deficit and inject enough liquidity so that no generation faces binding credit constraints. When consumption taxes are available, it is likely that the same conditions that ensure weak irrelevance of deficit restrictions also ensure that all credit constraints can be weakened locally.

The present paper along with Ghiglino and Shell [7] indicates that there can be limits on the government's ability to avoid the restrictions on its deficit. Of course, as Kotlikoff [9] and others have argued, there is still ample scope for the government to evade (as opposed to avoid) the deficit restrictions by altering the timing of receipts and disbursements, by guaranteeing private loans, and so forth.

We stress again that to say the deficit restriction is irrelevant is not to say that the deficit does not matter. It is likely to matter if individuals condition their expectations on the deficit. For the analysis to be complete we must consider the role of expectations in "selecting" equilibria and we must also extend perfect-foresight expectations to rational expectations (that include sunspots). We believe that this is important.

We chose to use the OG model as the vehicle for our analysis because we think that it is both more tractable and more appropriate for this problem than the model with infinite-lived consumers. Nonetheless we do not believe that our basic results would be altered if we had employed the model with infinite-lived dynasties. (Of course, the present problem requires a model with many commodities and many consumers. The dynasties model has only recently been generalized to accommodate these heterogeneities.) On the other hand, the infinite time-horizon in either the OG or the dynasties model does make the analysis of local irrelevance much more complicated. Hence the finite version of this model deserves immediate investigation, to serve at least as a benchmark. The finite version of the model would also relate better to the existing optimal taxation literature. Finally, the model needs to be extended to include not only many consumption goods and many consumers but also to include production with many inputs and many outputs. Heterogeneity is crucial in this problem.

Appendix: Rank computations

For notational convenience, we focus attention on the case $n \geq 2$. Assume also that the number of consumers for whom the constraint is binding is constant though their identity might change from one generation to the next. Let the consumers for whom their credit constraint is binding be $h = 1, \ldots, r$ while for the remaining, $h = r+1, \ldots n$, their credit constraint is not binding. First, consider a consumer of generation 0. It is clear from Ghiglino and Shell [7] that the set of free parameters left after imposing the condition of constant individual demands to these consumers is a set of dimension $l - n$. Let us then consider as free the last $l - n$ prices $p^{1,n+1}, \ldots, p^{1,l}$.

Second, consider the consumers of generation t, $(t = 1, 2, \ldots)$, with the constraint that the prices $p^{t1}, \ldots, p^{tn}$ are already fixed (from previous-period conditions).

Using the relationship between the prices in periods t and $t+1$, the system of equations associated to a given demand can be written as

$$\begin{aligned}
\hat{p}^t + \hat{\tau}_t^t &= (p^{t1} + \tau_t^{t1}) R_t^t \\
p^{t+1} + \tau_t^{t+1} &= (p^{t1} + \tau_t^{t1}) R_t^{t+1} \\
p^t \omega_{th}^t - \overline{b}_{th}^t &= (p^{t1} + \tau_t^{t1}) W_{th}^t \quad \text{for } h = 1, \ldots, r \\
p^t \omega_{th}^t + p^{t+1} \omega_{th}^{t+1} &= (p^{t1} + \tau_t^{t1}) W_{th} \quad \text{for } h = 1, \ldots, n
\end{aligned}$$

$$\sum_{h=1}^{n} \sum_{i=1}^{l} \tau_t^{ti} f_{th}^{ti}(R_t^t, R_t^{t+1}, W_{th}) + \\ + \tau_{t-1}^{ti} f_{t-1,h}^{ti}(R_{t-1}^{t-1}, R_{t-1}^t, W_{t-1,h}) = \delta^t,$$

where the quantities $R_t^t \in \mathbb{R}^{l-1}$, $R_{t-1}^t \in \mathbb{R}^l$, W_{th}^t, W_{th} and δ^t are fixed. The system of $2l - 1 + n + r + 1 = 2l + n + r$ equations becomes linear in $3l$ unknowns, $p^{t,n+1}, \ldots, p^{t,l}, p^{t+1,1}, \ldots, p^{t+1,n}$, τ_t^t and τ_t^{t+1}.

Introduce the vectors $P_0^t \in \mathbb{R}^{l-n}$, $P_1^{t+1} \in \mathbb{R}^n$ and $J_s \in \mathbb{R}^s$ defined by

$$P_0^t = \begin{bmatrix} p^{t,n+1} \\ p^{t,n+2} \\ \vdots \\ p^{t,l} \end{bmatrix}, P_1^{t+1} = \begin{bmatrix} p^{t+1,1} \\ p^{t+1,2} \\ \vdots \\ p^{t+1,n} \end{bmatrix} \text{ and } J_s = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

In matrix form, the system can be written as $A_t z_t = b_t$ with

$$A_t = \begin{bmatrix}
0 & 0 & -\underline{R}_t^t & I_{n-1} & 0 & 0 & 0 \\
I_{l-n} & 0 & -\overline{R}_t^t & 0 & I_{l-n} & 0 & 0 \\
0 & I_n & -\underline{R}_t^{t+1} & 0 & 0 & I_n & 0 \\
0 & 0 & -\overline{R}_t^{t+1} & 0 & 0 & 0 & I_{l-n} \\
\varpi_t^t & 0 & -W_t^t \cdot J_r & 0 & 0 & 0 & 0 \\
\omega_t^t & \omega_t^{t+1} & -W_t \cdot J_n & 0 & 0 & 0 & 0 \\
0 & 0 & \sum_{h=1}^n f_{th}^{t1} & \sum_{h=1}^n f_{th}^{\hat{t}} & & 0 & 0
\end{bmatrix}_{2l+n+r \times 3l},$$

$$\underline{R}_t^t = \begin{bmatrix} R_t^{t2} \\ R_t^{t3} \\ \vdots \\ R_t^{tn} \end{bmatrix}_{n-1 \times 1}, \overline{R}_t^t = \begin{bmatrix} R_t^{t,n+1} \\ R_t^{t,n+2} \\ \vdots \\ R_t^{tl} \end{bmatrix}_{l-n \times 1}, \underline{R}_t^{t+1} = \begin{bmatrix} R_t^{t+1,1} \\ R_t^{t+1,2} \\ \vdots \\ R_t^{t+1,n} \end{bmatrix}_{n \times 1},$$

$$\overline{R}_t^{t+1} = \begin{bmatrix} R_t^{t+1,n+1} \\ R_t^{t+1,n+2} \\ \vdots \\ R_t^{t+1,l} \end{bmatrix}_{l+n \times 1}, \varpi_t^t = \begin{bmatrix} \omega_{t1}^{t,n+1} & \omega_{t1}^{t,n+2} & \ldots & \omega_{t1}^{tl} \\ \omega_{t2}^{t,n+1} & \omega_{t2}^{t,n+2} & \ldots & \omega_{t2}^{tl} \\ \vdots & \vdots & \vdots & \vdots \\ \omega_{tr}^{t,n+1} & \omega_{tr}^{t,n+2} & \ldots & \omega_{tr}^{tl} \end{bmatrix}_{r \times l-n},$$

$$\omega_t^t = \begin{bmatrix} \omega_{t1}^{t,n+1} & \omega_{t1}^{t,n+2} & \dots & \omega_{t1}^{tl} \\ \omega_{t2}^{t,n+1} & \omega_{t2}^{t,n+2} & \dots & \omega_{t2}^{tl} \\ \vdots & \vdots & \vdots & \vdots \\ \omega_{tn}^{t,n+1} & \omega_{tn}^{t,n+2} & \dots & \omega_{tn}^{tl} \end{bmatrix}_{n\times l-n}, \omega_t^{t+1} = \begin{bmatrix} \omega_{t1}^{t+1,1} & \omega_{t1}^{t+1,2} & \dots & \omega_{t1}^{t+1,n} \\ \omega_{t2}^{t+1,1} & \omega_{t2}^{t+1,2} & \dots & \omega_{t2}^{t+1,n} \\ \vdots & \vdots & \vdots & \vdots \\ \omega_{tn}^{t+1,1} & \omega_{tn}^{t+1,2} & \dots & \omega_{tn}^{t+1,n} \end{bmatrix}_{n\times n}$$

and

$$z_t = \begin{bmatrix} P_1^t \\ P_0^{t+1} \\ \tau_t^{t1} \\ \tau_t^{t2} \\ \vdots \\ \tau_t^{t+1,l} \end{bmatrix}.$$

Let also $W_t^t \in \mathbb{R}^n$ and $W_t \in \mathbb{R}^n$ be the vectors of individual wealths. The rank of the matrix A_t is equal to the rank of the matrix

$$\begin{bmatrix} 0 & 0 & -\underline{R}_t^t & I_{n-1} & 0 \\ I_{l-n} & 0 & -\overline{R}_t^t & 0 & I_{l-n} \\ \varpi_t^t & 0 & -W_t^t \cdot J_n & 0 & 0 \\ \omega_t^t & \omega_t^{t+1} & -W_t \cdot J_n & 0 & 0 \\ 0 & 0 & \sum_{h=1}^n f_{th}^{t1} & & \sum_{h=1}^n \hat{f}_{th}^t \end{bmatrix}_{l+n+r\times 2l}$$

plus l. Some tedious manipulations similar to those performed in Ghiglino and Shell [7], show that generically the above matrix has maximal rank. Then, for $l = n + r$ the A_t matrix has full rank $3l$. In this case the system has always a solution. The same can be said for $n + r < l$.

Suppose now that $l+1 = n+r$. Then the A_t matrix is a $3l+1\times 3l$ matrix which has generically maximal rank $3l$. Consider the square $3l+1$ matrix associated to the augmented system, (A_t, b_t) and let us prove that Rank $(A_t, b_t) = 3l + 1$. Indeed, the last coordinate of b_t is a function of δ_t that can be written as

$$\delta^t - \sum_{h=1}^{n}\sum_{i=1}^{l} \tau_{t-1}^{ti} f_{it-1}^{ti}.$$

The determinant of (A_t, b_t) is a first degree polynomial expression in δ^t. Therefore, to prove that the relevant matrix has full rank for an open and dense set of values of δ^t it is enough that the coefficient of δ^t in the polynomial expression is nonzero, which can be seen to be generically true. Since $Rank(A) < Rank(A, b)$, the solution set is empty. This is the borderline case so the same result holds also whenever $n + r > l + 1$.

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