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General equilibrium with nonconvexities and money

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Abstract

In a general-equilibrium economy with nonconvexities, there are sunspot equilibria with good welfare properties; sunspots can ameliorate the effects of the nonconvexities. For these economies, we show that agents act *as if* they have quasi-linear utility functions. We use this result to construct a new model of monetary exchange along the lines of Lagos and Wright, where trade occurs in both centralized and decentralized markets, but instead of quasi-linear preferences we assume general preferences but with indivisible labor. This suggests that modern monetary theory is more robust than one might have thought. It also constitutes progress on the classic problem of integrating monetary economics and general-equilibrium theory.

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1. Introduction

We study nonconvex economies, in particular economies in which some goods are indivisible. We have two major goals. First, extending Shell and Wright [32], we show that in the presence of indivisibilities there exist *sunspot equilibria* without the usual assumptions needed to generate such equilibria in convex economies, and that these equilibria have good welfare properties because sunspots allow “convexification,” similar to the way lotteries work in the indivisible labor

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economy of Rogerson [29]. [1] Second, we emphasize something not appreciated in the existing literature on nonconvexities, sunspots, and lotteries: in these economies, as long as the optimizing choices of agents are interior, then they act *as if* they have quasi-linear preferences.

It is true that it has been noted previously of the Rogerson model that, when labor is indivisible, under certain additional assumptions that include additive separability between consumption and leisure, agents act as if utility is linear in leisure. Our result is more general. The fact that for more general specifications of underlying preferences agents act as if they have quasi-linear preferences is useful for a variety of reasons. For one thing, it means that, for the divisible goods in the economy, wealth effects vanish. This has many implications, including the law of demand (the demand for each of the divisible goods is unambiguously decreasing in its own price). Here we will emphasize the usefulness of these results for monetary theory.

A model of monetary exchange with micro-foundations based on search theory has been developed by Lagos and Wright [20], hereafter LW. The LW model is tractable because it gets around the problem of having to keep track of the distribution of money holdings as a state variable. It works by allowing agents to trade periodically in centralized markets (CMs), where they can adjust their cash positions by buying and selling other goods, as well as sometimes forcing them to trade in decentralized markets (DMs), where money is essential. If agents have quasi-linear utilities then, given interior solutions, they all take the same amount of money out of the CM, and hence the distribution of money entering the DM is degenerate. This makes the framework relatively easy to analyze, and hence one can extend and apply it in a number of ways. [2]

Although for some questions one would obviously like to allow for endogenous non-degenerate distributions of money holdings, it is useful to have a benchmark without this complication, and to this extent the LW model is helpful. One might object, however, that quasi-linear utility is very special. Our results show that one does not actually need quasi-linearity: for general preferences, as long as some goods are indivisible, and again given interior solutions, all agents take the same amount of money out of the CM. Thus, we provide an alternative set of assumptions that leads to a simple model of monetary exchange with explicit micro-foundations. [3]

We make an effort to describe the CM in the model in a fairly general way—there are few restrictions other than those in standard general-equilibrium (GE) theory. This generality comes at little cost, and shows that modern monetary theory is not as special as one might think based on previous presentations (in earlier discussions of the LW model, e.g., the CM has a single consumption good, agents are homogeneous, and so on). Indeed, our CM looks much like the

[1] A sunspot equilibrium is one in which extrinsic uncertainty (a random variable with no impact on preferences, endowments, or technologies) affects the allocation. In strictly convex economies, sunspot equilibria are necessarily inefficient, because random allocations are dominated by the average allocation. When some goods are indivisible, however, the average may not be feasible. For some recent papers on nonconvexities, lotteries, and sunspots, see Prescott and Shell [26] and papers cited therein, Garratt et al. [12,13], and Kehoe et al. [16].

[2] LW provide examples and references to other applications. An alternative approach is provided by Shi [34]. Faig [10] tries to integrate the two models, and gives some results related to those derived below. For models that are less tractable, precisely because one has to keep track of the relevant distribution, see Green and Zhou [14], Zhou [40], Molico [23], Camera and Corbae [6], Taber and Wallace [36], or Zhu [41,42]. Earlier search-based models, such as Kiyotaki and Wright [17,18], Aiyagari and Wallace [1], Shi [33], or Trejos and Wright [37], were also very simple, but only because they avoided the issue by assuming agents could only hold $m \in \{0, 1\}$ units of money. In this symposium, Zhu [43] uses the births and deaths in an overlapping-generation economy to provide relatively simple dynamics.

[3] We usually interpret the indivisible good as labor. Although this is not necessary for any of the results, it is a common interpretation in macroeconomics. In addition to Rogerson [29], a sample of well-known papers adopting the indivisible labor model includes Hansen [15], Cooley and Hansen [8], and Christiano and Eichenbaum [7]. Some more recent examples include Sargent and Ljungqvist [31].

textbook Arrow–Debreu model, and we can appeal to some standard results in GE. Moreover, given that we proceed to combine this with micro-based monetary theory, one might say that we have made progress on the classic problem of integrating money and GE theory. We think progress comes not from the effort to force money into GE, but by bringing GE into monetary economics.

The paper is organized as follows. In Section 2, we discuss indivisibilities and sunspots in GE without money. We show that agents not at corner solutions act as if they have quasi-linear utility: their demands for divisible goods are independent of wealth, and indirect utility functions are linear in wealth. We provide conditions that guarantee interiority. In Section 3, we consider monetary economies. We begin with a brief review of LW, then present our model, and compare the results to those of LW. In Section 4, we conclude. Some technical results are in the appendix.

2. GE with nonconvexities

2.1. Equilibrium: definition

We begin with a static GE model.[4] There is a measure space (I, Ω, α) of consumers, where $I = [0, 1]$, Ω a σ-algebra of subsets of I, and α the uniform distribution over I.[5] There are K firms indexed by $k = 1, \ldots, K$. There are $J + 1$ commodities: J standard consumption goods indexed by $j = 1, \ldots, J$, and one indivisible good. We call the indivisible good leisure, following some examples in macro, although this label means little for now. By saying leisure is indivisible, we mean that it must either be consumed in a single unit or not at all.

Agent i starts with 1 unit of leisure, and an arbitrary endowment of the other goods $\mathbf{e}^i \in \mathbb{R}^J_+$, where $\mathbf{e}^i : I \to \mathbb{R}^J_+$ is I-measurable and $\bar{\mathbf{e}} = \int \mathbf{e}^i \, di$. Consumer i has preferences represented by a von Neumann–Morgenstern utility function $U^i(\mathbf{c}, h)$, where $\mathbf{c} \in \mathbb{R}^J_+$ is consumption and $h \in \{0, 1\}$ is labor, which equals 1 minus leisure. The consumption set for each agent is denoted by $C = \mathbb{R}^J_+ \times \{0, 1\}$. We assume U^i is twice continuously differentiable, strictly increasing in $\mathbf{c}$, strictly decreasing in h, and strictly concave. To ease the presentation, assume $U_j(\mathbf{c}, h) \to \infty$ as $c_j \to 0$ for all j, where U_j is the partial derivative with respect to c_j, guaranteeing $c_j > 0$.

Consumption goods are produced by firms using labor as the only input. Firm k has a technology represented by production function $\mathbf{f}^k(n^k) = \left[f_1^k\left(n^k\right), \ldots, f_J^k\left(n^k\right)\right] \in \mathbb{R}^J_+$, where $f_j^k\left(n^k\right)$ is output of good j. Assume $\mathbf{f}^k$ is continuously differentiable, increasing, and concave. It is possible that $f_j^k\left(n^k\right) = 0$ for all n for some j—i.e., each firm k does not necessarily produce every good—but for any good it does produce f_j^k strictly increasing and concave. Profit for firm k is Π^k, and the share of Π^k paid to consumer i is $\eta_k^i \in \mathbb{R}_+$ where $\int \eta_k^i \, di = 1$. Thus the total dividend income Δ^i for consumer i is given by $\Delta^i = \sum_k \eta_k^i \Pi^k$.

[4] By *static*, we do not mean the economy is timeless, since as usual one can interpret goods as indexed by dates. We simply mean that there is a single market that convenes before any production and consumption take place. Later we consider sequential-market models.

[5] We take this specification from Aumann [4,5], who first studied equilibrium with a continuum of agents. We do not actually need a continuum here, but it is adopted because in the monetary models discussed below, as in much of the literature, when combined with random matching it generates anonymity. It is worth mentioning that we could get away with a finite number of agents (for the GE results, and also for the monetary results as long as we have some other way to motivate anonymity) because we use sunspots as opposed to lotteries; the latter generally need the law of large numbers while the former do not; see, e.g., Shell and Wright [32], Prescott and Shell [26], and the citations in [26].

Consumers are heterogeneous, but for simplicity we assume there are only a finite number of types; that is, $I = \cup_{\tau=1}^{T} I_\tau$, where $U^i = U^\tau$, $e^i = e^\tau$, and $\eta_k^i = \eta_k^\tau$ for all $i \in I_\tau$. Also, for simplicity, there is no intrinsic uncertainty: all of the fundamentals are deterministic. However, there is extrinsic uncertainty, represented by the probability space (S, Σ, π), where $S = [0, 1]$ is the set of states representing "sunspot activity", Σ the Borel sets on S, and π the uniform distribution over S. For what we do here, the choice of the uniform distribution is without loss in generality; see Garratt et al. [12]. Although the realization of $s \in S$ does not affect preferences, technology, or endowments, in principle it can affect individual's behavior and equilibrium outcomes: sunspots can matter.

Given indivisible goods, having allocations potentially depend on extrinsic uncertainty allows for "convexification", that can lead to efficiency gains over nonrandomized allocations. [6] We formalize this by assuming complete Arrow–Debreu markets in sunspot-state-contingent commodities. Thus, the commodity space is the set of π-measurable functions from S into C. Similarly, $n^k(s)$ is firm k's employment rule, a π-measurable function from S into $\mathbb{R}_+$. Let $\mathbf{p}(s) = [p_1(s), \ldots, p_J(s)] \in \mathbb{R}_+^J$ be the price vector of consumption goods and $w(s) \in \mathbb{R}_+$ the price of labor in state s. [7] For all $\tilde{S} \subset S$, $\int_{\tilde{S}} p_j(s)\, ds$ is the cost of a unit of good j if event $\tilde{S}$ occurs. Let $[\mathbf{c}^i(s), h^i(s)]$ list a point in commodity space for every consumer i, and $[n^k(s)]$ an employment rule for every firm k.

Definition 1. An equilibrium is a list $\{[\mathbf{c}^i(s), h^i(s)], [n^k(s)], [\mathbf{p}(s), w(s)]\}$ satisfying:

(i) Given $[\mathbf{p}(s), w(s)]$, $[\mathbf{c}^i(s), h^i(s)]$ solves

$$W^i = \max_{\mathbf{c}^i(s), h^i(s)} \int_S U^i[\mathbf{c}^i(s), h^i(s)]\, ds \tag{1}$$

subject to

$$\int_S \left[\mathbf{p}(s)\mathbf{c}^i(s) - w(s)h^i(s) - \mathbf{p}(s)\mathbf{e}^i - \Delta^i\right] ds \leqslant 0 \tag{2}$$

for all i.

(ii) Given $[\mathbf{p}(s), w(s)]$, $n^k(s)$ solves

$$\Pi^k = \max_{n^k(s)} \int_S \left\{\mathbf{p}(s)\mathbf{f}^k[n^k(s)] - w(s)n^k(s)\right\} ds \tag{3}$$

for all k.

(iii)

$$\sum_k n^k(s) - \int_I h^i(s)\, di = 0 \tag{4}$$

[6] One can define competitive equilibrium without sunspots in the model. By the First Welfare Theorem, which does not require convexity, if it exists such an equilibrium is Pareto optimal within the set of nonrandomized allocations. It is easy to provide robust examples, however, where it is Pareto dominated by randomized allocations, including sunspot equilibrium allocations (see, e.g., Shell and Wright [32]).

[7] We restrict attention to price systems that have an inner-product representation; see Stokey and Lucas [35, Chapter 15], e.g., for a discussion.

and

$$\int_I \mathbf{c}^i(s)\,di - \sum_k \mathbf{f}^k(n^k) - \bar{\mathbf{e}} = 0 \tag{5}$$

for $s \in S$.

Garratt et al. [12, Theorem 1] show that, in this kind of model, every sunspot equilibrium allocation can be supported by prices, when adjusted for probabilities, that are constant across states.[8] Therefore, in the following, we can set $[\mathbf{p}(s), w(s)] = (\mathbf{p}, w)$ for all $s \in S$. Based on this it is immediate that the solution to the firm problem in (3) is constant across all states (to be accurate, almost surely with respect to π, but to ease the presentation we describe results as holding in all states).

Lemma 2. $n^k(s) = n^k$ *for all k and* $s \in S$.

Proof. The result follows directly from the strict concavity of f_j^k in any good j that firm k produces. □

Something similar is true for consumers, except that in general we must distinguish between states where they are employed and those where they are not. Let $S_1^i = \left\{s \in S : h^i(s) = 1\right\}$ be the set of states where i is employed and $S_0^i = S \backslash S_1^i$, where we assume $S_1^i, S_0^i \in \Sigma$. Also, let $\ell^i = \pi(S_1^i)$ be the probability that i is employed. Then consumer choices are constant across states (again, almost surely) in each of the labor market regimes.

Lemma 3. $\mathbf{c}^i(s) = \mathbf{c}_1^i$ *for all i and* $s \in S_1^i$ *and* $\mathbf{c}^i(s) = \mathbf{c}_0^i$ *for all i and* $s \in S_0^i$.

Proof. Given $[\mathbf{p}(s), w(s)] = (\mathbf{p}, w)$, we can rewrite the consumer problem as

$$\max \int_{S_1} U^i[\mathbf{c}_1^i(s), 1]\,ds + \int_{S_0} U^i[\mathbf{c}_0^i(s), 0]\,ds \tag{6}$$

subject to

$$\mathbf{p}\int_{S_1} \mathbf{c}^i(s)\,ds + \mathbf{p}\int_{S_0} \mathbf{c}^i(s)\,ds - w\ell^i - \mathbf{p}\mathbf{e}^i - \Delta^i \leqslant 0, \tag{7}$$

where the maximization is over the sets S_1^i and S_0^i, as well as $\mathbf{c}_1^i(s)$, which is consumption in state $s \in S_1$, and $\mathbf{c}_0^i(s)$, which is consumption in state $s \in S_0$. The result now follows from the strict concavity of U^i. □

Lemma 3 implies that the budget constraint in (6) can be reduced to $\ell^i U^i(\mathbf{c}_1^i, 1) + (1 - \ell^i)U^i(\mathbf{c}_0^i, 0)$, and the maximand in (7) to $\ell^i \mathbf{p}\mathbf{c}_1^i + (1-\ell^i)\mathbf{p}\mathbf{c}_0^i - w\ell^i - x^i \leqslant 0$, where $x^i = \mathbf{p}\mathbf{e}^i + \Delta^i$ is non-labor income or wealth (implicitly, x^i depends on $\mathbf{p}$). Clearly, i cares about the probability

[8] This is an intuitive no-arbitrage-profits result: since fundamentals are state invariant, a good delivered if s occurs should have the same price as a good delivered if s' occurs, given s and s' occur with equal probability. This result does not necessarily obtain if the state space is finite; see Garratt et al. [13].

that he works, $\ell^i = \pi(S_1^i)$, but not about which states are in S_1^i. Also, while $\mathbf{c}_1^i$ does not equal $\mathbf{c}_0^i$, in general, it does for some specifications: the following says that if some commodities enter U^i separably from h, the demand for these commodities is the same whether or not i is employed. As a special case, if $U^i(\mathbf{c}^i, h) = u^i(\mathbf{c}^i) + v^i(h)$, then $\mathbf{c}_1^i = \mathbf{c}_0^i$.

Lemma 4. *Suppose we can partition* $\mathbf{c}^i = (\hat{\mathbf{c}}^i, \tilde{\mathbf{c}}^i)$, *so that* $U^i(\mathbf{c}^i, h) = u^i(\tilde{\mathbf{c}}^i) + v^i(\hat{\mathbf{c}}^i, h)$. *Then* $\tilde{\mathbf{c}}^i(s) = \tilde{\mathbf{c}}^i$ *for* $s \in S$.

Proof. This follows directly from strict concavity. □

Given Lemmas 2 and 3, we can summarize the decisions of all firms by (n^k) and consumers by $(\mathbf{c}_1^i, \mathbf{c}_0^i, \ell^i)$, which allows us to present a much simpler definition of equilibrium.[9] First, one more piece of notation: let $I_1(s) = \{i \in I : h^i(s) = 1\} = \{i \in I : s \in S_1^i\}$ be the set of agents who are employed, and $I_0(s) = I \backslash I_1(s)$ the set unemployed, for each s, where we assume $I_1(s), I_0(s) \in \Omega$. We need to include $I_1(s)$ in our list of equilibrium objects because we need to know who is working (and not just the measure of employed agents) in each state.

Definition 5. An equilibrium is now a list $\{(\mathbf{c}_1^i, \mathbf{c}_0^i, \ell^i), (n^k), (\mathbf{p}, w), I_1(s)\}$ satisfying:

(i) Given $(\mathbf{p}, w)$, $(\mathbf{c}_1^i, \mathbf{c}_0^i, \ell^i)$ solves

$$W^i = \max_{\mathbf{c}_1^i, \mathbf{c}_0^i, \ell^i} \left\{ \ell^i U^i(\mathbf{c}_1^i, 1) + (1 - \ell^i) U^i(\mathbf{c}_0^i, 0) \right\} \tag{8}$$

subject to

$$\ell^i \mathbf{p}\mathbf{c}_1^i + (1 - \ell^i)\mathbf{p}\mathbf{c}_0^i - w\ell^i - x^i \leqslant 0 \tag{9}$$

for all i.

(ii) Given $(\mathbf{p}, w)$, n^k solves

$$\Pi^k = \max_{n^k} \left\{ \mathbf{p}\mathbf{f}^k(n^k) - wn^k \right\} \tag{10}$$

for all i.

(iii)

$$\sum_k n^k = \alpha\,[I_1(s)]\,, \tag{11}$$

$$\bar{\mathbf{e}} + \sum_k \mathbf{f}^k(n^k) = \int_{I_1(s)} \mathbf{c}_1^i\, di + \int_{I_0(s)} \mathbf{c}_0^i\, di \tag{12}$$

for $s \in S$.

(iv)

$$\ell^i = \pi\{s : I_1(s) \ni i\} \tag{13}$$

for all i.

[9] It is simpler mainly because the firm problem has been reduced to choosing n^k and the consumer problem to choosing $(\mathbf{c}_1^i, \mathbf{c}_0^i, \ell^i)$, which are finite-dimensional objects.

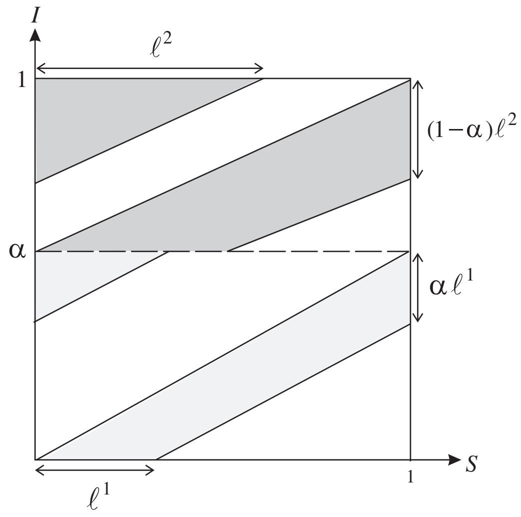

Fig. 1. Allocation rule.

Labor demand on the left-hand side of (11) is constant across states; hence so is labor supply on the right-hand side. This means we have to construct $I_1(s)$ so that the same measure of workers are employed for all s, and the last consistency condition says that we need to do so in such a way that each agent i is working with his chosen probability ℓ^i. For example, suppose we have homogeneous consumers, so $\ell^i = \bar{\ell}$ for all i. Then in equilibrium $\sum_k n^k = \bar{\ell}$. We have to construct $I_1(s)$ so that a fraction $\bar{\ell}$ of consumers are working in every state, and they are all working in a fraction $\bar{\ell}$ of the states.

This type of construction can be done by generalizing the method in Shell and Wright [32]. Consider an example with two types of consumers: the set $[0, \alpha]$ are type 1 and wish to work with probability ℓ^1, while the rest are type 2 and wish to work with probability ℓ^2. Set $S_1^i = [\frac{i}{\alpha}, \frac{i}{\alpha} + \ell^1]$ modulo ℓ^1 for $i \in [0, \alpha]$, and $S_1^i = [\frac{i-\alpha}{1-\alpha}, \frac{i-\alpha}{1-\alpha} + \ell^2]$ modulo ℓ^2 for $i \in (\alpha, 1]$. Fig. 1 shows as the shaded area pairs (s, i) such that i is employed in state s. Then every consumer is working with the desired probability, and the measure of $I_1(s)$ (total employment) is $\alpha\ell^1 + (1-\alpha)\ell^2$ for all s. However, there is a simpler alternative when we have a continuum of consumers. If type τ want to work with probability ℓ^τ, in each state s set $h^\tau = 1$ with probability ℓ^τ, and by the law of large numbers a measure ℓ^τ will be working in each state. [10]

2.2. Equilibrium: properties

Now that we have defined equilibrium, we provide some results. First note that (11)–(12) can be rewritten after manipulation as

$$\sum_k n^k - \int_I \ell^i \, di = 0 \tag{14}$$

[10] Again, we do not need the law of large numbers for sunspot equilibrium, as one does for lottery equilibrium, but it does simplify things slightly.

and

$$\int_I [\ell^i \mathbf{c}_1^i + (1-\ell^i)\mathbf{c}_0^i]\, di - \bar{\mathbf{e}} - \sum_k \mathbf{f}^k(n^k) = 0. \tag{15}$$

This suggests that our economy has a reduced form that looks like a model with no sunspots, where agents simply trade a probability of working ℓ^i for wages, and use it to buy consumption, in the spirit of lottery models such as Rogerson [29].

A particularly simple case is the one where $U^i(\mathbf{c}^i, h^i) = u^i(\mathbf{c}^i) + v^i(h^i)$ is separable in h, since then Lemma 4 implies $\mathbf{c}_1^i = \mathbf{c}_0^i = \mathbf{c}^i$. Without loss in generality, set $v^i(1) = 0$ and $v^i(0) = A^i > 0$. Then the consumer problem can be simplified further to

$$W^i = \max_{\mathbf{c}^i, \ell^i} \left\{ u^i(\mathbf{c}^i) - A^i \ell^i \right\} \tag{16}$$

subject to

$$\mathbf{p}\mathbf{c}^i - w\ell^i - x^i \leqslant 0. \tag{17}$$

In this case, it is as if the consumer had a utility function that was linear in ℓ^i. Assuming an interior solution, the first-order condition for any commodity j is $u_j^i(\mathbf{c}^i) = A^i p_j/w$. This immediately implies $\partial c_j/\partial x^i = 0$, $\partial c_j/\partial p_j = A^i p_j/w u_{jj}^i < 0$, and $\partial W^i/\partial x^i = A^i/w$. Hence, in this case, it is obvious that wealth effects are 0, demand curves slope downward, and the indirect utility function is linear in x^i.

We now show these results are general: we do not require U^i to be separable in h^i, although we do still need interiority of the consumer choice problem.

Proposition 6. *Suppose* $\ell^i \in (0, 1)$ *and* $w - \mathbf{p}\mathbf{c}_1^i + \mathbf{p}\mathbf{c}_0^i \neq 0$. *Then* $\partial \mathbf{c}_{hj}^i/\partial x^i = 0$, $h = 0, 1$ *and for all* i, j.

Proof. Consider the Lagrangian

$$\begin{aligned} W = {} & \ell U(\mathbf{c}_1, 1) + (1-\ell)U(\mathbf{c}_0, 0) \\ & + \lambda \left[w\ell + x - \ell \mathbf{p}\mathbf{c}_1 - (1-\ell)\mathbf{p}\mathbf{c}_0 \right], \end{aligned} \tag{18}$$

where $\lambda > 0$ is the multiplier, and we omit the index i where there is no risk of confusion. In the appendix we show that there is a unique solution to this problem, and assuming $\ell \in (0, 1)$ it uniquely satisfies the first-order conditions:

$$\mathbf{c}_1 : U_j(\mathbf{c}_1, 1) - \lambda p_j = 0 \quad \text{for all } j, \tag{19}$$

$$\mathbf{c}_0 : U_j(\mathbf{c}_0, 0) - \lambda p_j = 0 \quad \text{for all } j, \tag{20}$$

$$\ell : U(\mathbf{c}_1, 1) - U(\mathbf{c}_0, 0) + \lambda(w - \mathbf{p}\mathbf{c}_1 + \mathbf{p}\mathbf{c}_0) = 0, \tag{21}$$

and

$$\lambda : w\ell + x - \ell \mathbf{p}\mathbf{c}_1 - (1-\ell)\mathbf{p}\mathbf{c}_0 = 0. \tag{22}$$

Notice that x does not appear in (19)–(21). By the Implicit Function Theorem, these $2J+1$ equations determine $(\mathbf{c}_1, \mathbf{c}_0, \lambda)$ independent of x, as long as

$$\begin{bmatrix} \mathbf{H}_1 & \mathbf{0} & -\mathbf{p}^T \\ \mathbf{0} & \mathbf{H}_0 & -\mathbf{p}^T \\ \mathbf{0} & \mathbf{0} & w - \mathbf{pc}_1 + \mathbf{pc}_0 \end{bmatrix}$$

is nonsingular, where $\mathbf{H}_h$ is the Hessian matrix with (i, j) term $U_{ij}(c_h, h)$ and $\mathbf{p}^T$ is the transpose of $\mathbf{p}$. By strict concavity of U, we have $|\mathbf{H}_h| < 0$ and so nonsingularity is equivalent to $w - \mathbf{pc}_1 + \mathbf{pc}_0 \neq 0$. □

In the above result we rule out the possibility $w - \mathbf{pc}_1 + \mathbf{pc}_0 = 0$, which is equivalent to $U(\mathbf{c}_1, 1) = U(\mathbf{c}_0, 0)$ by (21). It is well known that this possibility arises for a very special utility function in the case of one good, $U(c, h) = u[c + v(h)]$.[11] More generally, consider $U(\mathbf{c}, h) = u[c_J + v(\mathbf{c}^{-J}, h)]$ where $\mathbf{c}^{-J} = (c_1, \ldots, c_{J-1})$ (i.e., a concave transformation of a utility function that is linear in some good, where we assume without loss in generality that this good is c_J). The first-order conditions (19)–(22) still hold, but now notice that, for good J, (19) and (20) tell us $u'\left[c_{1J} + v(\mathbf{c}_1^{-J}, 1)\right] = u'\left[c_{0J} + v(\mathbf{c}_0^{-J}, 0)\right]$, and so

$$c_{1J} + v(\mathbf{c}_1^{-J}, 1) = c_{0J} + v(\mathbf{c}_0^{-J}, 0). \tag{23}$$

Hence in this case $U(\mathbf{c}_1, 1) = U(\mathbf{c}_0, 0)$ and (21) implies $w - \mathbf{pc}_1 + \mathbf{pc}_0 = 0$. Now we cannot solve (19)–(21) for $(\mathbf{c}_1, \mathbf{c}_0, \lambda)$ independent of x. Indeed, (22) implies $\mathbf{pc}_0 = x$. However, (19)–(20) tell us that, for $j \neq J$, we have

$$u'[c_{1J} + v(\mathbf{c}_1, 1)]\, v_j(\mathbf{c}_1, 1) = \lambda p_j \text{ implies } v_j(\mathbf{c}_1, 1) = p_j/p_J, \quad j = 1, 2, \ldots, J-1$$

and

$$u'[c_{0J} + v(\mathbf{c}_0, 0)]\, v_j(\mathbf{c}_0, 0) = \lambda pj \text{ implies } v_j(\mathbf{c}_0, 0) = p_j/p_J, \quad j = 1, 2, \ldots, J-1.$$

We can solve these for $\mathbf{c}_1^{-J} = \mathbf{c}_1^{-J}(\mathbf{p})$ and $\mathbf{c}_0^{-J} = \mathbf{c}_0^{-J}(\mathbf{p})$. Then (23) implies

$$c_{1J} - c_{0J} = v[\mathbf{c}_0^{-J}(\mathbf{p}), 0] - v[\mathbf{c}_1^{-J}(\mathbf{p}), 1].$$

Now we conclude two things. First, normalizing $p_J = 1$,

$$w = \mathbf{p}(c_1 - \mathbf{c}_0) = \mathbf{p}^{-J}[\mathbf{c}_1^{-J}(\mathbf{p}) - \mathbf{c}_0^{-J}(\mathbf{p})] + v[\mathbf{c}_0^{-J}(\mathbf{p}), 0] - v[\mathbf{c}_1^{-J}(\mathbf{p}), 1],$$

which says that w must be a particular function of $\mathbf{p}$.[12] Second, from $\mathbf{pc}_0 = x$ we have

$$c_{0J} = x - \mathbf{p}^{-J}\mathbf{c}_0^{-J};$$

[11] See, e.g., [9,30].

[12] This may look to be nongeneric, in partial equilibrium, but it occurs naturally in GE. Suppose that consumers are homogeneous, $J = 1$, and there is a representative firm with $f'(0) = \infty$ and $f'(1) = 0$. Then $\ell = n \in (0, 1)$ in equilibrium so w will adjust to satisfy the relevant condition, which with $J = 1$ is $w/p = v(0) - v(1)$. That is, the real wage exactly compensates workers for lost leisure.

in this case it is c_{0J} that adjusts with x to satisfy the budget equation, and all other variables are independent of x. Given that we have explained how to handle the above special case, to conserve space, from now on we ignore it and assume $w - \mathbf{pc}_1^i + \mathbf{pc}_0^i \neq 0$.

Proposition 7. *Under the conditions in Proposition* 6, $\partial \mathbf{c}_{hj}^i / \partial p_j < 0$, $h = 0, 1$ *and for all i and j.*

Proof. Obvious. □

Proposition 8. *Under the conditions in Proposition* 6, $\partial W^i / \partial x^i = \lambda^i$ *is independent of* x^i.

Proof. We can rearrange (18) as

$$W = U(\mathbf{c}_0, 0) + \lambda x - \lambda \mathbf{pc}_0 + \ell \left[U(\mathbf{c}_1, 1) - U(\mathbf{c}_0, 0) + \lambda(w - \mathbf{pc}_1 + \mathbf{pc}_0) \right].$$

By (21), the term in brackets vanishes. From Proposition 6, $\mathbf{c}_0$ and λ are independent of x, and the result follows. □

We now provide something along the lines of an aggregation theorem.[13] Start with an economy where consumers have homogeneous preferences and wealth, and consider any equilibrium. Now change the distribution of wealth. Then there is an equilibrium where prices, consumption, and aggregate employment are exactly the same as in the homogeneous-wealth economy. The only requirement is interiority for ℓ^i, which (as we discuss below) can be guaranteed by assumptions on primitives, including some bounds on the extent of wealth heterogeneity.

Proposition 9. *Assume* $U^i = U$ *for all i. Let* $\left\{ \hat{\mathbf{c}}_1, \hat{\mathbf{c}}_0, \hat{\ell}, (\hat{n}^k), \hat{\mathbf{p}}, \hat{w} \right\}$ *be an equilibrium when* $x^i = \hat{x}$ *for all i, satisfying the conditions in Proposition* 6. *Give each consumer i a transfer* t^i *in units of the numeraire good, so that* $x^i = \hat{x} + t^i$, *where* $\int_I t^i \, di = 0$. *Then there exist* $\overline{x} > 0$ *and* $\underline{x} < \overline{x}$, *constructed in the proof, with the following property: if* $x^i \in (\underline{x}, \overline{x})$ *for all i, then an equilibrium exists with* $\ell^i \in (0, 1)$ *for all i and it has the same* $(\hat{\mathbf{p}}, \hat{w})$ *and* $(\mathbf{c}_1^i, \mathbf{c}_0^i) = (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_0)$ *for all i. Although* ℓ^i *may differ with i,* $\int_I \ell^i \, di = \hat{\ell}$.

Proof. In the homogeneous-wealth economy, from the budget equation,

$$\hat{\ell} = \frac{\hat{\mathbf{p}}\hat{c}_0 - \hat{x}}{\hat{w} + \hat{\mathbf{p}}\hat{c}_0 - \hat{\mathbf{p}}\hat{c}_1} \in (0, 1) \tag{24}$$

by assumption. Now consider the economy with transfers, and set $(p, w) = (\hat{p}, \hat{w})$. From (10), $n^k = \hat{n}^k$ and $\Pi^k = \hat{\Pi}^k$. From Proposition 6, if ℓ^i is interior for all i then $(\mathbf{c}_1^i, \mathbf{c}_0^i) = (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_0)$. From the budget equation,

$$\ell^i = \frac{\hat{\mathbf{p}}\hat{c}_0 - x^i}{\hat{w} + \hat{\mathbf{p}}\hat{c}_0 - \hat{\mathbf{p}}\hat{c}_1}. \tag{25}$$

Integrating across agents, $\int_I \ell^i \, di = \hat{\ell}$. Since all aggregate quantities are the same, markets clear.

[13] To be clear, these results concern a "representative agent" economy aggregated from an economy with heterogeneous wealth, but not with heterogeneous preferences.

It remains to give conditions on the distribution such that $\ell^i \in (0,1)$ for all i. Suppose $\hat{w} + \mathbf{p}\hat{\mathbf{c}}_0 - \mathbf{p}\hat{\mathbf{c}}_1 > 0$. Then we have

$$\ell^i > 0 \text{ if and only if } x^i < \overline{x} = \hat{\mathbf{p}}\hat{c}_0$$

and

$$\ell^i < 1 \text{ if and only if } x^i > \underline{x} = \hat{\mathbf{p}}\hat{c}_1 - \hat{w}.$$

Now suppose $\hat{w} + \mathbf{p}\hat{\mathbf{c}}_0 - \mathbf{p}\hat{\mathbf{c}}_1 < 0$. Then we have

$$\ell^i > 0 \text{ if and only if } x^i > \underline{x} = \hat{\mathbf{p}}\hat{c}_0$$

and

$$\ell^i < 1 \text{ if and only if } x^i < \overline{x} = \hat{\mathbf{p}}\hat{c}_1 - \hat{w}.$$

Note that in both cases $\overline{x} > 0$ and $\underline{x} < \overline{x}$. As long as $x^i \in (\underline{x}, \overline{x})$ for all i, then $\ell^i \in (0,1)$ for all i. Note that $x^i \in (\underline{x}, \overline{x})$ for all i is possible because $\hat{x} \in (\underline{x}, \overline{x})$ by (24). This completes the proof. □

One can generalize Proposition 9 so that it holds without the assumption of homogeneous preferences.

Corollary 10. *Suppose there are T consumer types, with $U^i = U^\tau$ for all $i \in I_\tau$, where $\cup_{\tau=1}^T I_\tau = I$. Suppose there is an equilibrium when $x^i = \hat{x}^\tau$ for all $i \in I_\tau$ satisfying the conditions in Proposition* 6. *Give each consumer i a transfer t_i so that now $x^i = \hat{x}^\tau + t^i$, where $\int_{I_\tau} t^i di = 0$ for all τ. Then there exist $\overline{x}^\tau > 0$ and $\underline{x}^\tau < \overline{x}^\tau$ with the following property: if $x^i \in (\underline{x}^\tau, \overline{x}^\tau)$ for all $i \in I_\tau$ and all τ, then an equilibrium exists with $\ell^i \in (0,1)$ for all i and it has the same prices, the same consumption, and the same total employment for every type.*

Proof. Obvious. □

These results say that, as long as it is not too disperse, the wealth distribution does not matter for consumption or aggregate employment. When wealth differs across agents, rich agents will work less and poor agents more, but nothing else changes. This is useful in the monetary economy studied below, where it implies that, even if agents enter the market with different amounts of money, they exit with the same. To preview how this works, we present an example where we put money in the utility function directly, although we commit this sin only for purposes of illustration. (In the next section, the value of money will be derived from first principles.)

Thus, in addition to leisure, there are two goods, consumption and money: $\mathbf{c} = (c, \hat{m})$. The endowment is $\mathbf{e}^i = (0, m^i)$ with $\int m^i di = M$ (here m is money brought into the market and $\hat{m}$ is money taken out). We normalize the price of money to 1, so p is the nominal price of c. Money cannot be produced; c is produced according to $c = Bn$, so that the real equilibrium wage in units of c is B.[14] Suppose $U(c, \hat{m}, h) = \log c - v(h) + V(\hat{m})$, with $v(1) - v(0) = A > 1$. Since U is

[14] The fact this production function is not strictly concave does violate assumptions made above, but this causes no problems, since we can solve the example explicitly.

separable, we know $c_0 = c_1 = c$ and $\hat{m}_0 = \hat{m}_1 = \hat{m}$. Hence, the consumer problem is

$$W(m) = \max_{c,\hat{m},\ell} \left\{ \log c + V(\hat{m}) - A\ell \right\}$$

subject to

$$c - B\ell - \frac{m - \hat{m}}{p} \leqslant 0.$$

Or, substituting from the budget constraint at equality, we have

$$W(m) = \max_{c,\hat{m}} \left\{ \log c + V(\hat{m}) - \frac{A}{B}\left(c - \frac{m - \hat{m}}{p}\right) \right\}.$$

The first-order conditions for an interior solution yield $c = B/A$ and $V'(\hat{m}) = A/Bp$; hence, c and $\hat{m}$ are indeed independent of m. Given c, the technology implies $n = c/B = 1/A \in (0, 1)$. From the budget equation, we have

$$\ell = \frac{1}{B}\left(c - \frac{m - \hat{m}}{p}\right) = \frac{1}{A} - \frac{(m - M)V'(M)}{A}$$

after substituting the market clearing conditions, $\hat{m} = M$ and $p = A/BV'(M)$. Clearly, ℓ is decreasing in m, and $\ell \in (0, 1)$ if and only if $m \in (\underline{m}, \overline{m})$, where

$$\underline{m} = M - \frac{A - 1}{V'(M)} \quad \text{and} \quad \overline{m} = M + \frac{1}{V'(M)}.$$

This example is particularly easy due to separability, but in Appendix C we work out a case where U is not separable. In any case, as one can see, it is not hard to solve examples and construct $(\underline{m}, \overline{m})$ explicitly.

2.3. Equilibrium: existence

We close this section with a discussion of existence in the general model. Define excess demand for labor and goods by

$$N(\mathbf{p}, w) \equiv \sum_k n^k - \int_I \ell^i \, di \tag{26}$$

and

$$\mathbf{Z}(\mathbf{p}, w) \equiv \int_I [\ell^i \mathbf{c}_1^i + (1 - \ell^i)\mathbf{c}_0^i] \, di - \bar{\mathbf{e}} - \sum_k \mathbf{f}^k(n^k). \tag{27}$$

We claim the following is true: (i) As we show in the appendix, even though W^i is not strictly quasi-concave, there is a unique solution to the consumer problem $(\mathbf{c}_1^i, \mathbf{c}_0^i, \ell^i)$ and it is continuous in $(\mathbf{p}, w)$. Also, profit maximization determines n^k as a continuous function of $(\mathbf{p}, w)$. Hence, excess demand is a continuous function of $(\mathbf{p}, w)$. (ii) It is clear that excess demand is homogeneous of degree 0. (iii) Integrating the budget equations over agents, Walras' Law holds: $wN(\mathbf{p}, w) + \mathbf{p}\mathbf{Z}(\mathbf{p}, w) = 0$. (iv) $Z(\mathbf{p}, w)$ and $N(\mathbf{p}, w)$ can be bounded below, as long as we bound the production function $\mathbf{f}^k$ for all k. (v) $\max\left[Z_j(\mathbf{p}_n, w_n), N(\mathbf{p}_n, w_n)\right] \to \infty$ for any sequence $(\mathbf{p}_n, w_n) \to (\mathbf{p}, w)$ with $p_j = 0$ or $w = 0$, as long as U^i and $\mathbf{f}^k$ are strictly increasing.

Properties (i)–(v) allow us to apply a standard existence result, such as Proposition 17.C.1 in [22]. Basically, by reducing the model to something that resembles a standard GE economy we can show that an equilibrium exists using textbook methods—the one detail being that W^i may not generally be strictly quasi-concave, but we can show directly that there is a unique solution to the consumer problem.

Proposition 11. *Given assumptions in the text, there is* $(\mathbf{p}, w)$ *such that* $\mathbf{Z}(\mathbf{p}, w) = \mathbf{0}$ *and* $N(\mathbf{p}, w) = 0$.

Of course, existence does not guarantee $\ell^i \in (0, 1)$ for all i. Since many of our results about the properties of equilibrium depend on it, it would be good to have some additional conditions to guarantee interiority. One way one might imagine proceeding is to put curvature restrictions on technology. Thus, if we assume that, for at least one firm k and good j, $\partial f_j^k(n)/\partial n \to \infty$ as $n \to 0$, then clearly aggregate labor demand satisfies $\ell = \sum_k n^k > 0$ in any equilibrium. And if we assume that for every firm k and good j, $\partial f_j^k(n)/\partial n \to 0$ as $n \to \bar{n}^k$ where $\sum_k \bar{n}^k \leqslant 1$, then $\ell = \sum_k n^k < 1$ in any equilibrium. However, this assumption would contradict property (v) that we used for existence. Therefore, we take a different track.

To illustrate the logic, consider the case with one good $J = 1$, and one firm $K = 1$, and homogeneous consumers. If $n = 1$ in equilibrium then the utility of a representative consumer is $U[f(1) + e, 1]$. To rule out such an equilibrium assume that

$$U[f(1) + e, 1] < U[f(1) + e - f'(1), 0] \tag{28}$$

is obtained. If (28) holds, then the consumer would be better off choosing $\ell = 0$ instead of $\ell = 1$. Condition (28) holds provided $f'(1) < \bar{\omega}$, where $\bar{\omega} > 0$ solves $U[f(1) + e, 1] = U[f(1) + e - \bar{\omega}, 0]$. Hence, in a homogeneous consumer economy, in any equilibrium we have $\ell \in (0, 1)$ provided (28) holds. By Proposition 9, if wealth is heterogeneous there is still an equilibrium where $\ell^i \in (0, 1)$ for all i, as long as wealth is not too heterogeneous.

One could generalize this logic to J goods as follows. Define $\hat{W}(1)$ by

$$\hat{W}(1) = \max_{\mathbf{c}_1} U(\mathbf{c}_1, 1)$$

subject to

$$\mathbf{p}[\mathbf{c}_1 - \mathbf{f}(1) - \mathbf{e}] = 0,$$

where $\mathbf{p}$ is a solution to $\mathbf{c}_1 = \mathbf{f}(1) + \mathbf{e}$. Define next $\hat{W}(0)$ by

$$\hat{W}(0) = \max_{\mathbf{c}_0} U(\mathbf{c}_0, 0)$$

subject to

$$\mathbf{p}[\mathbf{c}_0 - \mathbf{f}(1) - \mathbf{e} + \mathbf{f}'(1)] = 0.$$

To guarantee that $n = 1$ is not an equilibrium, we can impose $\hat{W}(1) < \hat{W}(0)$ for any $\mathbf{p}$ such that $\mathbf{c}_1 = \mathbf{f}(1) + \mathbf{e}$. This simply requires that $\mathbf{pf}'(1)$ is not too large.

3. Monetary theory

3.1. The LW model

We begin with a review of LW, to provide the basic environment, notation, etc. There is a [0, 1] continuum of infinitely-lived agents. Time is discrete, and each period is divided into two subperiods. In the first subperiod there is a frictionless CM. In the second subperiod there is a DM with two frictions: a double-coincidence problem, detailed below, and anonymity, which precludes private credit arrangements. These frictions make money essential. [15]

There is a single consumption good c in the CM. Agents have 0 endowment of this good, but can produce it with technology $c = h$. Assume for now that agents can only make spot trades in the CM—i.e., they cannot move resources across periods except by carrying money between markets—but we argue below that this is without loss of generality. In the DM, there is one good q. Each agent faces the following possibilities in the DM: with probability $\sigma \leqslant \frac{1}{2}$ he wants to consume and derives utility $u(q)$; with the same probability he has the ability to produce at disutility cost $\psi(q)$; and with probability $1 - 2\sigma$ he can neither consume nor produce.

Agents who want to consume and those who can produce meet bilaterally and anonymously in the DM, where they trade using money. [16] Let q^* be the quantity that solves $u'(q) = \psi'(q)$. Assume agents discount between the CM and DM at rate β_1, and between the DM and the next CM at rate β_2, and let $\beta = \beta_1\beta_2$. There is a stock of money that changes according to $M_{+1} = (1+\gamma)M$, where the subscript $+1$ indicates next period, and new money is injected (or withdrawn if $\gamma < 0$) via lump sum transfers (or taxes) in the CM. This completes the description of the basic environment.

An agent's wealth in the CM is $x^i = m^i + \gamma M$, but since γM is constant across agents we use m^i as the individual state variable. Thus, $W(m)$ and $V(m)$ are the value functions in the CM and DM; they are not indexed by i since agents are identical except for their current money balances. Then we have

$$W(m) = \max_{c,h,\hat{m}} \left\{ U(c,h) + \beta_1 V(\hat{m}) \right\} \tag{29}$$

subject to

$$pc + \hat{m} - ph - m - \gamma M \leqslant 0. \tag{30}$$

The solution in general depends on m. Hence, a distribution of m across agents entering the CM induces a distribution of $\hat{m}$ across agents exiting the CM and entering the DM.

If $U(c, h)$ is linear in either c or h, however, LW show that $\hat{m}$ is independent of m, so the distribution across agents entering the DM is degenerate at $\hat{m} = M(1+\gamma)$, and they also show that W is linear with $\partial W/\partial m = 1/p$. There are two caveats: the distribution of m across agents in the CM cannot be too disperse, since we need interior solutions, and one has to check that V is strictly concave. Assumptions on primitives to guarantee both can be found in LW.

[15] See Kocherlakota [19] and Wallace [38] for recent discussions of the essentiality of money, especially the role of anonymity. The ideas go back to Ostroy [24]. See also the survey by Ostroy and Starr [25]. We also emphasize that it is not important for the CM and DM to meet sequentially. One can also assume that they meet simultaneously, as long as agents cannot be in both places at the same time; see Williamson [39].

[16] In this presentation of the model no agent can both produce and consume, but this is easy to relax. In LW, all agents can do both, but there are many specialized goods and agents match randomly; this means that whether one consumes or produces depends on whom one meets, and some direct barter meetings are possible.

In the DM, in each match between a consumer and producer, they bargain over the quantity of goods q and amount of money d to swap, according to the generalized Nash solution with bargaining power for the consumer θ. The deal is constrained by $d \leqslant \hat{m}$. LW show that in equilibrium $d = \hat{m}$ and $q = q(\hat{m})$, where $q(\cdot)$ is given by the solution to $\beta_2 \hat{m}/p_{+1} = g(q)$, where g is defined by

$$g(q) = \frac{\theta\psi(q)u'(q) + (1-\theta)u(q)\psi'(q)}{\theta u'(q) + (1-\theta)\psi'(q)}. \tag{31}$$

We go into more detail on bargaining in the next subsection, where the derivation of (31) will be clear; for now we simply note that it is the price in the next CM, p_{+1}, that is relevant for q in this DM. [17] Given these results, the DM value function satisfies

$$\begin{aligned} V(\hat{m}) = \sigma &\left\{u[q(\hat{m})] + \beta_2 W_{+1}(0)\right\} \\ &+\sigma\left\{-\psi\left[q(M_{+1})\right] + \beta_2 W_{+1}(\hat{m} + M_{+1})\right\} \\ &+(1-2\sigma)\beta_2 W_{+1}(\hat{m}). \end{aligned} \tag{32}$$

The model is solved as follows. Given quasi-linearity, say $U = u(c) - h$, substitute h from the CM budget equation into W, and take the first-order condition with respect to $\hat{m}$:

$$\beta_1 V'(\hat{m}) = 1/p. \tag{33}$$

The envelope condition from (32) is $V'(\hat{m}) = \sigma u'(q)q'(\hat{m}) + (1-\sigma)\beta_2/p_{+1}$. Or, since $q'(\hat{m}) = \beta_2/p_{+1}g'(q)$, from the bargaining solution we have

$$V'(\hat{m}) = \frac{\beta_2}{p_{+1}}\left[1 - \sigma + \sigma\frac{u'(q)}{g'(q)}\right]. \tag{34}$$

Combining (33) and (34), we have

$$\frac{1}{p} = \frac{\beta}{p_{+1}}\left[1 - \sigma + \sigma\frac{u'(q)}{g'(q)}\right]. \tag{35}$$

Finally, the bargaining solution implies $1/p_{+1} = g(q)/\beta_2 M(1+\gamma)$ and lagging this yields $1/p = g(q_{-1})/\beta_2 M$, so (35) becomes

$$g(q_{-1}) = g(q)\frac{\beta}{1+\gamma}\left[1 - \sigma + \sigma\frac{u'(q)}{g'(q)}\right]. \tag{36}$$

Given a sequence for γ, any strictly positive and bounded solution to this difference equation in q constitutes a monetary equilibrium. (LW define equilibrium more formally, but it is clear that once one has the path for q one can recover all of the other endogenous variables.) If γ is constant, it makes sense to consider a steady state, where q is constant. Then the inflation rate

[17] Note that (31) is the solution for m only below a threshold m^*, while for $m \geqslant m^*$ we have $(d, q) = (m^*, q^*)$; LW prove that $m < m^*$ in any equilibrium, so we can ignore this detail. Also, note that Nash bargaining in the DM is not a key part of the theory: versions exist with different bargaining solutions (see Aruoba, Rocheteau and Waller [27]), with price taking or price posting (see Rocheteau and Wright [28]), and, when one allows for some multilateral meetings, with auctions (see Galenianos and Kircher [11]).

is γ, the real interest rate is ρ defined by $\beta = 1/(1+\rho)$, the nominal interest rate i is given by $i = (1+\rho)(1+\gamma) - 1$, and thus (36) reduces to

$$1 + \frac{i}{\sigma} = \frac{u'(q)}{g'(q)}. \tag{37}$$

This is the basic model.[18]

We close this section by arguing that, in any equilibrium, the assumption of only spot trades in the CM is not restrictive. First note that no trades can be made for delivery in any meeting of a future DM, since in this market meetings are anonymous (hence, anyone who was supposed to deliver in this market would simply renege). For similar reasons no trades can be made in the CM contingent on events in any future DM (no one sees what happens to other agents in the DM). So we are left with trades made in the current CM for delivery in some future CM. But in equilibrium no one partakes of these trades, for the following reason.

Suppose we open a market for Arrow securities that deliver 1 unit of purchasing power (money) in the next CM. Let $W(m, b)$ now be the value function for an agent entering the CM with money m and security holdings b, and let $\hat{\mathbf{b}}$ be a vector of assets purchased in that period. We assume that these Arrow securities are *not* tangible assets that can be traded bilaterally in the DM—they are simply promises of purchasing power to be delivered in the next CM—which is a strong but logically consistent assumption.[19] Then, extending the earlier results, one can show that $(\hat{m}, \hat{\mathbf{b}})$ is independent of (m, b). Hence, all agents choose the same portfolio, and the market clears at $\hat{\mathbf{b}} = 0$. Since $\hat{\mathbf{b}} = 0$ in equilibrium, we can shut down these asset markets with no loss in generality.

3.2. A new model

We now consider a model similar to LW, except for two main differences: instead of quasi-linearity, we allow a general utility function, but we assume that labor is indivisible. We also allow for other extensions such as a general vector of consumption goods $\mathbf{c}$, an arbitrary endowment vector $\mathbf{e}$ that may differ over time, profit-maximizing firms, and so on, but we continue to assume for now homogeneous preferences: $U^i = U(\mathbf{c}, h)$ for all i. The key assumption is indivisible labor, and, given this, we now consider sunspot equilibria. As in the previous section, we assume only spot trades, but argue below that this is again without loss in generality.

In terms of state-contingent commodities, the CM consumer problem is

$$W^i = \max_{\mathbf{c}^i(s), h^i(s), \hat{m}^i(s)} \int_S \left\{ U[\mathbf{c}^i(s), h^i(s)] + \beta_1 V^i[\hat{m}^i(s)] \right\} ds$$

subject to

$$\int_S [\mathbf{p}(s)\mathbf{c}^i(s) + \hat{m}^i(s) - w(s)h^i(s) - \mathbf{p}(s)\mathbf{e}^i - m^i - \gamma M - \Delta^i]\, ds \leqslant 0.$$

[18] We do not dwell on substantive results here, except to mention the following. Under standard assumptions, a monetary equilibrium exists if and only if $i \geqslant 0$. For $i > 0, q < q^*$, so equilibrium is inefficient. Welfare is maximized at the Friedman Rule, $i = 0$, but if $\theta < 1$ then we have $q < q^*$ even at $i = 0$.

[19] One justification for this is the following: even if these securities were tangible objects that could in principle be traded in the DM, if we assume they can be counterfeited costlessly while fiat currency cannot be, there can be a role for the latter as a medium of exchange.

This is formally equivalent to the consumer problem in Definition 1, even though money is not a standard commodity here. The only detail to worry about is that one has to show V^i is well behaved—in particular, strictly concave—which can be done following the methods in LW. Therefore, we can focus on equilibria with $[\mathbf{p}(s), w(s)] = (\mathbf{p}, w)$. Moreover, we know that $[\mathbf{c}^i(s), \hat{m}^i(s)] = (\mathbf{c}_1^i, \hat{m}_1^i)$ for all s such that $h^i(s) = 1$ and $[\mathbf{c}^i(s), \hat{m}^i(s)] = (\mathbf{c}_0^i, \hat{m}_0^i)$ for all s such that $h^i(s) = 0$, by Lemmas 2 and 3. Also, given $\hat{m}^i$ and h^i enter separably, we additionally know that $\hat{m}_1^i = \hat{m}_0^i = \hat{m}^i$, by Lemma 4.

Given these results, the problem reduces to

$$W^i(x^i) = \max_{\mathbf{c}_1^i, \mathbf{c}_0^i, \ell^i, \hat{m}^i} \left\{ \ell^i U(\mathbf{c}_1^i, 1) + (1 - \ell^i) U(\mathbf{c}_0^i, 0) + \beta_1 V^i(\hat{m}^i) \right\}$$

subject to

$$\ell^i \mathbf{p}\mathbf{c}_1^i + (1 - \ell^i)\mathbf{p}\mathbf{c}_0^i + \hat{m}^i - w\ell^i - x^i \leqslant 0,$$

where $x^i = \mathbf{p}\mathbf{e}^i + m^i + \gamma M + \Delta^i$. Assuming an interior solution, the first-order conditions are

$$c_{1j}^i : U_j(\mathbf{c}_1^i, 1) - \lambda^i p_j = 0 \quad \text{for all } j, \tag{38}$$

$$c_{0j}^i : U_j(\mathbf{c}_0^i, 0) - \lambda^i p_j = 0 \quad \text{for all } j, \tag{39}$$

$$\ell^i : U(\mathbf{c}_1^i, 1) - U(\mathbf{c}_0^i, 0) + \lambda^i (w + \mathbf{p}\mathbf{c}_0^i - \mathbf{p}\mathbf{c}_1^i) = 0, \tag{40}$$

$$\hat{m}^i : \beta_1 V'(\hat{m}^i) - \lambda^i = 0, \tag{41}$$

and

$$\lambda^i : w\ell^i + x^i - \ell^i \mathbf{p}\mathbf{c}_1^i - (1 - \ell^i)\mathbf{p}\mathbf{c}_0^i - \hat{m}^i = 0. \tag{42}$$

Observe that (38)–(40) constitute $2J+1$ equations in $2J+1$ unknowns. Under the assumptions of Proposition 6, and in particular $w - \mathbf{p}\mathbf{c}_1^i + \mathbf{p}\mathbf{c}_0^i \neq 0$, these equations can be solved for $(\mathbf{c}_1^i, \mathbf{c}_0^i, \lambda^i)$ independent of $\hat{m}^i$ and ℓ^i, as a function of $(\mathbf{p}, w)$ but not x^i. Because the only way consumers differ here is with respect to x^i, $(\mathbf{c}_1^i, \mathbf{c}_0^i, \lambda^i) = (\mathbf{c}_1, \mathbf{c}_0, \lambda)$ is the same for all i. Given λ, (41) can be solved for $\hat{m}^i$ independent of ℓ^i, as a function of $(\mathbf{p}, w)$ but not x^i. Notice that in (41) we did not index V' by i, implying $\hat{m}^i = \hat{m}$ for all i; this follows from the following lemma.

Lemma 12. *Under the assumptions of Proposition* 6, $W^i(x^i)$ *and* $V^i(\hat{m}^i)$ *depend on* i, *but* $\partial W^i(x^i)/\partial x^i = \lambda$ *and* $\partial V^i(\hat{m}^i)/\partial \hat{m}^i = V'(\hat{m}^i)$ *do not depend on* i.

Proof. Consider $W^i(x)$. We have

$$\frac{\partial W^i}{\partial x} = [U(\mathbf{c}_0, 0) - U(\mathbf{c}_1, 1) + \lambda(w + \mathbf{p}\mathbf{c}_0 - \mathbf{p}\mathbf{c}_1)] \frac{\partial \ell^i}{\partial x} + \lambda.$$

The first term vanishes by (40), so $\partial W^i/\partial x^i = \lambda$, which is independent of i and x^i. We show the other result below, after we have described V^i in more detail; see (47). □

We formalize the analysis of the first-order conditions up to this point as follows.

Lemma 13. *Under the assumptions of Proposition* 6, *we have* $(\mathbf{c}_1^i, \mathbf{c}_0^i, \hat{m}^i, \lambda^i) = (\mathbf{c}_1, \mathbf{c}_0, \hat{m}, \lambda)$ *for all* i, *independent of* x^i.

Proof. Follows from the discussion in the text. □

Given $(\mathbf{c}_1, \mathbf{c}_0, \hat{m}, \lambda)$, (42) implies that

$$\ell^i = \frac{\mathbf{pc}_0 + \hat{m}^i - x^i}{w + \mathbf{pc}_0 - \mathbf{pc}_1} = \frac{\mathbf{pc}_0 + M - m^i - \mathbf{pe}^i - \Delta^i}{w + \mathbf{pc}_0 - \mathbf{pc}_1}, \tag{43}$$

so ℓ^i is linearly decreasing in x^i and hence in m^i. But aggregate labor supply $\bar{\ell}$ is given by

$$\bar{\ell} = \bar{\ell}(\mathbf{p}, w) = \frac{\mathbf{pc}_0 - \mathbf{p}\bar{\mathbf{e}} - \bar{\Delta}}{w + \mathbf{pc}_0 - \mathbf{pc}_1} \tag{44}$$

and thus depends only on average real wealth, $\mathbf{p}\bar{\mathbf{e}} + \bar{\Delta}$. This means that aggregate demand for the J consumption goods,

$$\mathbf{D}(\mathbf{p}, w) = \bar{\ell}(\mathbf{p}, w)\mathbf{c}_1(\mathbf{p}, w) + [1 - \bar{\ell}(\mathbf{p}, w)]\mathbf{c}_0(\mathbf{p}, w),$$

does not depend on the distribution of wealth.

We can now define market clearing by

$$\mathbf{Z}(\mathbf{p}, w) \equiv \mathbf{D}(\mathbf{p}, w) - \Sigma_k \mathbf{f}^k[n^k(\mathbf{p}, w)] - \bar{e} = 0,$$
$$N(\mathbf{p}, w) \equiv \sum_k n^k(\mathbf{p}, w) - \ell(\mathbf{p}, w) = 0, \tag{45}$$

and

$$\hat{M}(\mathbf{p}, w) \equiv \hat{m}(\mathbf{p}, w) - M(1 + \gamma) = 0.$$

There are $J+2$ equations and we only determine $J+1$ prices $(\mathbf{p}, w)$, as we have already normalized the price of money to 1. Note that Walras' Law holds: integrating the budget equations over agents, if the goods and labor markets clear, then the money market clears. Existence of a solution to (45), which is an equilibrium in the CM, taking V^i as given, is established as in Proposition 11. Notice that CM equilibrium is determined independent of the value of q, which does not appear in (45); this is referred in Aruoba and Wright [3] to as the *neoclassical dichotomy*.[20]

To determine properties of V^i we now proceed to the DM, beginning with bargaining. Again we use the generalized Nash solution. Consider a meeting between agents i and i' where the former is the consumer and the latter the producer. For the consumer, his payoff is $u(q) + \beta_2 W^i_{+1}(x^i_{+1} - d)$ and his threat point $\beta_2 W^i_{+1}(x^i_{+1})$, where $x^i_{+1} = \mathbf{p}_{+1}\mathbf{e}^i_{+1} + \hat{m}^i + \gamma_{+1}M_{+1} + \Delta^i_{+1}$ is his wealth in the next CM if he does not trade. For the producer, his payoff is $-\psi(q) + \beta_2 W^{i'}_{+1}(x^{i'}_{+1} + d)$ and his threat point is $\beta_2 W^{i'}_{+1}(x^{i'}_{+1})$. Given that W^i_{+1} and $W^{i'}_{+1}$ both have slope λ_{+1} by Lemma 12, where λ_{+1} is the same for all consumers, the bargaining solution reduces to

$$\max_{q,d}[u(q) - \beta_2 d\lambda_{+1}]^{\theta}[\beta_2 d\lambda_{+1} - \psi(q)]^{1-\theta} \tag{46}$$

subject to the constraint $d \leqslant \hat{m}^i$.

[20] One implication of this dichotomy is that monetary policy (the path for M) does not affect the values of $(\mathbf{c}_1, \mathbf{c}_0, \bar{\ell})$ determined in the CM.

As in LW, one can show that in any equilibrium the constraint holds with equality: $d = \hat{m}^i$. Substituting this into (46), the first-order condition with respect to q is

$$\theta\left[-\psi(q) + \beta_2 \hat{m}^i \lambda_{+1}\right] u'(q) = (1-\theta)\left[u(q) - \beta_2 d \hat{m}^i \lambda_{+1}\right] \psi'(q).$$

This can be rearranged into $\beta_2 \hat{m}^i \lambda_{+1} = g(q)$, where $g(\cdot)$ is the same as in (31) above. Since $\hat{m}^i = M_{+1}$ for all i, in equilibrium, $q = q(M_{+1})$ is the same in every trade, and the DM value function satisfies

$$\begin{aligned} V^i(\hat{m}^i) = {} & \sigma\left\{u[q(\hat{m}^i)] + \beta_2 W_{+1}(x^i_{+1} - \hat{m}^i)\right\} \\ & +\sigma\left\{-\psi(q) + \beta_2 W_{+1}(x^i_{+1} + M_{+1})\right\} \\ & +(1-2\sigma)\beta_2 W_{+1}(x^i_{+1}). \end{aligned}$$

Notice that V^i is indexed by *i*, because $x^i_{+1} = \mathbf{p}_{+1}\mathbf{e}^i_{+1} + \hat{m}^i + \gamma_{+1} M_{+1} + \Delta^i_{+1}$ can differ across individuals. However, as claimed in Lemma 12, the derivative

$$\frac{\partial V^i}{\partial \hat{m}^i} = \sigma u'(q) q'(\hat{m}^i) + (1-\sigma)\beta_2 \lambda_{+1} = \left[1 - \sigma + \sigma \frac{u'(q)}{g'(q)}\right] \beta_2 \lambda_{+1} \tag{47}$$

does not depend on *i*, where we get $q'(\hat{m}^i) = \beta_2 \lambda_{+1} u'(q)/g'(q)$ from the bargaining solution. Inserting (47) into (41), we have

$$\lambda = \beta\left[1 - \sigma + \sigma \frac{u'(q)}{g'(q)}\right] \lambda_{+1}. \tag{48}$$

Using $\beta_2 \hat{m}^i \lambda_{+1} = g(q)$, and market clearing $\hat{m}^i = M(1+\gamma)$, (48) becomes

$$g(q_{-1}) = g(q) \frac{\beta}{1+\gamma}\left[1 - \sigma + \sigma \frac{u'(q)}{g'(q)}\right]. \tag{49}$$

Observe that (49) is identical to (36). Hence, in terms of the DM, the new model has *exactly the same* predictions as LW. Of course, the CM differs across the two models, because different commodities are being traded. Still, it is the case that the assumption of only spot trades in the CM is without loss in generality. This is perhaps less clear here because we have arbitrary endowments across agents, so one might think they would want to borrow or lend; but as long as they are at an interior solution, they would just as soon increase or decrease ℓ^i, and we can shut down the asset markets with no loss in generality.

Before closing this discussion, we reconsider the maintained assumption $\ell^i \in (0, 1)$ for all *i*, using the ideas in Proposition 9. In equilibrium all agents enter each CM with one of the three values of m: $m = 0$ if they consumed in the previous DM, $m = 2M$ if they produced in the previous DM, or $m = M$ if they did neither. So from (43), ℓ^i takes on one of the three values:

$$\ell^i = \begin{cases} \dfrac{\mathbf{pc}_0 + M - \mathbf{pe}^i - \Delta^i}{w + \mathbf{pc}_0 - \mathbf{pc}_1} & \text{if } m = 0, \\ \dfrac{\mathbf{pc}_0 - \mathbf{pe}^i - \Delta^i}{w + \mathbf{pc}_0 - \mathbf{pc}_1} & \text{if } m = M, \\ \dfrac{\mathbf{pc}_0 - M - \mathbf{pe}^i - \Delta^i}{w + \mathbf{pc}_0 - \mathbf{pc}_1} & \text{if } m = 2M. \end{cases} \tag{50}$$

As in the proof of Proposition 9, there are two cases: $w + \mathbf{pc}_0 - \mathbf{pc}_1 > 0$, or, equivalently from (40), $U(\mathbf{c}_0^i, 0) > U(\mathbf{c}_1^i, 1)$; and $w + \mathbf{pc}_0 - \mathbf{pc}_1 < 0$, or, equivalently, $U(\mathbf{c}_0^i, 0) < U(\mathbf{c}_1^i, 1)$. For brevity we present only the former case.[21]

Then, for a given i, (50) implies

$$\ell^i > 0\ \forall i \text{ if and only if } \mathbf{pe}^i + \Delta^i < \mathbf{pc}_0 - M \text{ for all } i \tag{51}$$

and

$$\ell^i < 1\ \forall i \text{ if and only if } \mathbf{pe}^i + \Delta^i > M - w + \mathbf{pc}_1 \text{ for all } i. \tag{52}$$

Or, to put this in real terms, using the bargaining solution to eliminate $M = g(q_{-1})/\beta_2\lambda$ allows us to write

$$\ell^i > 0 \text{ for all } i \text{ if and only if } g(q_{-1}) < \beta_2\lambda(\mathbf{pc}_0 - \mathbf{pe}^i - \Delta^i) \equiv \Gamma_1^i\ \forall i \tag{53}$$

and

$$\ell^i < 1 \text{ for all } i \text{ if and only if } g(q_{-1}) < \beta_2\lambda(\mathbf{pe}^i + \Delta^i - \mathbf{pc}_1 + w) \equiv \Gamma_2^i\ \forall i. \tag{54}$$

This yields bounds on $g(q_{-1})$, Γ_1^i, and Γ_2^i, which are independent of monetary considerations—i.e., they take on the same values in the nonmonetary economy where $M = 0$. The bounds are strictly positive if and only if $\ell^i \in (0, 1)$ in the nonmonetary economy. If $g(q_{-1}) < \min\left\{\Gamma_1^i, \Gamma_2^i\right\}$ for all i, we are done. But it is easy to see that g is increasing, and that $q < q^*$ in any equilibrium, as in LW, which can be used to guarantee that $g(q_{-1}) < \min\left\{\Gamma_1^i, \Gamma_2^i\right\}$.

Hence, one can impose conditions that guarantee interiority.[22] We summarize the above results as follows:

Proposition 14. *In the model of this section, $\ell^i \in (0, 1)$ for all i as long as $g(q^*) < \min\left\{\Gamma_1^i, \Gamma_2^i\right\}$ for all i, where Γ_1^i and Γ_2^i are defined in* (53) *and* (54). *Given this, the equilibrium q sequence satisfies* (49), *which is the same as the equilibrium condition in the basic LW model.*

Finally, we have an easy existence result. Due to the dichotomy, a monetary equilibrium consists of two independent components: we need a CM equilibrium, $\mathbf{Z}(\mathbf{p}, w) = \mathbf{0}$ and $N(\mathbf{p}, w) = 0$, at every date; and we need a path for $q > 0$ satisfying (49) at every date. The existence of the former is established, as we said above, as in Proposition 11; and the existence and properties of the latter follow from previous analyses in Lagos and Wright [20,21].

Proposition 15. *Given our assumptions, there are $(\mathbf{p}, w)$ paths such that $\mathbf{Z}(\mathbf{p}, w) = \mathbf{0}$ and $N(\mathbf{p}, w) = 0$ at every date, plus a path for $q > 0$ satisfying* (49).

4. Conclusion

We have presented a framework within which one can analyze GE with nonconvexities. In nonconvex economies, randomization can be desirable. We showed how to support random allocations as competitive equilibria by using sunspots. These equilibria have interesting properties,

[21] The other case is similar. Again we ignore the special intermediate case $U(\mathbf{c}_0^i, 0) = U(\mathbf{c}_1^i, 1)$.

[22] Intuitively, we need to have the DM to be "not too important", in the sense that q is not too big, because otherwise the value of money is too high and this either forces some people to $\ell = 1$ (those with no money trying to acquire $\hat{m}$), or forces some people to $\ell = 0$ (those with lots of money trying to spend down to $\hat{m}$).

including the property that agents act as if they have quasi-linear utility. This means that, given they choose interior solutions, there are zero wealth effects, which leads to strong results. In monetary economics, this provides an alternative to the framework in Lagos and Wright [21]. The alternative is based on indivisible labor and sunspots rather than on the quasi-linearity of underlying preferences.

We made an effort to provide a somewhat general presentation of the results, in order to show that monetary theory is actually more robust than one might have thought based on previous papers. This also helps reconcile somewhat two seemingly disparate literatures: GE theory and monetary economics. Our presentation, however, is not as general as one might like. In particular, the dichotomy result is convenient for several reasons, including the fact that it allows us to establish easily the existence of monetary equilibrium, but it is also quite special. For instance, it does not hold when preferences are not separable between $(\mathbf{c}, h)$ and q (see, e.g., [2]). Analyzing this more general case may be useful for the theory, as well as for the applications.

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Appendix A. Second-order conditions

Here, we check the second-order conditions for a strict local maximum to the consumer's problem, assuming $\ell \in (0, 1)$. The first-order conditions for an interior solution are given by (19)–(22). The bordered Hessian evaluated at any point where they are satisfied is

$$\mathbf{B} = \begin{bmatrix} 0 & -\ell\mathbf{p} & -(1-\ell)\mathbf{p} & w - \mathbf{pc}_1 + \mathbf{pc}_0 \\ -\ell\mathbf{p}^T & \ell\mathbf{H}_1 & 0 & 0 \\ -(1-\ell)\mathbf{p}^T & 0 & (1-\ell)\mathbf{H}_0 & 0 \\ w - \mathbf{pc}_1 + \mathbf{pc}_0 & 0 & 0 & 0 \end{bmatrix}.$$

For a maximum the last $2J - 1$ leading principal minors $|\mathbf{B}_3|, |\mathbf{B}_4|, \ldots, |\mathbf{B}_{2J+2}|$ must alternate in sign, with $|\mathbf{B}_{2J+2}| < 0$. To begin,

$$|\mathbf{B}_{2J+2}| = -(w - \mathbf{pc}_1 + \mathbf{pc}_0)^2 \ell(1-\ell)|\mathbf{H}_1||\mathbf{H}_0| < 0,$$

since $|\mathbf{H}_h| < 0$ by the strict concavity of U. Second, consider

$$\mathbf{B}_{1+j} = \ell \begin{bmatrix} 0 & -\mathbf{p}_j \\ -\mathbf{p}_j^T & \mathbf{H}_{1j} \end{bmatrix}$$

with $j \in \{2, \ldots, J\}$. Here, $\mathbf{p}_j = (p_1, \ldots, p_j)$ and $\mathbf{H}_{hj}$ is the submatrix of $\mathbf{H}_h$ defined by deleting all but the first j columns and rows. By the strict concavity of U, $|\mathbf{B}_{1+j}|$ has the same sign as $(-1)^j$.

Consider next

$$\mathbf{B}_{1+J+j} = \begin{bmatrix} 0 & -\ell\mathbf{p} & -(1-\ell)\mathbf{p}_j \\ -\ell\mathbf{p}^T & \ell\mathbf{H}_1 & 0 \\ -(1-\ell)\mathbf{p}_j^T & 0 & (1-\ell)\mathbf{H}_{0j} \end{bmatrix}$$

with $j \leqslant J$. Then

$$\left|\mathbf{B}_{1+J+j}\right| = (1-\ell)U_{jj}(\mathbf{c}_0, 0)\left|\mathbf{B}_{J+j}\right| - (1-\ell)^2 p_j^2 \ell(1-\ell)\left|\mathbf{H}_1\right|\left|\mathbf{H}_{0j-1}\right|.$$

By induction, $\left|\mathbf{B}_{1+J+j}\right|$ has the same sign as $(-1)^{J+j}$. To see this, note that $|\mathbf{B}_{J+2}|$ has the same sign as $(-1)^{J+1}$ and $|\mathbf{H}_1|\left|\mathbf{H}_{0j-1}\right|$ has the same sign as $(-1)^{J+j-1}$. Therefore any point that satisfies the first-order conditions is a strict local maximum.

Appendix B. Global maximum

Here, we use the results in Appendix A to show that a solution to the first-order conditions constitutes the global maximum. We begin by breaking the problem into two steps. First, define the problem

$$\mathcal{V}(\ell) = \max_{\mathbf{c}_1,\mathbf{c}_0}[\ell U(\mathbf{c}_1, 1) + (1-\ell)U(\mathbf{c}_0, 0)]$$

subject to

$$\ell\mathbf{p}\mathbf{c}_1 + (1-\ell)\mathbf{p}\mathbf{c}_0 - \ell w - x \leqslant 0.$$

Since U is strictly concave, this problem has a unique solution $[\mathbf{c}_0(\ell), \mathbf{c}_1(\ell)]$. By the Theorem of the Maximum, $\mathcal{V}(\ell)$ is continuous and hence achieves a maximum over $\ell \in [0, 1]$.

Suppose there are two local maxima. Then by continuity $\mathcal{V}(\ell)$ also has a local minimum at some $\tilde{\ell} \in (0, 1)$. Then $[\mathbf{c}_0(\tilde{\ell}), \mathbf{c}_1(\tilde{\ell}), \tilde{\ell}]$ is a saddle-point of the problem in Appendix A, which contradicts the result that any solution to the first-order conditions is a local maximum. Hence, there is a unique maximizer of $\mathcal{V}(\ell)$, say $\hat{\ell}$, and $[\mathbf{c}_0(\hat{\ell}), \mathbf{c}_1(\hat{\ell}), \hat{\ell}]$ is the unique solution to the problem in Appendix A.

Appendix C. Nonseparable example

Suppose that $U(c, \hat{m}, h) = c^a(1+b-h)^{1-a} + V(\hat{m})$, where $0 < a < 1$ and $0 < b < (1-a)/a$. Now we do not have $c_0 = c_1$, although we still have $\hat{m}_0 = \hat{m}_1 = \hat{m}$. The consumer problem is

$$W = \max\left\{\ell c_1^a b^{1-a} + (1-\ell)c_0^a(1+b)^{1-a} + V'(\hat{m})\right\}$$

subject to

$$\ell c_1 + (1-\ell)c_0 - B\ell - \frac{m-\hat{m}}{p} \leqslant 0.$$

Substituting for $\hat{m}$ and taking the first-order conditions yields

$$c_1 : ac_1^{a-1}b^{1-a} = pV'(\hat{m}),$$

$$c_0 : ac_0^{a-1}(1+b)^{1-a} = pV'(\hat{m}),$$

and

$$\ell : c_1^a b^{1-a} - c_0^a (1+b)^{1-a} = pV'(\hat{m})[c_1 - c_0 - B].$$

These equations can be solved for $c_0 = abB/(1-a)$, $c_1 = a(1+b)B/(1-a)$, and $V'(\hat{m}) = a^a(1-a)^{1-a}B^{a-1}/p$. Thus c_1, c_0, and $\hat{m}$ are independent of m. Now $n = a(1+b) \in (0,1)$. From the budget equation, we have

$$\ell = a(1+b) + (M-m)a^{-a}(1-a)^a B^a V'(M)$$

after substituting $\hat{m} = M$ and p. Hence, $\ell \in (0,1)$ if and only if $m \in (\underline{m}, \overline{m})$, where

$$\underline{m} = M - \frac{a^a B^a (1-a-ab)}{(1-a)^a V'(M)} \quad \text{and} \quad \overline{m} = M + \frac{a^{1+a}(1+b)B^a}{(1-a)^a V'(M)}.$$

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