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Growth Dynamics and Returns to Scale: Bifurcation Analysis[1]

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We investigate the dependence of the dynamic behavior of an endogenous growth model on the degree of returns to scale. We focus on a simple (but representative) growth model with publicly funded inventive activity. We show that constant returns to reproducible factors (the leading case in the endogenous growth literature) is a bifurcation point, and that it has the characteristics of a transcritical bifurcation. The bifurcation involves the boundary of the state space, making it

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difficult to formally verify this classification. For a special case, we provide a transformation that allows formal classification by existing methods. We discuss the new methods that would be needed for formal verification of transcriticality in a broader class of models. *Journal of Economic Literature* Classification Numbers: O41, O30.

Key Words: returns to scale; transcritical bifurcation; boundary bifurcation; endogenous growth; inventive activity.

1. INTRODUCTION

The mathematical theory of bifurcations has been used to analyze and categorize the dynamic behavior of a wide variety of economic models. The area that has probably received the most intensive application is the study of business cycles and economic fluctuations, where bifurcation theory has been useful for proving the existence both of deterministic cycles[2] and of sunspot equilibria.[3] Other applications have been found throughout economics, ranging from growth and development to the tatonement process in general equilibrium to continuous-time macroeconometric models.[4] All of these applications have employed the standard theory of bifurcations, as presented by Guckenheimer and Holmes [15], Wiggins [27], and others, which deals with dynamical systems defined on open sets. In particular, this implies that all relevant trajectories, including steady states, must be contained in the interior of the state space. In many economic models, however, the state space contains a boundary. The standard neoclassical growth model, for instance, has a steady state with zero capital, on the boundary of the set of feasible capital stocks. As long as the bifurcation that one is interested in analyzing does not involve this steady state, there is no problem in applying the standard theory. Bifurcations that do involve this boundary steady state, however, cannot be analyzed by direct application of the standard techniques.

We study a growth model in which a bifurcation event occurs on the boundary of the state space. We use graphical methods to analyze this event and show that, under the appropriate transformation, it displays the signature pattern of a transcritical bifurcation. We then show that, in a special case, applying a change of coordinates allows us to formally verify this classification using standard (interior) techniques. Our method does not, however, allow us to formally classify the bifurcation in the general

[2] See, for example, Benhabib and Nishimura [6], Benhabib and Rustichini [7], Cartigny and Venditti [10], and Boyd and Smith [8].

[3] See, among others, Grandmont [13], Azariadis and Guesnerie [2] and Chiappori *et al.* [11].

[4] See Matsuyama [19] and Becker and Foias [5] on the first topic, Bala [3] on the second, and Barnett and He [4] on the third.

model. We conclude by discussing the problems that generalization presents and what approaches we think might be successful in addressing them.

Ours is an endogenous growth model with increasing returns to scale in production. Our interest is in how the dynamic behavior of the economy is affected by the degree of returns to scale in the set of reproducible factors of production. Much is already known about this problem in a variety of settings. When there are decreasing returns to the set of reproducible factors, our model behaves very much as in Solow [25] and Diamond [12]; there is an interior steady state to which all interior trajectories converge, and an unstable steady state at the origin. When there are exactly constant returns, the origin is typically the only steady state and all interior trajectories converge to a balanced growth path. In this case our model reduces to the *AK* model that has received so much attention in the endogenous growth literature.[5] When there are increasing returns to the set of reproducible factors, there is an interior steady state that is unstable. Trajectories starting high enough experience unbounded growth while trajectories that start too low decay to the origin, a phenomenon often referred to as a "poverty trap." Such morphogenetic models have been studied by Shell [22, 23], Azariadis and Drazen [1], and others. Our goal is to synthesize these results for the different cases by providing a complete analysis of how the global dynamics vary with the degree of returns to scale and by analyzing the bifurcation that divides these regimes.

Our analysis sheds light on the mathematical nature of a strong criticism of endogenous growth theory made by Solow [26] and others. The recent endogenous growth literature has concentrated almost exclusively on models that generate balanced growth paths. As indicated in the discussion above, the existence of a balanced growth path requires that there be exactly constant returns to the set of reproducible factors of production. This led Solow [26, p. 13] to remark that "the key hypothesis underlying at least the *AK* version of New Growth Theory is completely nonrobust. Nature must do exactly the right thing or else the theory evaporates one way or another." Our analysis formalizes the mathematics behind this criticism: the *AK* model is based on a bifurcation point in parameter space. The global dynamics are necessarily qualitatively different if the returns-to-scale parameter is even slightly different from its bifurcation value.

In the next section, we describe our model and how it relates to others in the endogenous growth literature. In Section 3, we provide an analysis of the equilibrium trajectories generated by the model. In Section 4, we give a graphical analysis of the bifurcation, while Section 5 contains the formal bifurcation analysis for a special case of the model. In Section 6, we

[5] See, for example, McGrattan [20] and the references therein.

conclude by discussing the difficulties involved in generalizing the formal result.

2. THE MODEL

Our analysis is in discrete time. Had we cast the model in continuous time, for sufficiently strong increasing returns our production technology would have allowed infinite output after a finite amount of time. By using discrete-time analysis, we avoid this possibility. A second important modelling decision regards the lifetime of the consumers in the model. Much of the literature on endogenous growth uses infinitely lived agents, but we choose to work with an overlapping-generations model where each consumer lives for two periods. This allows us to avoid the possibility that an individual agent may, under strongly increasing returns, a-ttain unbounded levels of lifetime utility. It also serves to simplify the consumption side of the model considerably, allowing us to concentrate on the production side.

2.1. *Production.* Production in our model takes place using three inputs: technology, physical capital, and labor. We assume that technology is a nonrival input and that production exhibits exactly constant returns to scale in labor and physical capital. This second assumption is justified by a standard replication argument: since technology is nonrival, doubling capital and labor should exactly double output. We assume that technological progress occurs through the allocation of resources to inventive activity.[6] We also assume that technology is nonexcludable; once a new technology is invented, it is immediately available to all producers. One interpretation of technology in our model is the output of basic scientific research. These assumptions allow us to work with perfect competition in labor, capital, and product markets. In this way, the model here is much simpler than those in Romer [21], Grossman and Helpman [14], and others, where technological advances are at least partially protected by patent systems. This simplicity allows us to present our bifurcation analysis using closed-form expressions for the savings function, etc., but it may not be critical for our qualitative results.

It is worth emphasizing that these assumptions imply that production in our model necessarily displays increasing returns to the *total* set of factors of production: technology, capital, and labor. Our assumptions imply that payments to capital and labor will exhaust output, so that in competitive equilibrium profits will necessarily be zero. Competitive, profit maximizing firms will not undertake any investment in technological progress, because they are unable to internalize any of the return from this investment. As a

[6] This is in the tradition of Shell [22, 23, 24].

consequence, any investment in inventive activity must be publicly financed. We assume the presence of a government that imposes taxes on labor income and invests the revenue in inventive activity.

We use A to denote the stock of technology, K the stock of physical capital, and L the stock of labor (all nonnegative quantities). Total output Y is given by the function $F(A, K, L; \lambda)$, where the nonnegative scalar λ measures the degree of returns to scale in production. We begin by placing the following restrictions on F:

(A1) $F(0, K, L; \lambda) = F(A, 0, L; \lambda) = F(A, K, 0; \lambda) = 0$ for any A, K, L and λ

(A2) $F(A, K, L; \lambda)$ is strictly increasing in (A, K, L) on $\mathbb{R}^4_{++}$

(A3) F is three-times continuously differentiable on $\mathbb{R}^4_{++}$ and continuous on $\mathbb{R}^4_{+}$

(A4) F is homogeneous of degree one and concave in (K, L)

(A5) F is homogeneous of degree λ in (A, K)

(A6) The cross-partial derivative F_{AL} is positive for all A, K, L and λ in $\mathbb{R}^4_{++}$

The first two assumptions are standard. The third guarantees that the equilibrium difference equation will be twice continuously differentiable, and thereby allows us to use the results of smooth bifurcation theory. The fourth assumption can, as mentioned above, be justified using a standard replication argument. The fifth allows us to talk unambiguously about the degree of returns to scale in technology and capital, the reproducible factors of production in this model. Our primary interest, therefore, is in how the equilibrium dynamical system varies with the parameter λ. The final assumption is a restriction on the form of technological change; increases in technology must increase the marginal product of labor. We will see below that this implies that private saving is an increasing function of the level of technology.

We employ a simple one-sector technology in which output can be converted into either new capital or new technology. Each period a fraction μ of the capital stock is lost to depreciation, as is a fraction ρ of the stock of technology. As in Shell [22], this depreciation of A is taken to be a reduced-form representation of some underlying process where there are frictions in the transfer of knowledge from one generation to the next (see Jovanovic and Nyarko [18] for an explicit model of this process).

Using lower-case letters to denote per capita variables, we have

$$y = f(A, k; \lambda) \equiv F\left(A, \frac{K}{L}, 1; \lambda\right).$$

The intensive production function f inherits the following properties from F: on $\mathbb{R}^3_{++}$ it is both strictly increasing in (A, k) and C^3, it has $f(0, k; \lambda) = f(A, 0; \lambda) = 0$, and it is homogenous of degree λ in (A, k).

Firms take the level of A as given and choose k to maximize profits. The first-order conditions of a firm's problem are therefore completely standard: they require that the wage for labor and the rental rate for capital equal the marginal products of these factors. Denoting these by w and r, respectively, we have

$$w_t = f(A_t, k_t; \lambda) - k_t f_k(A_t, k_t; \lambda) \equiv w(A_t, k_t; \lambda) \tag{2.1}$$

$$r_t = f_k(A_t, k_t; \lambda). \tag{2.2}$$

Note that the wage function w is homogeneous of degree λ in (A, k), while the rental rate r is homogeneous of degree $(\lambda - 1)$. Both are twice continuously differentiable functions of (A, k, λ).

The government taxes labor income in order to finance inventive activity. For simplicity, we assume that wages are taxed at a rate that is constant over time and given by τ. Since all inventive activity is publicly financed, total expenditure on inventive activity at time t is given by $\tau w_t L_t$.

2.2. *Consumers.* Each consumer lives for two periods and consumes the single commodity in both periods. Let c_t^s denote the consumption in period s of the consumer born in period t, and let c_t be the vector of consumptions (c_t^t, c_t^{t+1}). Each consumer has preferences over consumption represented by the utility function $u(c_t) = (c_t^t)^\gamma \, (c_t^{t+1})^{1-\gamma}$. In addition, each consumer is endowed with one unit of labor when young, which she supplies inelastically, earning the wage w_t. The problem of a consumer born in period t is then to choose a savings function s_t to solve

$$\max (c_t^t)^\gamma \, (c_t^{t+1})^{1-\gamma}$$

subject to

$$c_t^t = (1-\tau)\, w_t - s_t$$

$$c_t^{t+1} = s_t(r_{t+1} + 1 - \mu).$$

The savings function that solves this problem is given by

$$s_t = (1-\gamma)(1-\tau)\, w_t. \tag{3}$$

The fact that saving is independent of the rate of return is a consequence of the assumptions of log-linear preferences and no old-age income. In this case, the income and substitution effects associated with a change in the rate of return exactly cancel, leaving the optimal level of saving unchanged. This results in a substantial simplification of the equilibrium law of motion for the stock of physical capital. The form of the savings function implies that total investment in physical capital depends on payments to labor and not on payments to capital. This combines with assumption (A6) to generate a positive relationship between the level of technology and the level of private savings.

We analyze the model with no population growth. It is worth noting that there is a scale effect in this model. Since the use of technology is nonrival, it is the total level of technology A, not the per-capita level, that determines per-capita output. Therefore, it is the total level of investment in inventive activity $\tau w_t L$ that affects future per-capita output. *Ceteris paribus*, an economy with a larger population will accumulate more technology and thus grow faster. This scale effect is found in many models with nonrival technology.[7] We do not focus on this feature. Since we take L to be fixed, there is no further loss in normalizing it to unity and studying an economy with a single agent per generation.

3. EQUILIBRIUM

The market-clearing conditions for this economy are (1), (2), and (3), together with

$$k_{t+1} = s_t.$$

Notice that this requires savings by the young consumer in period t to equal the entire period $t+1$ capital stock, not just new investment. This implies that existing capital crowds out new investment completely. The depreciation parameter μ therefore has no effect on the accumulation of capital, although it does affect the level of old-age consumption.

Combining these equations shows that the dynamics of this economy are governed by the system

$$A_{t+1} = \tau w(A_t, k_t; \lambda) + (1-\rho) A_t$$
$$k_{t+1} = (1-\gamma)(1-\tau)\, w(A_t, k_t; \lambda).$$

The remainder of the paper is dedicated to analyzing this dynamical system. We begin with the steady-state analysis.

3.1. *Existence of Steady States.* It follows directly from (A1) that there always exists a trivial steady state at the origin. If ρ is positive, no other points on the axes are stationary. Trajectories on the k axis jump to the origin, while trajectories on the A axis decay to the origin asymptotically.

[7] A nice discussion and list of references is contained in Jones [16].

An interior steady state, where we have $A_{t+1} = A_t = A$ and $k_{t+1} = k_t = k$, must satisfy

$$A = \frac{\tau}{\rho} w(A, k; \lambda)$$

$$k = (1-\gamma)(1-\tau)\, w(A, k; \lambda).$$

This implies that at any steady state, the ratio of technology to physical capital must be given by

$$\frac{A}{k} = \frac{\tau}{\rho(1-\gamma)(1-\tau)} \equiv \sigma, \tag{4}$$

a constant independent of λ. This ray from the origin plays a pivotal role in the analysis that follows. It is interesting to note how the different parameters affect the slope of this ray. Since taxation funds technology formation and depresses capital accumulation, higher levels of τ increase the A/k ratio and lead to more technology-intensive steady states. Higher levels of ρ have the opposite effect, retarding the formation of technology and leading to more capital-intensive steady states. Higher levels of γ mean that consumers place a higher weight on consumption in the first period of life, and hence save less. This depresses capital accumulation and leads to more technology-intensive steady states, since the taxes funding technology formation are on total labor income, not consumption.

The fact that any steady state must fall on this ray allows us to reduce the analysis of steady states to a one-dimensional problem, a major simplification. Steady state levels of the capital stock are given by the solutions to

$$g(k) \equiv (1-\gamma)(1-\tau)\, w(\sigma k, k; \lambda) - k = 0. \tag{5}$$

Using the homogeneity of w, this can be written as

$$g(k) = (1-\gamma)(1-\tau)\, w(\sigma, 1; \lambda)\, k^{\lambda} - k = 0.$$

The derivative of the function g is then given by

$$g'(k) = (1-\gamma)(1-\tau)\, w(\sigma, 1; \lambda)\, \lambda k^{\lambda-1} - 1. \tag{6}$$

This implies that we have

$$\lim_{k \to 0} g'(k) = \left\{ \begin{matrix} \infty \\ -1 \end{matrix} \right. \quad \text{for} \quad \left. \begin{matrix} \lambda < 1 \\ \lambda > 1 \end{matrix} \right\}$$

and

$$\lim_{k\to\infty} g'(k) = \begin{cases} -1 & \text{for } \lambda < 1 \\ \infty & \text{for } \lambda > 1 \end{cases}.$$

Since g is continuous and $g(0)=0$, an interior steady state must exist for every value of λ, except possibly unity. By looking at the second derivative of g, we can verify that the interior steady state is also unique in all of these cases. Differentiating (6), we have

$$g''(k) = (1-\gamma)(1-\tau)\, w(\sigma, 1; \lambda)\, \lambda(\lambda-1)\, k^{\lambda-2},$$

so that

$$g''(k) \begin{cases} <0 & \text{for } \lambda < 1 \\ >0 & \text{for } \lambda > 1 \end{cases}.$$

When λ is less than unity, g is strictly concave, and when λ is greater than unity, g is strictly convex. In either case, the interior steady state is unique. Thus we have established the following proposition.

PROPOSITION 1. *When $\lambda \neq 1$, there exists a unique interior steady state. The origin is a steady state for all values of λ.*

When λ is equal to unity, g is linear. In this case, there exists an interior steady state if and only if g is equal to zero everywhere, in which case there are a continuum of steady states. Dividing (5) through by $((1-\gamma)(1-\tau))^{-1} k$ and using the homogeneity of w shows that this occurs when

$$w\left(\frac{\tau}{\rho}, (1-\tau)(1-\gamma); 1\right) = 1. \tag{7}$$

If (7) holds, then every point along the ray where $A=\sigma k$ is a steady state. This condition will reappear later in the analysis and is important for determining the qualitative properties of the system as λ is varied. It is a condition solely on the parameters of the model, but the form of the equation follows directly from the consumer's savings function. Recall that the consumer will save $(1-\gamma)(1-\tau)\, w(A_t, k_t; \lambda)$. Condition (7) comes from equating these savings to the current level of the capital stock k_t, evaluating this along the $A=\sigma k$ ray, and dividing the variable k_t out using the homogeneity of w. When this condition holds, the consumer's savings rule will exactly replenish the capital stock whenever $A=\sigma k$, which is to say that every point on this ray is a steady state.

If λ is equal to one and (7) does not hold, then no interior steady states exist. If, instead, we have

$$w\left(\frac{\tau}{\rho}, (1-\tau)(1-\gamma); 1\right) > 1, \tag{8}$$

then g' is positive along the $A = \sigma k$ ray and both A and k are growing along the ray. In this case, the consumer's savings rule generates an increase in the capital stock whenever $A = \sigma k$. We refer to an economy for which (8) holds as a "high-savings" economy. It is important to bear in mind that consumers have the same savings rule whether an economy is "high-savings" or not. The term simply refers to a set of parameter values that, when inserted in the decision rule, generate positive net investment along the $A = \sigma k$ ray. If the inequality in (8) is reversed, g' is negative and both A and k are decaying along the ray. We refer to this as the "low-savings" case. It is clear that the parameter values at which (7) is satisfied represent a very special case; the global dynamics when $\lambda = 1$ holds are qualitatively different for high savings and low savings economies.

3.2. *Characterization of Steady States.* The derivative of the function g defined in (5) above is also useful for characterizing interior steady states. Let T and D denote the trace and determinant, respectively, of the Jacobian matrix J for our dynamical system. The eigenvalues of J are given by the roots of the characteristic polynomial $p(x)$, where

$$p(x) = x^2 - Tx + D.$$

It is straightforward to establish that (*i*) both T and D are nonnegative and (*ii*) the roots of this polynomial are always real and distinct. This implies that both eigenvalues are nonnegative and that the stability of a steady state is determined by the sign of the expression

$$p(1) = -\rho g'(k^*).$$

If we have $g'(k^*) > 0$, so that $p(1)$ is negative, the two eigenvalues must lie on opposite sides of unity and therefore the steady state is a saddle point. If we instead have $g'(k^*) < 0$, so that $p(1)$ is positive, the two eigenvalues must lie on the same side of unity. It is straightforward to show that in this case the trace of J is necessarily less than 2. Since the trace is equal to the sum of the eigenvalues, each eigenvalue must be less than unity and hence the steady state is locally stable. Thus we have a simple result: a steady state (A^*, k^*) is a sink if $g'(k^*)$ is negative and is a saddle point if $g'(k^*)$ is positive. The case of $g'(k^*) = 0$ is a degenerate one to which we will return later.

It is worth reemphasizing the ability of the univariate function g to determine the behavior of the steady states of the bivariate dynamical system. If $g(k)$ is positive, both A and k are increasing at points along the $A=\sigma k$ ray (although the trajectories need not stay on the ray). When $g(k)$ is zero, we have a steady state, and when $g(k)$ is negative, both A and k are falling at points along the ray. In addition, we have seen that if k^* is a steady state value and $g'(k^*)$ is negative, then (A^*, k^*) is locally stable. If instead $g'(k^*)$ is positive, then (A^*, k^*) is a saddle point. From the analysis of the previous section, it is clear that the sign of $g'(k^*)$ is determined by the magnitude of the parameter λ. When λ is below unity, g is strictly concave and hence at the interior root $g'(k^*)$ is negative. When λ is above unity, g is strictly convex and at the interior root $g'(k^*)$ is positive. Combining this with the above results yields the following proposition.

PROPOSITION 2. *When λ is less than unity, the unique interior steady state is a sink, and when λ is greater than unity it is a saddle point. When λ is equal to unity, an interior steady state, if it exists, is nonisolated.*

4. GRAPHICAL ANALYSIS

It should be clear at this point that the dynamics of our system undergo a qualitative change when λ passes through unity. In this section, we use graphical methods to analyze this event. In order to give exact expressions for the figures that we draw, we specialize to log-linear production,[8]

$$f(A, k) = (A^{1-\alpha} k^{\alpha})^{\lambda},$$

with $0<\alpha<1$ and $0 \leqslant \lambda < 1/\alpha$. This function is clearly homogeneous of degree λ in (A, k). It is straightforward to show that, in this case, the wage function is given by

$$w(A, k) = (1-\lambda\alpha)(A^{1-\alpha} k^{\alpha})^{\lambda}$$

and the equilibrium dynamical system is given by

$$A_{t+1} = \tau(1-\lambda\alpha)(A_t^{1-\alpha} k_t^{\alpha})^{\lambda} + (1-\rho) A_t$$

$$k_{t+1} = (1-\gamma)(1-\tau)(1-\lambda\alpha)(A_t^{1-\alpha} k_t^{\alpha})^{\lambda}.$$

[8] In this case our model resembles a two-dimensional version of that in Section 5 of Jones and Manuelli [17]. That section studies the global dynamics of an overlapping-generations model with log-linear production and a nonconvexity generated by externalities.

4.1. *Phase Diagrams.* We now proceed to construct the phase diagram for this system. The level of technology is unchanging when $A_{t+1} = A_t$, which occurs when either $A = 0$ or when we have

$$A = \left(\frac{\tau(1-\lambda\alpha)}{\rho} \right)^{1/(1-\lambda(1-\alpha))} k^{\lambda\alpha/(1-\lambda(1-\alpha))}. \tag{9}$$

Likewise, the level of capital is unchanging whenever $k_{t+1} = k_t$, which occurs when either $k = 0$ or when we have

$$A = \left(\frac{1}{(1-\gamma)(1-\tau)(1-\lambda\alpha)} \right)^{1/\lambda(1-\alpha)} k^{(1-\lambda\alpha)/\lambda(1-\alpha)}. \tag{10}$$

The curves described in (9) and (10) are therefore the interior *nullclines* of the system.

When $\lambda < 1$ holds, the curve defined in (9) is strictly concave and the curve defined in (10) is strictly convex. Both begin at the origin, and we know from above that there is a unique interior crossing that represents a locally stable steady state. It is fairly straightforward to show that this steady state attracts all interior trajectories and that there is no oscillating behavior in this system. The phase diagram is given in Fig. 1.

As λ increases toward unity, the steady state values of A and k can either increase or decrease. Letting A^* and k^* denote these values, we can solve for the interior steady state in terms of parameters,

$$A^* = \left[(1-\lambda\alpha) \left(\frac{\tau}{\rho} \right)^{1-\lambda\alpha} ((1-\gamma)(1-\tau))^{\lambda\alpha} \right]^{1/(1-\lambda)}$$

$$k^* = \left[(1-\lambda\alpha) \left(\frac{\tau}{\rho} \right)^{\lambda(1-\alpha)} ((1-\gamma)(1-\tau))^{1-\lambda(1-\alpha)} \right]^{1/(1-\lambda)}.$$

Since the steady state must, for any value of λ, lie along the $A = \sigma k$ ray, we know that A^* and k^* must move in the same direction in response to a parameter change. It is straightforward to see that, for λ close enough to unity, each of these expressions is increasing in λ if and only if

$$(1-\alpha) \left(\frac{\tau}{\rho} \right)^{1-\alpha} ((1-\gamma)(1-\tau))^{\alpha} > 1 \tag{11}$$

holds, that is, if and only if the high-savings condition defined in (8) holds. When this is the case, both A^* and k^* diverge to infinity as λ approaches unity. When the inequality in (11) is reversed, both A^* and k^* converge to zero as λ goes to unity. In either case, the steady state approaches one of the boundaries of the state space.

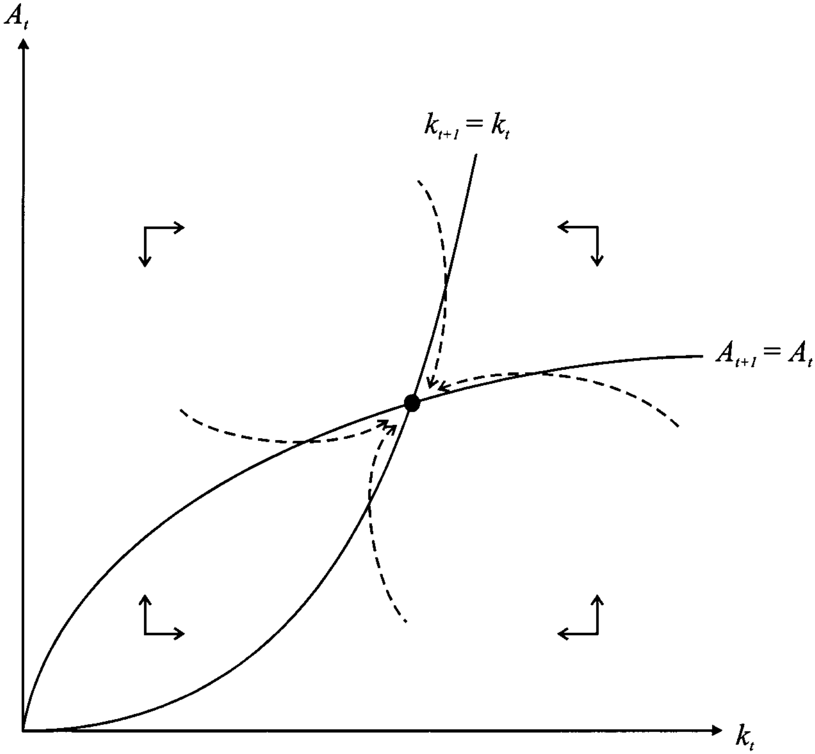

FIG. 1. Phase diagram for $\lambda < 1$.

Suppose the high-savings condition holds. As λ approaches unity, the nullclines "open up" in such a way that their crossing point (the interior steady state) diverges to infinity. When we have $\lambda = 1$, the nullclines are linear, with the curve given in (9), where A is constant, being the steeper of the two. The $A = \sigma k$ ray now represents a balanced growth path to which all interior trajectories converge. The growth rate along this path is given by

$$\frac{A_{t+1} - A_t}{A_t} = \frac{k_{t+1} - k_t}{k_t} = (1-\alpha)\left(\frac{\tau}{\rho}\right)^{1-\alpha}[(1-\gamma)(1-\tau)]^{\alpha} - 1, \quad (12)$$

which is positive by the high savings condition. This situation is depicted in Fig. 2a.

If, instead, the economy is characterized by low savings, the nullclines "close" as λ approaches unity and the interior steady state approaches the origin. When $\lambda = 1$ holds, the nullclines are again linear, but now the curve given by (10), where k is stationary, is the steeper of the two. There is still

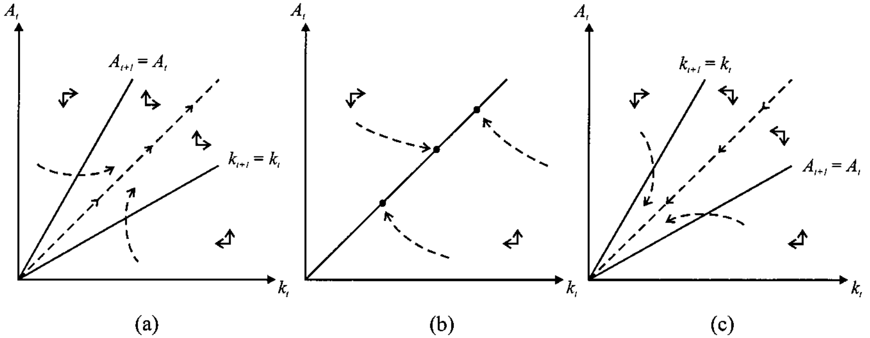

FIG. 2. Phase diagram for $\lambda = 1$.

a balanced growth path with the growth rate given by (12), only now this rate is negative, so that all trajectories decay to the origin. This situation is depicted in Fig. 2c.

The remaining case, where (11) holds with equality, is depicted in panel (b) of Fig. 2. In this case, the two nullclines completely overlap, and there is a continuum of steady states along this ray. All interior trajectories converge to some point on this ray.

As λ moves past unity, the curvature of the nullclines changes: the curve where A is stationary becomes convex, while the curve along which k is stationary becomes concave. There is once again a unique interior crossing which now represents a saddle point. If the economy is characterized by high savings, this steady state "emerges" from the origin as λ crosses unity. This case is depicted in panel (a) of Fig. 3. If the economy is low savings, the steady state instead "descends" from infinity, as shown in panel (b)

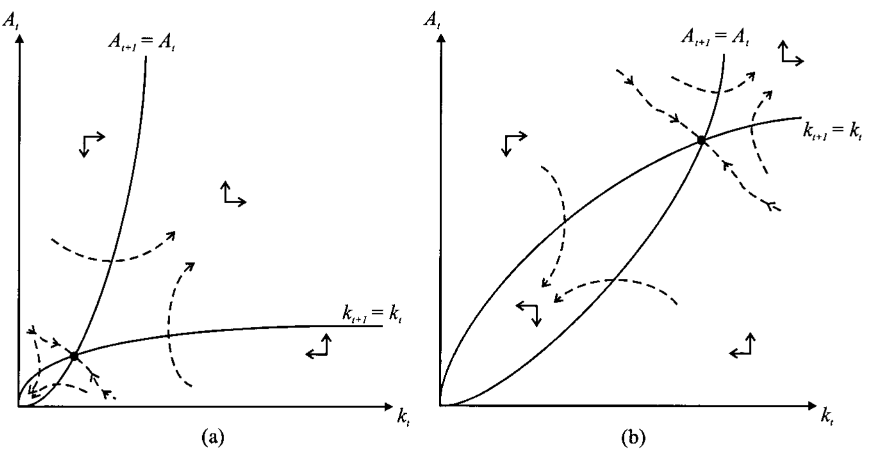

FIG. 3. Phase diagram for $\lambda > 1$.

of Fig. 3. The figure is drawn assuming that $\lambda(1-\alpha)<1$ holds. If λ is increased further, to $(1-\alpha)^{-1}$, the curve given by (9) becomes vertical; this is the case studied by Shell [22]. If λ increases further still, this curve becomes downward sloping. In all of these cases, the interior steady state remains a saddle point.

This collection of figures demonstrates that both high-savings and low-savings economies have the following pattern: (i) as λ approaches unity, the interior steady state converges to one of the boundaries of the state space (zero or infinity) and (ii) as λ crosses unity, the interior steady state emerges from the other boundary, with different stability properties. The steady state at the origin also changes stability properties as λ passes unity.

4.2. *Bifurcation Diagrams.* Another way to analyze the changes in steady states as the parameter λ is varied is to look at the bifurcation diagram. Since the steady state levels of A and k are related by (4), it makes no difference which variable we put on the vertical axis; we will use k. Consider first the case of a high-savings economy. When λ is less than unity, the economy has a unique interior steady state that is locally stable and an unstable steady state at the origin. The value of k at the interior steady state, k^*, may or may not be monotonically increasing in λ, depending

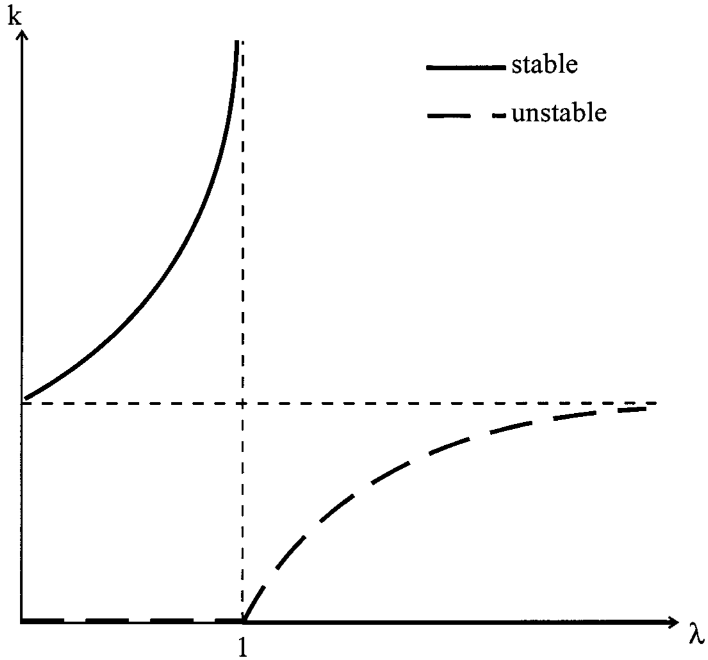

FIG. 4. Bifurcation diagram for a high-savings economy.

on parameter values, but it certainly diverges to infinity as λ goes to unity. For λ greater than unity, the origin is stable and the interior steady state, if we look at movements along the steady state ray, is unstable. This information is summarized in Fig. 4. For a low-savings economy, the analysis is reversed. The interior steady state is still locally stable for low values of λ, but it now converges to zero as λ approaches unity. As λ passes unity, an unstable steady state appears from infinity. The bifurcation diagram for this case is given in Fig. 5.

These diagrams clearly show the phenomenon described above: in both cases the interior steady state diverges to one of the boundaries as λ goes to unity and then emerges from the other boundary. As this happens, it exchanges stability properties with the boundary steady state. The defining characteristic of a transcritical bifurcation is that two fixed points collide, passing "through" each other and exchanging stability properties in the process (see, for example, Wiggins [27, pp. 256–257]). Hence the event we have here closely resembles a transcritical bifurcation, except that the interior steady state collides with one boundary and then emerges from the other.

Suppose that we were to equate these two boundaries, that is, suppose we were to "wrap" infinity back around so that it touches zero. The state

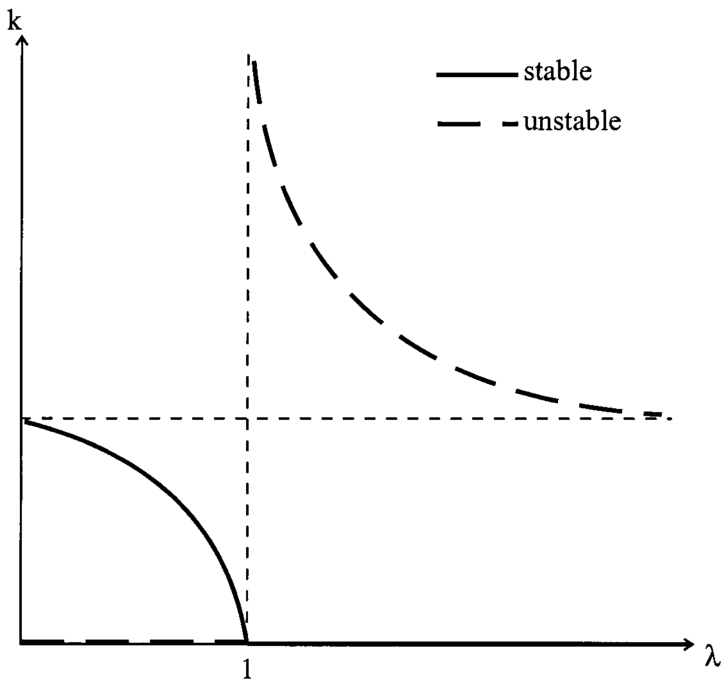

FIG. 5. Bifurcation diagram for a low-savings economy.

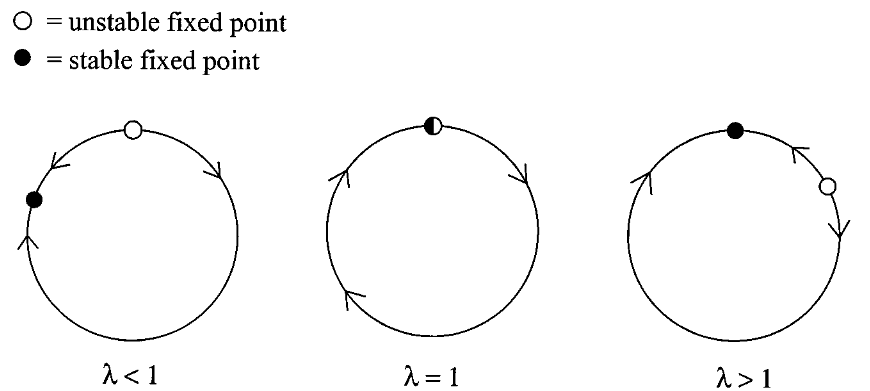

FIG. 6. The bifurcation on the circle: a high-savings economy.

space would then be a circle and the movement of the interior steady state would be *continuous* on the circle. This situation is depicted in Figs. 6 and 7. Figure 6 corresponds to the high-savings economy in Fig. 4. The point on "top" of the circle corresponds to a capital stock of zero and hence is a steady state for every value of λ. For obvious reasons we will refer to this as the "boundary" steady state even though all points are interior on the circle. Increases in k correspond to clockwise motion on the circle, and as k goes to infinity, we return to the top from the left side.

The left panel in Fig. 6 depicts the case where λ is less than unity, so that there is an interior steady state that attracts all trajectories except the boundary steady state. As λ approaches unity, the interior steady state diverges to infinity, which is to say that it approaches the (boundary) steady state at the top of the circle from the left. When λ is exactly unity, there is a unique steady state and all other trajectories experience unbounded

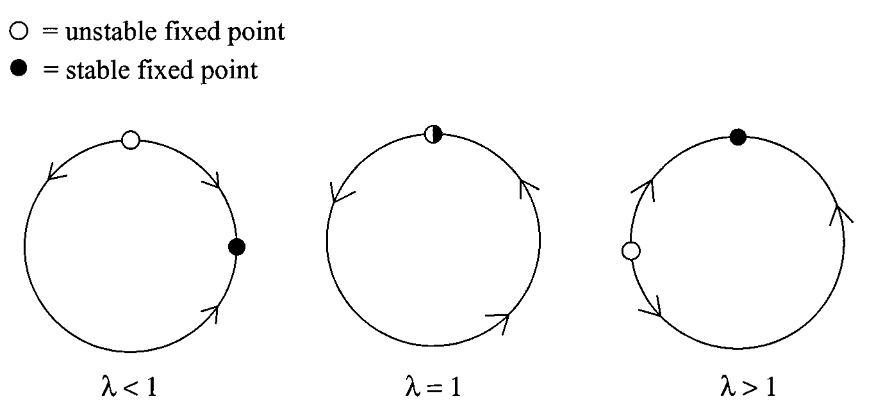

FIG. 7. The bifurcation on the circle: a low-savings economy.

growth. The steady state is half stable, with trajectories approaching it from the left and being repelled from it on the right. When λ is greater than unity, the interior steady state has continued on its path of clockwise motion and is now unstable, while the boundary steady state has become stable. We clearly see in this series of pictures the defining characteristic given above for the transcritical bifurcation: two fixed points colliding and exchanging stability properties.

For a low-savings economy, everything happens in reverse: the interior steady state moves in the counterclockwise direction as λ is increased. It again collides with the boundary steady state, and the two exchange stability properties. This is depicted in Fig. 7. Hence the event occurring when λ equals unity clearly conforms to the pattern of a transcritical bifurcation. What one would like to do next is to confirm this classification by applying a formal bifurcation classification theorem. This task is taken up in the next section.

5. FORMAL BIFURCATION ANALYSIS

For the formal analysis we return to using the general production function F. We begin by looking at a special case of the model, where technology depreciates completely every period (i.e., we have $\rho = 1$). This is perhaps not the most compelling value for the depreciation parameter on economic grounds, but it simplifies the dynamics in a crucial way. In this case, the dynamical system is given by

$$A_{t+1} = \tau w(A_t, k_t; \lambda)$$

$$k_{t+1} = (1-\gamma)(1-\tau)\, w(A_t, k_t; \lambda).$$

Notice that we now have $A_t = \sigma k_t$ for all periods t greater than zero. Trajectories in this system jump to the steady state ray, and then move along the ray. This means that the system is, in effect, one dimensional. To formalize this, note that we can rewrite the system as

$$k_{t+1} = (1-\gamma)(1-\tau)\, w(\sigma k_t, k_t; \lambda)$$

with $A_t = \sigma k_t$ holding for $t \geqslant 1$. Using the homogeneity of w, we can rewrite this as

$$k_{t+1} = (1-\gamma)(1-\tau)\, w(\sigma, 1; \lambda)\, k_t^{\lambda}.$$

Letting $a(\lambda)$ denote the term $(1-\gamma)(1-\tau)\, w(\sigma, 1; \lambda)$, which depends only on parameters, our dynamical system reduces to

$$k_{t+1} = a(\lambda)\, k_t^{\lambda}.$$

Our assumptions on F imply that a is twice continuously differentiable in λ.

5.1. *The Transformation.* In order to perform the formal bifurcation analysis, we must change coordinates so that our system is defined on a circle and hence the bifurcation occurs in the interior of the state space. The transformation of the state space must be done so that the dynamical system is at least C^2 in the new space. The method of transformation here consists of two steps, each of which is a strictly monotone transformation of variables. The first step creates a linear system on the whole real line. We define a new variable z_t by

$$z_t = \ln(k_t).$$

We then have

$$\begin{aligned} z_{t+1} &= \ln(k_{t+1}) \\ &= \ln(a(\lambda)) + \lambda \ln(k_t) \\ &= \ln(a(\lambda)) + \lambda z_t. \end{aligned}$$

The importance of this step is that it has eliminated the asymmetry between zero and infinity in the original system. We now have a linear system defined over the entire real line. The second step is to map this new system onto the interval $[-\frac{\pi}{2}, \frac{\pi}{2})$. By equating the two endpoints, this will give us a dynamical system on a circle with circumference π. The transformation we use is to define y_t by

$$y_t = \tan^{-1}(z_t).$$

This implies that

$$\begin{aligned} y_{t+1} &= \tan^{-1}(z_{t+1}) \\ &= \tan^{-1}(\ln(a(\lambda)) + \lambda z_t) \end{aligned}$$

or

$$y_{t+1} = \tan^{-1}(\ln(a(\lambda)) + \lambda \tan(y_t)).$$

One small problem is that $\tan(y_t)$ is not defined at $y_t = -\pi/2$. However, the limit of the right-hand side is defined, with

$$\lim_{y_t \to -\pi/2} [\tan^{-1}(\ln(a(\lambda)) + \lambda \tan(y_t))] = -\frac{\pi}{2}.$$

Therefore we can define the transition function $G(y_t, \lambda)$ in the following way

$$y_{t+1} = \left\{ \begin{matrix} \tan^{-1}(\ln(a(\lambda)) + \lambda \tan(y_t)) \\ -\dfrac{\pi}{2} \end{matrix} \right\} \quad \text{for} \quad y_t \left\{ \begin{matrix} \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \\ = -\dfrac{\pi}{2} \end{matrix} \right\}$$

$$\equiv G(y_t, \lambda).$$

This is the dynamical system on which we perform the bifurcation analysis. We first need to establish some properties of the function G. It is continuous, with

$$G(\cdot, \lambda)\colon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \qquad \text{for any } \lambda$$

and we have

$$\lim_{y \to -\pi/2} G(y, \lambda) = -\frac{\pi}{2} \qquad \text{and} \qquad \lim_{y \to \pi/2} G(y, \lambda) = \frac{\pi}{2}.$$

For the values $\lambda = 0.5$ and $a = 1.1$, the graph of G is given in Fig. 8. As expected, G intersects the 45° line (the dotted line in the figure) at $y = -\pi/2$, at a unique interior steady state, and again in the limit as y approaches $\pi/2$. For our analysis, we need to know that the dynamical system on the circle is at least C^2. The only potential problem is at the endpoints. Since the points $-\pi/2$ and $\pi/2$ are mapped into the same point on the circle, we need to know that we have

$$\lim_{y \to -\pi/2} h(y, \lambda) = \lim_{y \to \pi/2} h(y, \lambda),$$

where h represents each of the first- and second-order derivatives of G in (y, λ). The derivations are tedious and a summary can be found in the appendix.

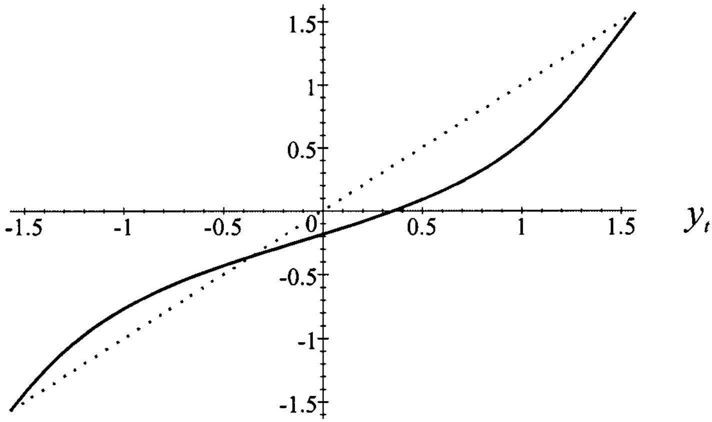

FIG. 8. Graph of $G(y_t, 0.5)$.

5.2. *Steady States on the Circle.* The system on the circle has two steady states. The first is at $y = -\pi/2$; we will continue to refer to this as the "boundary" steady state. The derivatives contained in the appendix determine the stability of this steady state. From (19), we have

$$\frac{\partial G}{\partial y}\left(-\frac{\pi}{2}, \lambda\right) = \frac{1}{\lambda},$$

demonstrating that the steady state is unstable when λ is less than unity, stable when λ is greater than unity, and nonhyperbolic when λ is equal to unity. Next, we need to analyze the "interior" steady state. This point is given by

$$y^* = \tan^{-1}\left(\frac{\ln(a(\lambda))}{1-\lambda}\right).$$

Evaluating the first derivative at this point yields

$$\frac{\partial G}{\partial y}(y^*, \lambda) = \lambda.$$

Therefore the interior steady state has the opposite stability properties of the boundary steady state. It is stable when λ is less than unity and unstable when λ is greater than unity. Notice that this exactly matches the stability properties of the interior steady state given in proposition 1. The interior steady state does not exist when $\lambda = 1$ holds, because it has collided with the boundary steady state. To see this, note that

$$\lim_{\lambda \to 1} \left[\tan^{-1} \left(\frac{\ln(a(\lambda))}{1-\lambda} \right) \right] = \left\{ \begin{matrix} -\dfrac{\pi}{2} \\ \dfrac{\pi}{2} \end{matrix} \right\} \quad \text{for} \quad a(1) \left\{ \begin{matrix} < \\ > \end{matrix} \right\} 1.$$

When the high-savings condition holds, the interior steady state converges to the boundary steady state from the "positive" side as λ goes to unity and reappears on the "negative" side, having exchanged stability properties with the boundary steady state. This process is exactly that depicted in Fig. 6, where $y = -\pi/2$ is the top of the circle and increases in y correspond to clockwise motion. When $a(1) < 1$ holds, so that we have a low-savings economy, the direction of movement of the interior steady state is reversed. This exactly matches what is shown in Fig. 7.

5.3. *Classifying the Bifurcation.* We next confirm that the event occurring when λ crosses unity is indeed a transcritical bifurcation. We do this using a set of sufficient conditions for a transcritical bifurcation given by Wiggins [27, p. 365].

Classification Theorem (Wiggins). *A one-parameter family of C^2 one-dimensional maps*

$$y \mapsto G(y, \lambda), \qquad y \in \mathbb{R}, \quad \lambda \in \mathbb{R}$$

having a nonhyperbolic fixed point, i.e.,

$$G(y^*, \lambda^*) = y^* \tag{14}$$

$$\frac{\partial G}{\partial y}(y^*, \lambda^*) = 1 \tag{15}$$

undergoes a transcritical bifurcation at $(y, \lambda) = (y^*, \lambda^*)$ if

$$\text{(i)} \quad \frac{\partial G}{\partial \lambda}(y^*, \lambda^*) = 0 \tag{16}$$

$$\text{(ii)} \quad \frac{\partial^2 G}{\partial \lambda \, \partial y}(y^*, \lambda^*) \neq 0 \tag{17}$$

and

$$\text{(iii)} \quad \frac{\partial^2 G}{\partial y^2}(y^*, \lambda^*) \neq 0. \tag{18}$$

In our system the bifurcation point is $(y^*, \lambda^*) = (-\frac{\pi}{2}, 1)$. Condition (14). requires that this point actually be a steady state, i.e., that we have

$$G\left(-\frac{\pi}{2}, 1\right) = -\frac{\pi}{2}.$$

This clearly holds. Condition (15). requires that the steady state be nonhyperbolic. From (19) we have

$$\frac{\partial G}{\partial y}\left(-\frac{\pi}{2}, 1\right) = 1$$

and this condition is satisfied. From the other derivatives in the appendix we have

$$\frac{\partial G}{\partial \lambda}\left(-\frac{\pi}{2}, \lambda\right) = 0$$

for all values of λ, and

$$\frac{\partial^2 G}{\partial y\, \partial \lambda}\left(-\frac{\pi}{2}, \lambda\right) = -\frac{1}{\lambda^2},$$

so that conditions (16) and (17) are clearly satisfied at λ^*. The final condition, (18), requires that the second derivative of G with respect to y also be nonzero. The expression for this derivative given by

$$\frac{\partial^2 G}{\partial y^2}\left(-\frac{\pi}{2}, \lambda\right) = \frac{2\ln(a(\lambda))}{\lambda^2}.$$

Therefore, we have

$$\frac{\partial^2 G}{\partial y^2}\left(-\frac{\pi}{2}, 1\right) = 2\ln(a(1)) \neq 0 \qquad \text{if} \quad a(1) \neq 1,$$

or whenever the economy is not on the high-savings/low-savings boundary. This establishes our main result.

THEOREM. *If*

$$(1-\gamma)(1-\tau)\, w(\sigma, 1; 1) \neq 1$$

holds, then the dynamical system defined in (13) *undergoes a transcritical bifurcation as* λ *passes through unity.*

6. COMMENTS ON GENERALIZING THE FORMAL RESULT

The case for which we are able to provide the formal bifurcation analysis (when technology depreciates completely every period) is special in at least two ways. First, and most obviously, it reduces the dimension of the dynamical system from two to one. Equally important, though, is the fact that it delivers an exact power law for the dynamical system. This allows the system to be mapped onto the circle in a smooth way. We now discuss the difficulties involved in trying to extend our formal result, first to other one-dimensional settings and then to fully two (or higher) dimensional models.

The importance of the exact power-law relationship between k_{t+1} and k_t can be seen by looking at a perturbed dynamical system such as

$$k_{t+1} = ak_t^{\lambda} + (1-\mu)\, k_t$$

with μ strictly between zero and one. (This law of motion obtains, for example, from a discrete-time Solow model with log-linear production and savings rate a.) The bifurcation diagrams for this system are qualitatively the same as the ones presented above. Furthermore, if our change of coordinates is used to generate a dynamical system on the circle, it will be qualitatively the same as in our power-law system. The event that occurs when λ crosses unity again has the signature properties of a transcritical bifurcation. In performing the formal bifurcation analysis, however, a problem arises—the system is not C^2 on the circle. In particular, there is a jump discontinuity in the derivative at the point where zero and infinity are "pasted" together. The fact that we were able to create a smooth system on the circle is a special property of our model; in general one-dimensional systems this is not possible. This means that the classification theorem we appealed to cannot be used for these other models. There do exist, however, techniques for analyzing bifurcations in nonsmooth systems (Budd [9] gives one example). It is possible that such techniques could be employed here, so that bifurcations such as the one just described could be formally classified. The methods that exist to study bifurcations in nonsmooth systems are typically problem specific, however, and how they can be adapted to economic models is an open question.

This nonsmoothness problem is even more pervasive in fully two (or higher) dimensional systems, since such models cannot have exact power law dynamics for both variables. If one did, it would simplify to a one-dimensional system as in Section 5.) Hence the type of nonsmooth methods mentioned above will be needed here as well. However, the added dimension brings with it another problem. For our general model with incomplete technological depreciation, the dynamics are defined on the nonnegative orthant of the plane. This can be mapped onto a sphere using a combination of a logarithmic change of coordinates and a stereographic projection. In this way, all of the points on the "boundary" of the plane are mapped into a single point, the "top" of the sphere. Here there is the potential for worse types of nonsmoothness than a simple jump discontinuity in the derivative. The directional derivative will likely be defined in every direction, but it might be different in every direction. In fact, this exact problem occurs at the boundary steady state of our general model on the plane, and hence is likely to reappear in any transformation of the system onto the sphere. Again, methods for analyzing bifurcations in nonsmooth systems seem to be needed here, but higher dimensions seem to require stronger tools because of the different types of nonsmoothness that may occur.

In spite of these nontrivial difficulties in extending the formal result, it seems clear from looking at diagrams that events resembling a transcritical bifurcation occur in a broader class of endogenous growth models. If the proper tools can be developed, we believe that extensions of our result will be helpful in organizing the analysis of a large class of endogenous growth models.

APPENDIX

For our analysis, we need to know that the dynamical system on the circle is at least C^2. The only potential problem is at the endpoints. Since the points $-\pi/2$ and $\pi/2$ are mapped into the same point on the circle, we need to know that

$$\lim_{y \to -\pi/2} h(y, \lambda) = \lim_{y \to \pi/2} h(y, \lambda),$$

where h represents each of the first- and second-order derivatives of G in (y, λ), that is,

$$h(y, \lambda) = \text{(i)}\ \frac{\partial G}{\partial y}, \quad \text{(ii)}\ \frac{\partial G}{\partial \lambda}, \quad \text{(iii)}\ \frac{\partial^2 G}{\partial y^2}, \quad \text{(iv)}\ \frac{\partial^2 G}{\partial \lambda^2}, \quad \text{and} \quad \text{(v)}\ \frac{\partial^2 G}{\partial y\, \partial \lambda}.$$

We will check these in order.

$$\frac{\partial G}{\partial y}(y_t, \lambda) = \lambda \frac{1 + \tan^2 y_t}{1 + (\ln a(\lambda) + \lambda \tan y_t)^2} > 0, \tag{i}$$

at the two boundary points, we have

$$\lim_{y \to \pm\pi/2} \frac{\partial G}{\partial y} = \frac{1}{\lambda}. \tag{19}$$

$$\frac{\partial G}{\partial \lambda}(y_t, \lambda) = \frac{\dfrac{a'(\lambda)}{a(\lambda)} + \tan y_t}{1 + (\ln a(\lambda) + \lambda \tan y_t)^2}. \tag{ii}$$

At the two boundary points, we have

$$\lim_{y_t \to \pm\pi/2} \frac{\partial G}{\partial \lambda} = 0.$$

The second-order derivatives are all lengthy expressions, so we will only give their values at the endpoints:

$$\lim_{y_t \to \pm\pi/2} \frac{\partial^2 G}{\partial y^2} = \frac{2 \ln(a(\lambda))}{\lambda^2}, \tag{iii}$$

$$\lim_{y_t \to \pm\pi/2} \frac{\partial^2 G}{\partial \lambda^2} = 0, \tag{iv}$$

and

$$\lim_{y_t \to \pm\pi/2} \frac{\partial^2 G}{\partial y\, \partial \lambda} = -\frac{1}{\lambda^2}. \tag{v}$$

The fact that all of these derivatives are the same as y_t approaches both $-\pi/2$ and $+\pi/2$ shows that our system is, in fact, C^2 on the circle.

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