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The Overlapping-Generations Model, I: The Case of Pure Exchange without Money*

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1. INTRODUCTION

The overlapping-generations model[1] was introduced by Samuelson [17] in 1958. Samuelson's seminal paper has several important themes.

He provides examples with nicely behaved agents, complete and costless markets, full information, and no externalities, in which Walrasian equilibria are not Pareto-optimal. This failure of the First Theorem of Welfare Economics was clarified in Shell [19]. It is shown that the "double infinity" (of consumers and commodities)—in particular, the assumption of an unbounded time horizon along with (dated) commodities for each period—suffices to render the First Theorem inapplicable to the general overlapping-generations model.

A second theme in [17] is that the limited opportunities for intertemporal exchange are a possible cause of inoptimality. Cass and Yaari [8] study this aspect, stressing that this source of inoptimality is not exclusive to dynamic models.

A third theme of the Samuelson article is that paper assets (e.g., money) created in consequence of the government's deficit can cure—or, at least, reduce—inoptimality. This theme has been explored in several articles, including [6, 19, 21, 22]. The tools of this research have been picked up by macroeconomic theorists and used to address the more traditional questions about government policy. Since the overlapping-generations model is the

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[1] Following Samuelson, the overlapping-generations model is often loosely referred to as the consumption-loans model. We eschew this usage, since it replaces the structural description with a description in terms of an attribute of some of the models. In particular, the latter description might incorrectly suggest that inefficiency in this class of models results only from imperfections in the borrowing and lending markets.

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only natural model in which money is permitted to serve as a store of value (with a positive price!), it is not surprising that it provides the most congenial setting for a study of monetary and fiscal policies—as evidenced, for example, by the recent research of Lucas [14] and his followers.

Our long-range research interests address all three of these themes. In the present paper, however, we study only an overlapping-generations model without money (or any other form of government debt). A major goal of this article is to provide a firm foundation for our forthcoming investigation of the monetary economy. In the sequel, we study equilibrium and optimality in a monetary model of overlapping generations with arbitrary (active or inactive) government deficit patterns over time.

In this article, we assume that markets in all the (dated) commodities are perfect and that agents possess perfect foresight. Therefore, even though the basic interest in the model stems from dynamic economics, it can also be interpreted as an extension of the general equilibrium model (see, e.g., Debreu [10] or Arrow and Hahn [1]) to a case where there are an infinite number of commodities *and* an infinite number of consumers.[2] A crucial feature of our model derives from the structure of generational overlap, by which each consumer is (at least indirectly) linked to each other consumer. Furthermore, since consumers live lives of only finite lengths, their budget constraints can be expressed in terms of simple inner products of price and quantities.[3] Hence, a system of prices can be described by an infinite sequence of nonnegative vectors. While aggregate wealth evaluated at these prices is possibly infinite, the individual consumer's wealth evaluated at the same prices is always finite.

We present our basic assumptions, definitions, and notation in Section 2. In Section 3, we establish the existence of a competitive equilibrium price sequence for the overlapping-generations economy. The equilibrium prices which we find are the limits of equilibrium prices for a sequence of finite, truncated economies.

Sections 4 and 5 provide the welfare analysis. Section 4 focuses on a concept of weak Pareto-optimality (or, equivalently, short-run Pareto-optimality). We show that every competitive allocation is weakly-Pareto-optimal, and that every weakly-Pareto-optimal allocation is a competitive allocation associated with some suitably assigned endowments. Not all

[2] The general-equilibrium model has been extended to allow for an infinite number of commodities (cf., e.g., Debreu [9]) or an infinite number of consumers (cf., e.g., Aumann [2]). Our analysis, however, requires the double infinity of commodities *and* consumers.

[3] If individuals lead lifetimes of infinite length, it is mathematically natural to express budget constraints as continuous linear functionals, not necessarily representable as inner products of prices and quantities; cf., e.g., Debreu [9]. The economic justification of this maneuver is questionable (cf. Shell [19, pp. 1007–1010]) and is avoided in our study where, because of finite lifetimes, only ordinary inner-product budget constraints are needed.

competitive allocations are Pareto-optimal. We study the relationship between weakly-Pareto-optimal allocations and Pareto-optimal allocations in Section 5. Our analysis is in terms of the price sequence shown to support a weakly-Pareto-optimal allocation. The central result of Section 5 is the complete characterization of Pareto-optimal allocations in terms of the supporting price sequences for economies in which consumers satisfy some mild uniformity conditions.

2. Assumptions, Definitions, and Notation

In order to simplify notation, we begin with a seemingly special case in which there are two-period lifetimes and the same number of commodities in each period. Later, we proceed to establish that our results easily extend to the very general case where lifetimes are finite, the number of commodities available in any one period is finite, and the number of consumers alive in any one period is finite.

For the simplified economy, in each period t ($t = 1, 2,...$), there is a finite (constant) number l of completely perishable commodities. There is no storage of commodities nor any other production process. Consumers, who are indexed by their order of birth $h = 0, 1,...$, are either present at the inception of the economy (in which case, they live out the balance of their lives during period 1) or are born at the beginning of some period t and live out the whole of their lives in periods t and $t + 1$. The analysis is first carried out for the simplest demographic pattern, i.e., each generation consists of a single consumer indexed uniquely by his birthdate.[4]

Let $x_t^{s,i}$ be consumption of commodity i ($i = 1,..., l$) in period s ($s = 1, 2,...$) by consumer t ($t = 0, 1,...$). Consumer t derives utility from consuming goods during his lifetime. His utility function can be represented as

$$u_t(x_t), \qquad t = 0, 1,...,$$

where

$$x_0 = x_0^1 = (x_0^{1,1},..., x_0^{1,l}) \in \mathbb{R}^l_{++} \qquad \text{for} \quad t = 0$$

and

$$x_t = (x_t^t, x_t^{t+1}) = (x_t^{t,1},..., x_t^{t,l}, x_t^{t+1,1},..., x_t^{t+1,l}) \in \mathbb{R}^{2l}_{++} \qquad \text{for} \quad t = 1, 2,... .$$

[4] In the notation of Cass *et al.* [6], we begin with the case in which there are at date t, two age-groups: an older generation, G_{t-1}, born in the preceding period, and a younger generation, G_t, born in the current period. Without loss in generality, we focus on the simplest population dynamics: $G_t = \{t\}$ for $t = 0, 1,...$.

When convenient, we also denote by x_0 and x_t the respective sequences

$$x_0 = (x_0^1, 0, \ldots, 0, \ldots) \qquad \text{for} \quad t = 0$$

and

$$x_t = (0, \ldots, 0, x_t^t, x_t^{t+1}, 0, \ldots) \qquad \text{for} \quad t = 1, 2, \ldots .$$

The utility functions $u_0(\cdot)$ and $u_t(\cdot, \cdot)$ are assumed to have strictly positive first-order partial derivatives (i.e., to be differentiably monotonic) and to be strictly quasiconcave (i.e., to exhibit diminishing marginal rates of commodity substitution). Furthermore, the closure of every indifference surface in $\mathbb{R}^l$ (resp. $\mathbb{R}^{2l}$) is assumed to be contained in the corresponding strictly positive orthant $\mathbb{R}^l_{++}$ (resp. $\mathbb{R}^{2l}_{++}$);[5] cf. Debreu [11, p. 611]. Each consumer has strictly positive endowments of the goods during his lifetime

$$\omega_0 = \omega_0^1 = (\omega_0^{1,1}, \ldots, \omega_0^{1,l}) \in \mathbb{R}^l_{++} \qquad \text{for} \quad t = 0$$

and

$$\omega_t = (\omega_t^t, \omega_t^{t+1}) = (\omega_t^{t,1}, \ldots, \omega_t^{t,l}, \omega_t^{t+1,1}, \ldots, \omega_t^{t+1,l}) \in \mathbb{R}^{2l}_{++} \qquad \text{for} \quad t = 1, 2, \ldots .$$

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and

$$\omega_t = (0, \ldots, 0, \omega_t^t, \omega_t^{t+1}, 0, \ldots) \qquad \text{for} \quad t = 1, 2, \ldots .$$

Each consumer can buy and sell on both a spot market and on a one-period futures market at perfectly foreseen prices. (This interpretation is natural because of the dynamic nature of the problem. From the general equilibrium viewpoint, however, it is unnecessarily restrictive. The model has the more general interpretation that the consumer can trade in every period, but only has tastes for, or endowments of, commodities "during his lifetime." In equilibrium, no pure arbitrage profits are available, so he will not gain from trades for those periods "during which he is not alive.")

Let $p^{t,i}$ denote the price of commodity i ($i = 1, \ldots, l$) in period t ($t = 1, 2, \ldots$). Denote by p^t the vector $(p^{t,1}, \ldots, p^{t,l}) \in \mathbb{R}^l_{++}$ and by p the price sequence $(p^1, p^2, \ldots)$. We choose the normalization $p^{1,1} = 1$ and thus restrict attention to the set S of sequences of *present* prices, $S = \{ p \mid p^{1,1} = 1 \}$. Each consumer

[5] These regularity assumptions are not essential for our proof of the existence of competitive equilibrium. The smoothness and positive closure assumptions, however, play important roles in our study of Pareto-optimality.

chooses his lifetime consumption profile rationally, that is, as the solution to the budget-constrained utility-maximization problem:

$$\left.\begin{array}{ll}\text{maximize} & u_0(x_0^1)\\ \text{subject to} & p^1 \cdot x_0^1 \leqslant p^1 \cdot \omega_0^1 = w_0\end{array}\right\} \text{ for } t = 0,$$
$$\left.\begin{array}{ll}\text{maximize} & u_t(x_t^t, x_t^{t+1})\\ \text{subject to} & p^t \cdot x_t^t + p^{t+1} \cdot x_t^{t+1} \leqslant p^t \cdot \omega_t^t + p^{t+1} \cdot \omega_t^{t+1} = w_t\end{array}\right\} \text{ for } t = 1, 2, \ldots, \tag{2.1}$$

where w_t is interpreted as "income" (or, more accurately, the present value of wealth) of consumer t ($t = 0, 1, \ldots$). Individual demand functions f_0 and f_t ($t = 1, 2, \ldots$) are defined by (2.1). The range of f_0 or f_t is always that of the appropriate demand, x_0 or x_t, but the domain depends on the interpretation. For example, if demand is determined by prices p and individual incomes w_t, then we have

$$\begin{array}{ll} f_0 \colon S \times \mathbb{R}_{++} \to \mathbb{R}^l_{++} & \text{for } t = 0, \\ f_t \colon S \times \mathbb{R}_{++} \to \mathbb{R}^{2l}_{++} & \text{for } t = 1, 2, \ldots . \end{array} \tag{2.2}$$

Let $x = (x_0, x_1, \ldots)$ be a strictly positive allocation sequence. We denote by X the set of all such sequences, i.e., $X = \mathbb{R}^l_{++} \times \mathbb{R}^{2l}_{++} \times \mathbb{R}^{2l}_{++} \times \cdots$ (infinite product). Let $W = \mathbb{R}^{\infty}_{++}$ be the set of strictly positive individual "income" sequences $(w_0, w_1, \ldots)$. We shall find it convenient to denote by the same symbols f_0 or f_t (i) the mapping of price sequences and "income" scalars into demand vectors as defined by the formulas in (2.2), and (ii) the mapping of price sequences and "income" sequences into demand sequences, $f_0 \colon S \times W \to X$ and $f_t \colon S \times W \to X$, defined in the obvious way from the formulas in (2.2), i.e., by completing demand vectors with the appropriate zeros in order to make sequences.

2.3. Definition. Let $\omega \in X$ be a sequence of strictly positive endowments. A *Walrasian equilibrium* associated with $\omega \in X$ is a sequence of strictly positive commodity prices $p \in S$ (and associated optimal consumption profiles $x \in X$) satisfying the market clearing equation

$$\sum_t f_t(p, p \cdot \omega_t) = \sum_t \omega_t.$$

Note that the seemingly infinite sums in Definition (2.1) are well defined since for each coordinate there are only a finite number of nonzero terms. The associated competitive allocation $x \in X$ consists of the lifetime consumption profiles $x_t = f_t(p, p \cdot \omega_t)$ for $t = 0, 1, \ldots$.

2.4. Definition. The *allocation* $x=(x_0, x_1,\ldots, x_t,\ldots)\in X$ is *Pareto-optimal* (PO) if there is no $y=(y_0, y_1,\ldots, y_t,\ldots)\in X$ with the property that

$$\sum_t y_t = \sum_t x_t$$

and

$$u_t(y_t)\geqslant u_t(x_t) \quad \text{with at least one strict inequality} \quad \text{for } t=0, 1,\ldots.$$

2.5. Definition. The *allocation* $x=(x_0, x_1,\ldots, x_t,\ldots)\in X$ is *weakly Pareto-optimal* (WPO) if there is no $y=(y_0, y_1,\ldots, y_t,\ldots)\in X$ with the property

$$\sum_t y_t = \sum_t x_t,$$

$$y_t = x_t \quad \text{except for a finite number of } t,$$

and

$$u_t(y_t)\geqslant u_t(x_t) \quad \text{with at least one strict inequality} \quad \text{for } t=0, 1,\ldots.$$

2.6. Lemma. *If x is PO, then x is also WPO.*

Proof. Obvious from Definitions (2.4) and (2.5). ∎

2.7. Definition. The *allocation* $x=(x_0, x_1,\ldots, x_t,\ldots)\in X$ is *short-run Pareto-optimal* (SRPO) if there are no $y=(y_0, y_1,\ldots, y_t,\ldots)\in X$ and $t'\geqslant 0$ with the property

$$\sum_t y_t = \sum_t x_t,$$

$$y_t = x_t \quad \text{for every } t\geqslant t',$$

and

$$u_t(y_t)\geqslant u_t(x_t) \quad \text{with at least one strict inequality} \quad \text{for } t=0, 1,\ldots.$$

2.8. Lemma. *The allocation x is SRPO if and only if x is WPO.*

Proof. Immediate from Definitions (2.5) and (2.7). ∎

Efficiency notions like SRPO are frequently encountered in the capital theory literature. We shall use the fact that SRPO and WPO are equivalent concepts, but we state our main results solely in terms of the WPO criterion.

3. Existence of Competitive Equilibrium

In this section, we establish the existence of a price sequence $p \in S$ at which supply and demand are equated for each commodity, i.e., we establish the existence of Walrasian equilibrium (Definition 2.3). First, we truncate the infinite economy and establish in a standard way the existence of a Walrasian equilibrium for the finite, truncated economy. Some Walrasian equilibrium for the full (infinite) overlapping-generations economy can be found as the limit of the equilibria for the sequence of truncated economies.

3.1. Definition. The *sequence* $p = (p^1, p^2,..., p^t, p^{t+1},...) \in S$ is a *t-equilibrium* if the following equations are satisfied:

$$\begin{aligned} &f_0^1(p^1, p^1 \cdot \omega_0^1) + f_1^1(p^1, p^2, p^1 \cdot \omega_1^1 + p^2 \cdot \omega_1^2) = \omega_0^1 + \omega_1^1, \\ &f_1^2(p^1, p^2, p^1 \cdot \omega_1^1 + p^2 \cdot \omega_1^2) + f_2^2(p^2, p^3, p^2 \cdot \omega_2^2 + p^3 \cdot \omega_2^3) = \omega_1^2 + \omega_2^2, \\ &\cdots \\ &f_{t-1}^t(p^{t-1}, p^t, p^{t-1} \cdot \omega_{t-1}^{t-1} + p^t \cdot \omega_{t-1}^t) \\ &\quad + f_t^t(p^t, p^{t+1}, p^t \cdot \omega_t^t + p^{t+1} \cdot \omega_t^{t+1}) = \omega_{t-1}^t + \omega_t^t. \end{aligned} \tag{3.2}$$

3.3. *Remark.* Clearly, if $p = (p^1, p^2,..., p^t, p^{t+1},...) \in S$ is a t-equilibrium price sequence, then so is $p' = (p'^1, p'^2,..., p'^t, p'^{t+1},...) \in S$ satisfying $p'^s = p^s$ for $s = 1,..., t+1$. In other words, the components $p^{t+2}, p^{t+3},...$ are completely indeterminate at a t-equilibrium.

The purpose of the following lemma is to put bounds on p^s which are valid for every t-equilibrium, i.e., which are independent of t as long as $s \leqslant t+1$. Without loss of generality, we can then impose these bounds on p^s ($s = 1,..., t, t+1,...$).

3.4. Lemma. *There are vectors $\alpha^s \in \mathbb{R}^l$ and $\beta^s \in \mathbb{R}^l$ ($s = 1, 2,...$) with the property that for every t-equilibrium $p = (p^1, p^2,..., p^t, p^{t+1},...) \in S$,*

$$0 < \alpha^s \leqslant p^s \leqslant \beta^s < +\infty, \tag{3.5}$$

for $s = 1,..., t+1$. These bounds are independent of the truncation t. Furthermore, if $p = (p^1,..., p^s,...) \in S$ is a Walrasian equilibrium (for the infinite system (2.2)*), then the bounds* (3.5) *hold for $s = 1, 2,...$.*

Proof. The proof is by induction on consumer $s = 0, 1,..., t$. Consider first the implications of the market behavior of consumer 0. Define the gradient grad $u_0(x_0) \in \mathbb{R}^l$ and the normalized gradient

$$\text{grad}_n\, u_0(x_0) = \frac{\text{grad}\, u_0(x_0)}{\partial u_0/\partial x_0^{1,1}}.$$

The normalized gradient belongs to $\mathbb{R}^l_{++}$ because of our regularity assumptions on utility functions. (See, e.g., Arrow and Hahn [1, pp. 29–30 and pp. 101–104], especially Theorem 4.8). At equilibrium, the choices of consumer 0 are restricted to the compact set $\{x_0 \in \mathbb{R}^l_{++} \mid u_0(x_0) \geqslant u_0(\omega_0)$ and $x_0 \leqslant \omega_0^1 + \omega_1^1 < +\infty\}$. The image of this compact set by the continuous mapping $x_0 \longmapsto \text{grad}_n\, u_0(x_0)$ is a compact subset of $\mathbb{R}^l_{++}$. Therefore, there exist $\alpha^1 \in \mathbb{R}^l$ and $\beta^1 \in \mathbb{R}^l$ with the property that

$$0 < \alpha^1 \leqslant \text{grad}_n\, u_0(x_0) \leqslant \beta^1 < +\infty.$$

Now, if $p \in S$ is a t-equilibrium with $t \geqslant 1$, then necessarily $p^1 = \text{grad}_n\, u_0(x_0)$.

Consider next the implications of the market behavior of consumer $s = 1, \ldots, t$. Conditions (2.1) and our regularity assumptions imply that

$$\text{grad}_n\, u_s(x_s) = \frac{\text{grad}\, u_s(x_s^s, x_s^{s+1})}{\partial u_s / \partial x_s^{s,1}} = \frac{1}{p^{s,1}}\,(p^s, p^{s+1}),$$

which belongs to $\mathbb{R}^{2l}_{++}$, is strictly positive and finite for $x_s \in \mathbb{R}^{2l}_{++}$. At equilibrium, the choices of consumer s are restricted to the compact set $\{x_s \in \mathbb{R}^l_{++} \times \mathbb{R}^l_{++} \mid u_s(x_s) \geqslant u_s(\omega_s)$ and $x_s \leqslant (\omega_{s-1}^s + \omega_s^s, \omega_s^{s+1} + \omega_{s+1}^{s+1}) < +\infty\}$. Therefore, by the same argument as above, if $p \in S$ is a t-equilibrium with $t \geqslant s$, then necessarily the vector $(1/p^{s,1})(p^{s+1}) \in \mathbb{R}^l_{++}$ belongs to a compact set. But, by the induction hypothesis, there exist $\alpha^{s,1} \in \mathbb{R}$ and $\beta^{s,1} \in \mathbb{R}$ such that $0 < \alpha^{s,1} \leqslant p^{s,1} \leqslant \beta^{s,1} < +\infty$. Hence, there exist $\alpha^{s+1} \in \mathbb{R}^l$ and $\beta^{s+1} \in \mathbb{R}^l$ with the property $0 < \alpha^{s+1} \leqslant p^{s+1} \leqslant \beta^{s+1} < +\infty$. The proof by induction is complete. ∎

Lemma (3.4) allows us to restrict attention to a compact subset of the set of price sequences.

3.6. Definition. Let $S^* \subset S$ be

$$S^* = \{p = (p^1, \ldots, p^t, \ldots) \in S \mid 0 < \alpha^t \leqslant p^t \leqslant \beta^t < +\infty \text{ for } t = 1, 2, \ldots\},$$

where α^t and β^t are the bounds defined in Lemma (3.4).

Next we define $\mathscr{W}(t)$ the set of t-equilibrium price sequences in S^*.

3.7. Definition. Let $\mathscr{W}(t)$ be the set

$$\{p \in S^* \mid p \text{ satisfies the system of Eqs. (3.2)}\}.$$

By construction, $\{\mathscr{W}(t)\}$, the sequence of equilibrium sets is nested and decreasing; this fact is formalized in the next lemma, which is motivated by Remark (3.3).

3.8. LEMMA.

$$\mathscr{W}(1) \supset \mathscr{W}(2) \supset \cdots \supset \mathscr{W}(t) \supset \cdots.$$

Proof. Obvious from the definition of t-equilibrium (Definition (3.1)). ∎

3.9. LEMMA. *$\mathscr{W}(t)$ is nonempty and compact in the product topology for $t = 1, 2,\ldots$.*

Proof. (a) Nonemptiness. Complete the system (3.2) by the equation

$$f_t^{t+1}(p^t, p^{t+1}, p^t \cdot \omega_t^t + p^{t+1} \cdot \omega_t^{t+1}) = \omega_t^{t+1}.$$

This new equation system represents a conventional pure-exchange, general-equilibrium system consisting of the first $t+1$ consumers, which economy we denote by $\mathscr{E}_t$. In particular, each of the consumers is resource-related to each of the other t consumers. Therefore, there exists a competitive equilibrium price vector $(p^1,\ldots, p^{t+1}) \in \mathbb{R}_{++}^{(t+1)l}$ (see, e.g., Arrow and Hahn [1, pp. 117–119]). Furthermore, it is obvious that p^s $(s = 1,\ldots, t+1)$ satisfies the bounds specified in Definitions (3.6) and (3.7). Finally, to complete the sequence $p \in \mathscr{W}(t)$, assign p^s $(s = t+2, t+3,\ldots)$ any value which satisfies the bounds in Definitions (3.6) and (3.7).

(b) Compactness. By Definition (3.6), S^* is the (infinite) product of the compact sets $\{p^t \in \mathbb{R}^l \mid \alpha^t \leqslant p^t \leqslant \beta^t\}$ for $t = 0, 1,\ldots$, hence is compact for the product topology by Tychonoff's theorem (see, e.g., Bourbaki [5, I, Sect. 9.5, Theorem 3, p. 88]). $\mathscr{W}(t)$ is a closed subset of S^* and hence is compact. ∎

3.10. PROPOSITION. *For the overlapping-generations economy described in Section 2, there always exists a competitive (Walrasian) equilibrium price sequence $p \in S$ associated with every sequence of endowments $\omega \in X$.*

Proof. We need to establish that the set

$$\mathscr{W} = \{p \in S \mid p \text{ solves the system of (Walrasian equilibrium) equations in Definition (2.1)}\}$$

is nonempty. Since $\mathscr{W}(\infty) = \bigcap_{t=1}^{\infty} \mathscr{W}(t)$ is included in $\mathscr{W}$, it suffices to establish that $\mathscr{W}(\infty)$ is nonempty. By Lemmas (3.8) and (3.9), $\mathscr{W}(\infty)$ is the intersection of a nonincreasing sequence of nonempty, compact sets and is therefore nonempty; cf. Bourbaki [5, I, Sect. 9.1, pp. 83–84]. ∎

The assumptions in Section 2 are far stronger than are needed to establish the existence of a Walrasian equilibrium price sequence. In the proof of Proposition (3.10), only two ingredients are crucial: (1) The existence of a sequence of finite truncated economies, in which each truncated economy has the property that every consumer is indirectly resource-related to every other consumer. (Cf. Arrow and Hahn [1, p. 117].) (2) The aggregate demand function for each commodity is sufficiently regular so that bounding properties on prices (such as those in Lemma 3.4) can be deduced.[6] In Proposition (3.11), we provide a generalization of Proposition (3.10). While the hypothesis of Proposition (3.11) is rather weak, it is quite far from the most general result directly derivable within this framework.

3.11. **Proposition.** *Consider a pure exchange, overlapping-generations economy, with a finite number of commodities in each period, in which lifetimes are finite and the number of consumers alive in any period is finite and positive. In period t ($t = 1, 2,...$), there is at least one living consumer who will also be alive in period $t + 1$. Each consumer has a utility function which is smooth, strictly-quasiconcave, and strictly increasing in the commodities available during his lifetime. Also, the closure of every indifference surface is contained in the corresponding positive orthant. Each consumer is endowed with strictly positive and finite amounts of the commodities available during his lifetime.* Under the above hypothesis, there exists a Walrasian equilibrium price sequence.

Proof. Under these assumptions, aggregate endowment of each (dated) commodity is positive and finite. Therefore, the regularity of utility functions and the structure of generational overlap allow bounds to be placed on the components of the price sequence which are valid for each truncated equilibrium as well as for the infinite equilibrium. The remainder of the proof parallels that given for Proposition (3.10), where $\mathscr{E}_t$ is replaced by $\mathscr{E}_t^*$ ($t = t^*, t^* + 1,...$), the truncated economy based on all consumers whose death date is t or earlier, and t^* is the first death date recorded in the infinite economy. Under the above hypothesis, for each $\mathscr{E}_t^*$, consumers are resource-related. ∎

It should be re-emphasized that our regularity assumptions on utility functions, viz., strict monotonicity, smoothness and strict quasi-concavity are not essential for the proof of existence of competitive equilibrium. Only indirect resource-relatedness and a mild (not necessarily uniform) bounding property on prices are required. Our proof of existence could easily accommodate upper semi-continuous demand correspondences, precisely as in

[6] In an unpublished paper [15], Okuno and Zilcha have examined the question of existence of equilibrium. They restrict attention to a model with a very particular monetary policy and do not distinguish between monetary and nonmonetary equilibria. They also adopt a stronger assumption of resource-relatedness than is normally used.

Debreu [10] or Arrow and Hahn [1]. These regularity assumptions—notably smoothness—are introduced because they play an important role in our welfare analysis in Section 5 and because they will play an intrinsic role in our forthcoming analysis of the topological structure of the set of monetary and nonmonetary equilibria and the set of Pareto-optimal allocations.

The assumption that indifference surfaces have positive closures is made to rule out complications which could arise at the boundary of the consumption set X. This and the assumption of strictly positive endowments would be unnecessary if X were the entire Euclidean space, rather than its positive orthant.

3.12. *Remark.* For the proof of Lemma (3.9), the equation system (3.2) was completed in a particular way, which was sufficient for establishing the existence of an equilibrium (Proposition (3.10)). If the method of completing the system (3.2) is generalized to

$$f_t^{t+1}(p^t, p^{t+1}, p^t \cdot \omega_t^t, p^{t+1} \cdot \omega_t^{t+1}) = z_t^{t+1},$$

where $z_t^{t+1} \in (0, \omega_t^{t+1} + \omega_{t+1}^{t+1})$, then *each* Walrasian equilibrium (Definition (2.3)) can be found as the limit of a truncated economy for *some* positive sequence $\{z_t^{t+1}\}$.

4. Welfare Analysis: Weak Pareto-Optimality

It is fair to say that in the decade following the publication of Samuelson's seminal article [17], there was substantial confusion about the welfare implications of his analysis. (See, e.g., [13].) The confusion was in part caused by the imprecision of the criteria for optimality[7] employed in [17]. It is therefore worthwhile to take some care with the basic aspects of the welfare analysis. The focus of this section is the WPO-criterion (Definition (2.5)). The next section will evaluate WPO-allocations (and thus Walrasian equilibrium allocations) in terms of the stronger PO-criterion (Definition (2.4)).

[7] In our reading of [17] (see especially pp. 479–480), three welfare concepts are implicit in the discussion: (1) What we call weak Pareto-optimality (WPO) (or equivalently SRPO); see Definitions (2.5) and (2.7)). Samuelson [17] refers to our WPO concept (Definition (2.5)) as "Pareto-optimality." (2) A Benthamite-type (or, Ramsey-type) social welfare function with a zero rate of planner's impatience. (3) What we call Pareto-optimality (PO) (see Definition (2.4)). Oddly enough, criteria (1) and (2) receive the most attention in [17]. This is consistent with Samuelson's approach in a later paper [18] on the Phelps–Koopmans theorem, where Samuelson seems not only to reject the PO concept for infinite programs, but also the weaker concept of intertemporal efficiency for infinite programs.

4.1. DEFINITION. Let $x = (x_0, x_1, \ldots, x_t, \ldots) \in X$ be an allocation. The *price sequence* $p = (p^1, p^2, \ldots, p^t, \ldots)$ is said to support x if and only if, for each t $(t = 0, 1, \ldots)$, the commodity bundle x_t maximizes $u_t(\cdot)$ subject to $p \cdot x_t \leqslant w_t$ for some sequence of incomes $(w_0, w_1, \ldots, w_t, \ldots) \in W$.

4.2. LEMMA. *Every competitive allocation $x \in X$ associated with the Walrasian equilibrium price sequence $p \in S$ and some endowment sequence $\omega \in X$ is supported by p. Conversely, every allocation $x \in X$ supported by a price sequence $p \in S$ is a competitive allocation associated with the Walrasian equilibrium price sequence p and some suitably assigned endowments $\omega \in X$.*

Proof. (a) Let $p \in S$ be an equilibrium price sequence associated with the endowment sequence $\omega = (\omega_0, \omega_1, \ldots, \omega_t, \ldots) \in X$. The commodity bundle $x_t = f_t(p, p \cdot \omega_t)$ maximizes $u_t(x_t)$ under the constraint $p \cdot x_t \leqslant p \cdot \omega_t$. Therefore, the competitive allocation $x = (f_0(p, p \cdot \omega_0), f_1(p, p \cdot \omega_1), \ldots, f_t(p, p \cdot \omega_t), \ldots)$ is supported by the price sequence p (with the sequence of incomes $w = (w_0, w_1, \ldots, w_t, \ldots) = (p \cdot \omega_0, p \cdot \omega_1, \ldots, p \cdot \omega_t, \ldots)$).

(b) Let the sequence $p \in S$ support the allocation $x = (x_0, x_1, \ldots, x_t, \ldots) \in X$. Therefore, $x_t = f_t(p, p \cdot x_t)$ for $t = 0, 1, \ldots$. Clearly, p is an equilibrium price sequence associated with the initial endowment sequence $\omega = x$, where x is also the competitive allocation sequence. ∎

4.3. LEMMA. *The allocation sequence $x \in X$ is weakly Pareto-optimal (WPO) if and only if there exists a price sequence $p \in S$ which supports x.*

Proof. (a) Consider the t-truncated economy $\mathscr{E}_t$ defined by the first $t + 1$ consumers. It is immediate from Lemma (2.8) that $x = (x_0, x_1, \ldots, x_t, \ldots) \in X$ is WPO if and only if for each $t = 0, 1, \ldots, (x_0, x_1, \ldots, x_t)$ is PO in the truncated economy $\mathscr{E}_t$.

Assume that x is WPO. We must show that there is a $p \in S$ which supports x. The proof is by induction on the truncation t. Assume that there exists a (unique) normalized price vector $(p^1, p^2, \ldots, p^t)$ supporting $(x_0, x_1, \ldots, x_{t-1})$ in $\mathscr{E}_{t-1}$. We must show that there is a (unique) $p^{t+1} \in \mathbb{R}^l_{++}$ such that $(p^1, p^2, \ldots, p^t, p^{t+1})$ supports $(x_0, x_1, \ldots, x_t)$ in $\mathscr{E}_t$.

If (p'^t, p'^{t+1}) supports x_t, then $(p'^t, p'^{t+1}) = \theta \operatorname{grad} u_t(x_t)$, where $\theta > 0$. Consider the reduced economy consisting only of the commodities of period t and two consumers: consumer $t - 1$ defined by the utility function $u_{t-1}(\bar{x}^{t-1}_{t-1}, \cdot)$, where $\bar{x}^{t-1}_{t-1}$ is fixed; and consumer t defined by the utility function $u_t(\cdot, \bar{x}^{t+1}_t)$, where $\bar{x}^{t+1}_t$ is fixed. The allocation (x^t_{t-1}, x^t_t) is clearly PO in the reduced economy. We then have that p'^t is colinear with $\operatorname{grad} u_t(\cdot, \bar{x}^{t+1}_t)$ evaluated at x^t_t, which, because the reduced economy is PO, is colinear with $\operatorname{grad} u_{t-1}(\bar{x}^{t-1}_{t-1}, \cdot)$ evaluated at x^t_{t-1}. But by the induction hypothesis, this is colinear with p^t. Therefore, p^t and p'' are colinear in $\mathbb{R}^l$, $p'' = \lambda p^t$ for some $\lambda > 0$. Define $p^{t+1} = p''^{+1}/\lambda$. Clearly the vector $(p^1, p^2, \ldots, p^t, p^{t+1})$ supports $(x_0, x_1, \ldots, x_t)$ in $\mathscr{E}_{t+1}$. The proof by induction is completed by observing that x_0 is supported in $\mathscr{E}_0$ by any p^1 colinear with grad $u_0(x_0)$. If we let $p^1 \in S^1 = \{p^1 \in \mathbb{R}^l \mid p^{1 \cdot 1} = 1\}$ be the (unique) normalized vector supporting x_0 in $\mathscr{E}_0$, then we have defined the (unique) $p \in S$ which supports the WPO allocation x.

(b) Let $x = (x_0, x_1, \ldots, x_t, \ldots)$ be supported by a price sequence $p = (p^1, p^2, \ldots, p^t, \ldots)$. Clearly, $x_t = f_t(p, p \cdot x_t)$. Assume that x is not WPO. Then by Lemma (2.2), there exists $y = (y_0, y_1, \ldots, y_t, \ldots) \in X$ with $y_t = x_t$ for $t \geq$ some t', $\sum_t y_t = \sum_t x_t$ and $u_t(y_t) \geq u_t(x_t)$ with at least one strict inequality. Then $p \cdot y_t \geq p \cdot x_t$ (since $x_t = f_t(p, p \cdot x_t)$) and there is strict inequality of $u_t(y_t) > u_t(x_t)$ for some t. Therefore,

$$p \cdot \sum_{t=0}^{t=t'-1} y_t > p \cdot \sum_{t=0}^{t=t'-1} x_t,$$

which contradicts the assumptions that $\sum_t y_t = \sum_t x_t$ and $y_t = x_t$ for $t = t'$, $t' + 1, \ldots$. ∎

4.4. PROPOSITION. *Every competitive (Walrasian) allocation is weakly Pareto-optimal. Conversely, every weakly Pareto-optimal allocation is a competitive allocation associated with some suitably assigned endowments.*

Proof. The proposition is a direct consequence of Lemmas (4.2) and (4.3). ∎

5. WELFARE ANALYSIS: PARETO-OPTIMALITY

In this section, we investigate the relationship between Pareto-optimal allocations and weakly Pareto-optimal allocations. We know that PO allocations are WPO (Lemma (2.1)). We show that the converse is not true.[8] More precisely, we characterize those WPO allocations which are PO (and, obviously, those which are not PO). Then we show that not all WPO allocations are PO. These conditions are stated in terms of the price sequence known to support a WPO allocation. The application of this section to studying the relationship between competitive allocations and PO allocations is immediate, so that no more will be said about it.

[8] That the converse is false can be established by any one of a host of examples drawn from the existing literature. There are regular examples (in, e.g., [8, 12, 17]) in which competitive equilibrium allocations are not PO. By Proposition (4.4), we can interpret these as examples of WPO allocations which are not PO.

Looking back to Sections 3 and 4, the simplicity of the analysis of existence of competitive equilibrium and its relationship with WPO allocations is quite striking. Largely for convenience, we employed regularity conditions on consumers. The only uniformity condition we used was on the structure of generational overlap, which was imposed to ensure resource-relatedness. It is noteworthy that while we assumed individual endowments to be positive and finite, it was not necessary to impose uniform bounds, nor were any uniformity assumptions imposed on utility functions. In the present section, however, we shall employ some mild uniformity assumptions on consumers. The assumptions will be discussed as they are introduced.

We consider a given allocation $x \in X$. We shall need to determine whether or not there is a sequence of feasible commodity transfers h which is Pareto-improving on the allocation x, i.e., such that $x + h$ is Pareto-superior to x. These concepts are clarified by the next two definitions.

5.1. DEFINITION. Let $x = (x_0, x_1, ..., x_t, ...) \in X$ be a given allocation. The *sequence of commodity transfers* $h = (h_0, h_1, ..., h_t, ...)$ is *feasible* if $(x + h)$ belongs to X and $\sum_t h_t = 0$.

Note that for our model feasibility implies that h takes the form $(h_0, h_1, ..., h_t, ...)$ where

$$h_0 = h_0^1 = -h_1^1 \qquad \text{for} \quad t = 0$$

and

$$h_t = (h_t^t, h_t^{t+1}) = (h_t^t, -h_{t+1}^{t+1}) \qquad \text{for} \quad t \geqslant 1.$$

5.2. DEFINITION. We say that the *sequence of commodity transfers h* is *Pareto-improving* upon the allocation x if h is feasible and $u_t(x_t + h_t) \geqslant u_t(x_t)$ with strict inequality for at least one t, $t = 0, 1, ...$.

Clearly, if the sequence $\{x_t\}$ is bounded from above and if h is Pareto-improving, then h is bounded and λh, where $\lambda \in (0, 1)$, is also Pareto-improving upon $x = \{x_t\}$. The strategy in characterizing Pareto optima will be either to construct a Pareto-improving sequence h or to show that there does not exist such a sequence. The characterization of Pareto-optimality will be stated in terms of the price sequence known to support a WPO allocation.

5.3. PROPOSITION. *Consider the WPO allocation x supported by p. Assume:*

(a) *Property B: the sequence $\{x_t\}$ is bounded from above;*

(b) *Condition P:* $\liminf_{t \to \infty} \| p^t \| = 0$.

Then, x is PO.

Okuno and Zilcha [16] have provided a result close to Proposition (5.3). Note that (5.3) does not require the convergence of $\sum_t p \cdot x_t$ but does require x to be bounded from above. If we have $\sum_t p \cdot x_t < +\infty$, this last requirement is not necessary. (Proof of assertion: Assume that there is an allocation $y \in X$ such that $u_t(y_t) \geqslant u_t(x_t)$ with at least one inequality ($t = 0, 1,...$) and $\sum_t y_t = \sum_t x_t$. Multiplying by p yields $\sum_t p \cdot y_t = \sum_t p \cdot x_t < +\infty$ by hypothesis. Obviously, $p \cdot y_t \geqslant p \cdot x_t$ with at least one inequality (for $t = 0, 1,...$) by the assumption that y is Pareto-superior to x, thus contradicting $\sum_t p \cdot x_t = \sum_t p \cdot y_t < +\infty$.)

Property B can be interpreted as restricting the long-run real growth rates of the economy to be nonpositive. Condition P can be interpreted as implying that the long-run interest rates are positive. Proposition (5.3) is thus in accord with results in capital theory (cf., e.g., Cass [5]): If the commodity rates of interest exceed the real rates of growth, then SRPO allocations are PO.

To establish Propostion (5.3), we first have to prove the next two lemmas.

5.4. LEMMA. *Let h be Pareto-improving upon the WPO allocation x. If t_0 denotes the smallest t ($t = 0, 1,...$) such that $h_t \neq 0$, then $h_t \neq 0$ for $t = t_0, t_0 + 1,...$.*

Proof. Assume that there is some $t_1 > t_0$ such that $h_{t_1} = 0$. Because of the particular structure of generational overlap, the condition $\sum_t h_t = 0$, splits into the conditions

$$\sum_{t=t_0}^{t=t_1} h_t = 0 \qquad \text{and} \qquad \sum_{t>t_1} h_t = 0.$$

Therefore, the sequence $(0,..., 0, h_{t_0},..., h_{t_1}, 0,...)$ would be feasible, and $u_t(x_t + h_t)$ would be no smaller than $u_t(x_t)$ for $t = 0, 1,...$. Thus, $(0,..., 0, \lambda h_{t_0},..., \lambda h_{t_1}, 0,...)$ would be Pareto-improving for $0 < \lambda < 1$, because of the strict quasiconcavity of utility functions. Since the sequence $(0,..., 0, \lambda h_{t_0},..., \lambda h_{t_1}, 0,...)$ has only a finite number of nonzero components, this is a contradiction to the hypothesis that x is WPO. ∎

5.5. LEMMA. *Let h be Pareto-improving upon the WPO allocation x supported by the price sequence $p = (p^1, p^2,..., p^t,...)$. Then, we have the following inequalities ($t = 1, 2,...$):*

$$p^{t+1} \cdot h_{t+1}^{t+1} \leqslant p^t \cdot h_t^t \leqslant \cdots \leqslant p^1 \cdot h_1^1 = -p^1 \cdot h_0^1 \leqslant 0,$$

the inequalities being strict for $t \geqslant t_0$, where t_0 is defined in Lemma (5.4).

Proof. First, $h_0 = (h_0^1, 0,...)$ and $u_0(x_0 + h_0) \geqslant u_0(x_0)$ which implies that $p^1 \cdot (x_0^1 + h_0^1) \geqslant p^1 \cdot x_0^1$ and thus $p^1 \cdot h_0^1 \geqslant 0$. This inequality is strict if $h_0^1 \neq 0$ because of the strict quasiconcavity of $u_0(\cdot)$.

We next prove that $p^{t+1} \cdot h_{t+1}^{t+1} \leqslant p^t \cdot h_t^t$ with strict inequality if $h_t \neq 0$. Since $u_t(x_t + h_t) \geqslant u_t(x_t)$, we have $p \cdot (x_t + h_t) \geqslant p \cdot x_t$, hence $p \cdot h_t \geqslant 0$, i.e., $p^t \cdot h_t^t + p^{t+1} \cdot h_t^{t+1} \geqslant 0$, and by Lemma (5.4), $p^{t+1} \cdot h_{t+1}^{t+1} \leqslant p^t \cdot h_t^t$. These inequalities are strict if $h_t \neq 0$, because of the strict quasiconcavity of $u_t(\cdot)$. An argument by induction, aided by Lemma (5.4), completes the proof of Lemma (5.5). ∎

Proof of Proposition 5.3. Assume that there is a Pareto-improving sequence h. Consider t_0 defined in Lemma (5.4) so that $h_{t_0} \neq 0$. From Lemma (5.5), we have that

$$p^t \cdot h_t^t < p^{t_0+1} \cdot h_{t_0+1}^{t_0+1} < p^{t_0} \cdot h_{t_0}^{t_0} = 0 \qquad \text{for} \quad t = t_0 + 2, t_0 + 3, ...$$

But, since h_t^t is bounded and $\lim \inf_{t \to \infty} \|p^t\| = 0$ by condition P, we have

$$\liminf_{t \to \infty} \|p^t \cdot h_t^t\| = 0,$$

which is a contradiction to Lemma (5.5). ∎

We next state our main welfare result, which provides a complete characterization of Pareto optima.

5.6. PROPOSITION. *Let $x = (x_0, x_1, ..., x_t, ...) \in X$ be a WPO allocation supported by the price sequence $p = (p^1, p^2, ..., p^t, ...)$. Assume*:

(a) *Property C: The Gaussian curvature*[9] *at every point on consumer t's indifference surface through x_t is uniformly bounded from above*;

(b) *Property C′: The Gaussian curvature of consumer t's indifference surface through x_t at every point $y = (y_0, ..., y_t, ...)$ such that $0 \leqslant y_t^{t+1} \leqslant x_t^{t+1} + x_{t+1}^{t+1}$ for $t = 0, 1, ...$ and $0 \leqslant y_t^t \leqslant x_t^t + x_{t-1}^t$ for $t = 1, 2, ...$ is uniformly bounded away from* 0;

(c) *Property G: There exist constants P and Q (independent of t) such that*

$$0 < P \leqslant \frac{p^{s,t}}{\|(p^t, p^{t+1})\|} \leqslant Q < +\infty$$

[9] See Debreu [11], especially pp. 612–613, for the definition of Gaussian curvature and its role in demand theory; also, see Spivak [20, pp. 7–14].

The role of some maintained uniform "curvature" assumption in characterizations of intertemporal efficiency is recognized in Cass's fundamental paper [5] and later formalized in Benveniste's pretty paper [3]. Both [3] and [5] focus on intertemporal consumption efficiency. Okuno and Zilcha [16] apply the Benveniste analysis to the overlapping-generations model. In none of these three papers, however, is the Gaussian measure of curvature utilized.

for $s = t, t+1$; $i = 1,..., l$; and $t = 1, 2,...$. Furthermore, there exist constants P' and Q' (independent of t) such that

$$0 < P' \leqslant \frac{p^{t,i}}{\|p^t\|} \leqslant Q' < +\infty$$

for $i = 1,..., l$; and $t = 1, 2,...$;

(d) *Property B: The sequence $\{x_t\}$ is bounded from above;*

(e) *Property B': The sequence $\{x_t\}$ is bounded from below by a strictly positive vector.*

Then, x is Pareto-optimal if and only if

$$\sum_t \frac{1}{\|p^t\|} = +\infty.$$

If we exclude "kinks" in indifference surfaces, then there is obviously an upper bound for the Gaussian curvature of a given indifference surface. Property C adds that there is a *uniform* bound on the Gaussian curvature of the respective indifference surfaces passing through $x_1,..., x_t,...$. Property C' is a uniform strengthening of strict quasiconcavity of utility functions, ruling out the possibility that indifference surfaces are arbitrarily close to being flat in the neighborhood of x_t. The uniform curvature Properties C and C' are essential for Proposition (5.6). See Cass [5, pp. 221–222]. Property C is needed for the "only if" part. Property C' is needed for the "if" part. Examples are provided in [19] where the interest rate is identically zero (and thus $\sum_t 1/\|p^t\|$ diverges) and in which competitive allocations are not PO; the examples in [19] are for the case where indifference curves are lines (which, of course, have zero Gaussian curvature).

Property G is a condition on the gradients of utility functions needed to avoid cases in which the steepness of indifference surfaces is either arbitrarily large or arbitrarily close to zero; it can be taken as a uniform strengthening of the bounds on prices provided in Lemma (3.4). Property B restricts the long-run real growth rates in the endowments of each of the l commodities to be nonpositive, while Property B' restricts the long-run real growth rates to be bounded from below by some negative number. The importance of Property G is that it implies that for any WPO allocation, long-run own-rates-of-interest for each of the l commodities must be "similar," i.e., the sequences $\{p^{i,t}\}$ and $\{p^{j,t}\}$ $(i, j = 1,..., l)$ must obey "similar" growth conditions.

Like Proposition (5.3), Proposition (5.6) is reminiscent of the results on intertemporal efficiency pricing in the capital theory literature (cf., e.g., Cass [5]). If the long-run interest rates are nonnegative, then $\sum_t 1/\|p^t\| = +\infty$, and the allocation is PO. More generally, if long-run growth rates do not exceed long-run interest rates, then Pareto-optimality obtains. If, on the other hand long-run interest rates are bounded below some negative number, $\sum_t 1/\|p^t\|$ converges, and the allocation is not Pareto-optimal. More generally, if long-run growth rates exceed long-run interest rates, then the allocation is not Pareto-optimal.

Note that, if conditions (a) through (e) are satisfied, then Proposition (5.6) implies Proposition (5.3): If $\liminf_{t\to\infty} \|p^t\| = 0$, then $1/\|p^t\|$ does not tend to zero and $\sum_t 1/\|p^t\|$ necessarily diverges.

The basic structure of the proof of Proposition (5.6) is revealed in the one-commodity case. Therefore, the next lemmas will be stated in the case $l = 1$, and used to prove Proposition (5.6) for that case. The extension to arbitrary l, which requires Property G, is provided after Lemma (5.10).

Assume $l = 1$. We then find it more convenient to set $-\varepsilon^t = h_t^t$, so that for feasible h, we can write

$$h_t = (0,\dots, 0, -\varepsilon^t, \varepsilon^{t+1}, 0,\dots) \qquad \text{for} \quad t = 1, 2,\dots .$$

Thus, ε^t can be interpreted as the amount of commodity "transferred" by consumer t to consumer $t-1$. We also set

$$\eta^t = p^{t+1} \cdot \varepsilon^{t+1} - p^t \cdot \varepsilon^t,$$

the present value of "transfers to" consumer t minus "transfers from" consumer t. Let $x = (x_0, x_1,\dots, x_t,\dots)$ be the allocation under consideration. We also associate with the sequences $\{\varepsilon^t\}$ and $\{p^t\}$ the sequence $\{\alpha^t\}$ defined by the formula

$$\alpha^t = \frac{((\varepsilon^t)^2 + (\varepsilon^{t+1})^2) \cdot ((p^t)^2 + (p^{t+1})^2)^{1/2}}{\eta^t}.$$

5.7. LEMMA. *If h is Pareto-improving upon x and if C' holds, then the sequence $\{\alpha^t\}$ is bounded from above.*

Proof. By Property C′, there is a lower bound $2/\rho$ of the Gaussian curvature at the points of the indifference curve of consumer t through x_t with coordinates $\leqslant(x_{t-1}^t + x_t^t, x_t^{t+1} + x_{t+1}^{t+1})$. It results from the convexity of preferred sets and from the definition of the lower bound $2/\rho$ that any circle of radius greater than $\rho/2$, tangent to the indifference curve at (x_t^t, x_t^{t+1}), and with center on the line perpendicular to the tangent to the indifference curve, lies below the indifference curve (except, of course, at x_t) in the feasible domain, i.e., the set of points with co-ordinates $\leqslant(x_{t-1}^t + x_t^t, x_t^{t+1} + x_{t+1}^{t+1})$. See Fig. (5.1). Therefore, any feasible point N which dominates the point $M = x_t = (x_t^t, x_t^{t+1})$ lies inside this circle. We consider the circle Γ of radius ρ. The point $N = (x_t^t - \varepsilon^t, x_t^{t+1} + \varepsilon^{t+1})$ lies in the interior of Γ if and only if

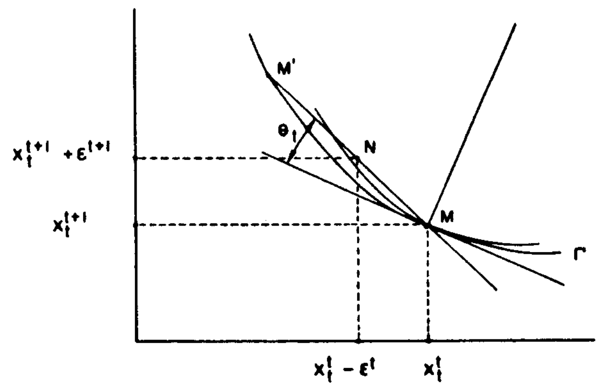

FIGURE 5.1

length $MN <$ length MM' where M' denotes the second intersection point of Γ with the line MN. Since $MM' = 2\rho \sin \theta_t$ where θ_t is the angle formed by MN and the tangent at M to Γ, we have

$$(\varepsilon^t)^2 + (\varepsilon^{t+1})^2 < (2\rho \sin \theta_t)^2.$$

Using the definition of the inner product yields

$$\begin{aligned}\eta^t &= p^{t+1} \cdot \varepsilon^{t+1} - p^t \cdot \varepsilon^t \\ &= \sin \theta_t((\varepsilon^t)^2 + (\varepsilon^{t+1})^2)^{1/2}((p^t)^2 + (p^{t+1})^2)^{1/2}.\end{aligned}$$

Eliminating $\sin \theta_t$ in these two relationships, we obtain

$$((\varepsilon^t)^2 + (\varepsilon^{t+1})^2)^{1/2} < 2\rho\eta^t \cdot \frac{1}{((\varepsilon^t)^2 + (\varepsilon^{t+1})^2)^{1/2}} \cdot \frac{1}{((p^t)^2 + (p^{t+1})^2)^{1/2}}.$$

Hence

$$\alpha^t = \frac{((\varepsilon^t)^2 + (\varepsilon^{t+1})^2)((p^t)^2 + (p^{t+1})^2)^{1/2}}{\eta^t} < 2\rho,$$

which proves Lemma (5.5). ∎

In the next lemma it is also assumed that $l = 1$.

5.8. LEMMA. *If there exist a positive sequence* $\{\eta^t\}$ *and a scalar* ε^1 *such that*

(a) $\{\varepsilon^t\}$ *is bounded;*

(b) $\{\alpha^t\}$ *is bounded,*

and if Properties B, B′, and C hold, then the sequence h defined by $\{\varepsilon^t\}$ *is Pareto-improving upon x.*

Proof. By Property C, there is an upper bound $1/2\rho$ of the Gaussian curvature on the indifference curve passing through x_t. It results from the convexity of preferred sets and from the definition of the upper bound $1/2\rho$ that any circle of radius less than 2ρ, tangent to the indifference curve at (x_t^t, x_t^{t+1}), and with center on the line perpendicular to the tangent to the indifference curve, lies entirely above the indifference curve (except, of course, at x_t). Therefore, any point strictly inside the circle Γ dominates (from the viewpoint of consumer t) the allocation $x_t = (x_t^t, x_t^{t+1})$. Take Γ to be the circle of radius ρ. The point $N = (x_t^t - \varepsilon^t, x_t^{t+1} + \varepsilon^{t+1})$ lies inside Γ (see Fig. 5.2), if length $MN <$ length MM'' where M'' denotes the second intersection point of Γ with the line MN. Since $MM'' = 2\rho \sin \theta_t$, where θ_t is the angle formed by MN and the tangent at M to Γ, this will be ensured if we have

$$(\varepsilon^t)^2 + (\varepsilon^{t+1})^2 < (2\rho \sin \theta_t)^2.$$

As before, using the definition of the inner product, η^t, the above condition amounts to

$$\alpha^t < 2\rho.$$

These results enable us to construct a Pareto-improving sequence h. Let $\{\eta^t\}$ and ε^1 satisfy the assumptions of Lemma (5.8) and let $h = (h_0, h_1, \ldots, h_t, \ldots)$ correspond to the sequence $\{\varepsilon^t\}$. Clearly, it results from Properties B and B′ that, if λ is small enough, then λh is feasible. Furthermore, α^t being linear in λ can be made arbitrarily small by taking λ small enough, hence smaller than

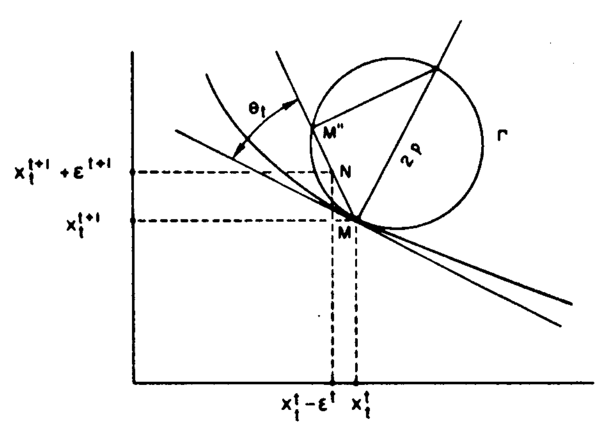

FIGURE 5.2

2ρ for every $t = 1, 2, \ldots$. The above results imply that λh is then feasible and Pareto-improving upon x.

For the next lemma, we continue to impose the restriction $l = 1$.

5.9. LEMMA. *Given the positive price sequence* $\{p^t\}$ *satisfying Property G, there exist a positive sequence* $\{\eta^t\}$ *and an* $\varepsilon^1 > 0$ *such that the associated sequences* $\{\alpha^t\}$ *and* $\{\varepsilon^t\}$ *are bounded, if and only if* $\sum_t 1/p^t < +\infty$.

Proof. First, we show that the condition $\sum_t 1/p^t < +\infty$ implies the existence of some positive sequence $\{\eta^t\}$ (and of an ε^1) such that the associated sequences $\{\varepsilon^t\}$ and $\{\alpha^t\}$ are bounded. Let us show that the sequences which satisfy $\eta^t = 1/p^t$ also satisfy the required properties for any arbitrarily given $\varepsilon^1 > 0$. We have

$$\varepsilon^t = \frac{p^1 \cdot \varepsilon^1}{p^t} + \frac{1}{p^t}\left(\sum_{l=1}^{t-1} \frac{1}{p^l}\right).$$

Convergence of $\sum 1/p^t$ implies that $1/p^t \to 0$, hence $p^t \to \infty$ as $t \to +\infty$. We then have that $(p^1 \cdot \varepsilon^1)/p^t \to 0$ and $(1/p^t) \to 0$, while

$$\sum_{l=1}^{t-1} \frac{1}{p^l} \to \sum_{l=1}^{t} \frac{1}{p^l}.$$

Consequently, $\varepsilon^t \to 0$ and is, therefore, bounded. Let $s_t = \sum_{l=1}^{t} 1/p_l - p^1\varepsilon^1$. Clearly, α^t is less than

$$s_t^2 \left\{\frac{1}{(p^{t+1})^2} + \frac{1}{(p^t)^2}\right\} p^t((p^t)^2 + (p^{t+1})^2)^{1/2},$$

and, hence α^t is less than

$$s_t^2(1 + u_t)\left(1 + \frac{1}{u_t}\right)^{1/2},$$

where $u_t = (p^t/p^{t+1})^2$. Therefore, $(\alpha^t)^2$ is less than

$$s_t^4(1 + u_t)^2\left(1 + \frac{1}{u_t}\right) = s_t^4\left(u_t^2 + 3u_t + 3 + \frac{1}{u_t}\right).$$

It results from Property G that the ratio of prices u_t belongs to an interval $[\alpha, \beta]$ where $\alpha > 0$ and $\beta < +\infty$. Consequently, $\{u_t^2 + 3u_t + 3 + 1/u_t\}$ is bounded, hence $\{(\alpha^t)^2\}$ and $\{\alpha^t\}$ are bounded.

We must now prove the converse, namely that the boundedness of $\{\varepsilon^t\}$ and $\{\alpha^t\}$ implies $\sum_t 1/p^t < +\infty$. By definition,

$$p^t \cdot \varepsilon^t = p^1 \cdot \varepsilon^1 + (\eta^1 + \cdots + \eta^{t-1}).$$

Hence,

$$\varepsilon^t > \frac{\eta^1 + \cdots + \eta^{t-1}}{p^t}.$$

We also have that

$$\alpha^{t-1} > \frac{(\varepsilon^t)^2 p^t}{\eta^{t-1}} > \frac{(\eta^1 + \cdots + \eta^{t-1})^2}{p^t \cdot \eta^{t-1}}.$$

Therefore, the boundedness of $\{\alpha^t\}$ implies that the sequence $\{\beta^t\}$ defined by

$$\beta^t = \frac{(\eta^1 + \cdots + \eta^{t-1})^2}{p^t \cdot \eta^{t-1}}$$

is bounded. The proof of Lemma (5.9) follows immediately from the next lemma (Lemma (5.10)). ▮

5.10. LEMMA. *Let $\{p^t\}$ and $\{\eta^t\}$ be positive sequences such that the sequence $\{\beta^t\}$ defined by*

$$\beta^t = \frac{(\eta^1 + \cdots + \eta^{t-1})^2}{p^t \cdot \eta^{t-1}}$$

is bounded. Then, $\sum_t 1/p^t$ converges.

Proof. There is by hypothesis a $K < +\infty$ such that $\beta^t < K$ for $t = 2, 3, \ldots$. Therefore, we have

$$\frac{1}{p^t} < \frac{K\eta^{t-1}}{(\sum_{l=1}^{t-1} \eta^l)^2}.$$

To establish that $\sum 1/p^t < +\infty$, it is, therefore, sufficient to show that the sum

$$\sum \gamma^t = \sum \frac{\eta^{t-1}}{(\sum_{l=1}^{t-1} \eta^l)^2}$$

converges. This is obvious if $\sum \eta^t < +\infty$. If, however, $\sum \eta^t = +\infty$, convergence of $\sum \gamma^t$ is established by comparison with the integral $\int_t^\infty (1/x^2)\, dx$, which converges. For t sufficiently large, the integral dominates the sum $\sum_{s=t}^{\infty} \gamma^s$, as is illustrated in Fig. 5.3. ▮

Proof of Proposition (5.6). It is clear that Lemma (5.10), along with Lemmas (5.7) and (5.8), provides a proof of Proposition (5.6) for the case $l = 1$. (Note, however, that ε^1 must be chosen to be sufficiently small.)

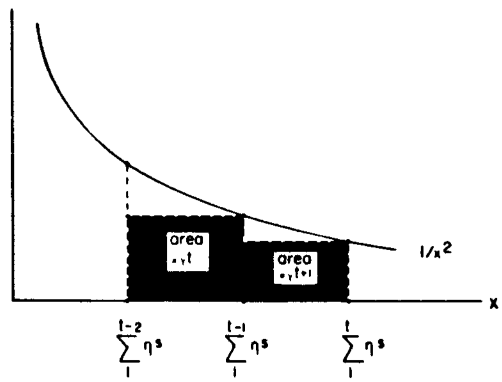

FIGURE 5.3

Property G allows us to extend the results to the general case, including $l \geqslant 2$. Pick an arbitrary commodity (say, commodity i). From Property G, we know that there are constants P' and Q' such that

$$\frac{1}{Q'} \cdot \frac{1}{\|p^t\|} \leqslant \frac{1}{p^{t,i}} \leqslant \frac{1}{P'} \cdot \frac{1}{\|p^t\|},$$

hence $\sum 1/p^{t,i}$ converges if and only if $\sum 1/\|p^t\|$ converges. Assume, therefore, that $\sum 1/\|p^t\| < +\infty$ and thus $\sum 1/p^{t,i} < +\infty$. We are then back to the results from the case $l = 1$; here we can find a Pareto-improving sequence of transfers using only commodity i, while holding allocation of all other commodities fixed.

Conversely, assume that there exists a (bounded) Pareto-improving sequence $h = \{h_t\}$. For consumer $t \geqslant 1$, consider the vector h_t and the price vector (p^t, p^{t+1}). These vectors define a plane in $\mathbb{R}^{2l}$. This plane intersects the indifference surface through x_t, thus defining a 1-dimensional curve in this plane. Clearly, this curve inherits the curvature Properties C and C′ of the original indifference surface. Except for the labelling of axes, we face the same situation as in Fig. (5.1), which was constructed for the case $l = 1$. This is shown in Fig. (5.4). The circle Γ has radius ρ (where $2/\rho$ is the lower bound on Gaussian curvatures at x), and is tangent at x_t to the curve resulting from the intersection of the plane of Fig. (5.4) with the indifference surface. The center of Γ lies on the line perpendicular to this curve at x_t. Therefore, Γ lies below this curve (except, of course, at x_t) in the relevant range.

As in the one-commodity case, we define the present value of net transfers to consumer t by $\eta^t = p \cdot h_t$ (where η^t is then positive). We have that $\eta^0 + \cdots + \eta^{t-1} = -p^t \cdot h_t^t$. Hence,

$$\|h_t^t\| \cdot \|p^t\| \geqslant \eta^0 + \cdots + \eta^{t-1}.$$

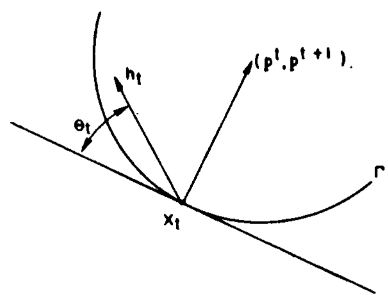

FIGURE 5.4

The fact that $x_t + h_t$ is preferred to x_t by consumer t along with our Gaussian curvature Property C′ implies that $\|h_t^t\| < 2\rho \sin \theta_t$, as in the case with $l = 1$.

Since we have that

$$\eta^t = p \cdot h_t = \|(p^t, p^{t+1})\| \cdot \|(h_t^t, -h_{t+1}^{t+1})\| \cdot \sin \theta_t,$$

we obtain

$$\eta^t < \frac{\|h_t\|^2 \cdot \|(p^t, p^{t+1})\|}{2\rho}$$

and hence

$$\frac{\|h_t\|^2 \cdot \|(p^t, p^{t+1})\|}{\eta^t} < 2\rho \qquad \text{for} \quad t \geqslant 1.$$

Since $h_t = (h_t^t, -h_{t+1}^{t+1})$, it follows that $\|h_t\| \geqslant \|h_{t+1}^{t+1}\|$. Furthermore, $\|(p^t, p^{t+1})\|$ is at least as large as $\|p^{t+1}\|$. Therefore, the sequence $\|h_{t+1}^{t+1}\|^2 \cdot \|p^{t+1}\|/\eta^t$ is bounded. Since $\|h_t^t\| \cdot \|p^t\| \geqslant \eta^0 + \cdots + \eta^{t-1}$, we deduce that the sequence

$$\beta^t = \frac{(\eta^0 + \cdots + \eta^t)^2}{\|p^{t+1}\| \, \eta^t}$$

is bounded. We conclude by applying Lemma (5.10). ∎

5.11. *Remark.* One can construct examples of allocations in economies satisfying the hypothesis of Proposition (5.6) such that $\sum_t 1/\|p_t\| < +\infty$. Hence, one can construct WPO allocations which are not PO. Begin by picking an allocation $x = \{x_t\}$ satisfying Properties B and B′. Then pick a price sequence $\{p^t\}$ such that $\sum_t 1/\|p^t\|$ converges. It is then straightforward to choose demand functions $\{f_t(p, w_t)\}$ consistent with Properties C, C′, and G such that

$$f_0(p^1, p^1 \cdot x_0^1) = x_0^1 \qquad \text{for} \quad t = 0$$

and

$$f_t(p^t, p^{t+1}, p^t \cdot x_t^t + p^{t+1} \cdot x_t^{t+1}) = (x_t^t, x_t^{t+1}) \qquad \text{for} \quad t = 1, 2, \ldots .$$

The allocation x is supported by p and is therefore WPO. But, by construction, x is not PO.

6. Concluding Remarks

The overlapping-generations model is genuinely dynamic. It thus departs from the basically atemporal character of most economic models in that it reflects both the open-endedness and the unidirectional nature of time. Therefore, the overlapping-generations approach provides a promising framework for the analysis of intertemporal allocation and the roles of monetary and fiscal policy. It is obvious that the present paper goes only a short way in developing a theory rich enough to be useful in policy analysis. Clearly, money (and other government debt), production, and private durable assets (including capital) must be incorporated into the theory. The structure of the set of competitive equilibria must be studied in depth.

We do not expect these extensions of the theory to be at all trivial to develop. For one thing, rather new mathematical techniques are likely to be required. There is a sense, however, in which these extensions are somewhat straightforward: The basic formulations involved are closely related to those of the currently extant general-equilibrium literature. (This relationship is reflected in the present study by the reliance on general-equilibrium techniques and notation.) The more difficult and subtle extensions which will be needed for the overlapping-generations model are likely to require new formulations, which go far beyond what is now available in the literature. Included are questions involving expectations formation, transaction costs, and price rigidities.[10]

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[10] An elaboration of some of these issues can be found in Cass and Shell [7].

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