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The Overlapping-Generations Model. III. The Case of Log-Linear Utility Functions*

YVES BALASKO

Université Paris I, CEPREMAP, Paris, France, and Center for Analytic Research in Economics and the Social Sciences, University of Pennsylvania, Philadelphia, Pennsylvania 19104

AND

KARL SHELL

Center for Analytic Resarch in Economics and the Social Sciences, University of Pennsylvania, Philadelphia, Pennsylvania 19104

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1. INTRODUCTION

Existence of competitive equilibria in the overlapping-generations model has been studied in some detail; see [4–6, 9]. We need to know more about the properties of these equilibria. How many are there? How do they depend on the basic parameters of the economy? We also need to compare the set of monetary competitive equilibria with the set of nonmonetary competitive equilibria. Are there "vastly more" monetary equilibria than nonmonetary equilibria?

Analysis of the properties of competitive equilibria is for the overlapping-generations model an important, but difficult, task. In his seminal article [8], Samuelson put it this way: "... if we take any finite stretch of time and write out the equilibrium conditions, we always find them containing discount rates from before the finite period and discount rates from afterward. We never seem to get enough equations; lengthening our time periods turns out always to add as many new unknowns as it supplies equations ..." Samuelson suggested that, rather than facing this difficulty directly, "We can try to cut the Gordian knot by our special assumption of stationariness ..."

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Indeed, assuming stationarity is just that—*cutting* the Gordian knot. An important goal of government macroeconomic policy is stabilization of the short-run fluctuations in an evolving and changing economic environment. All this is lost when we focus solely on the long-run steady state of a replicating economy. One of the defects of recent monetary theory has been its emphasis on the long run, a very peculiar point of view for the stabilization-oriented macroeconomist.

At any rate, stationary analyses are not appropriate to the overlapping-generations model which we considered in [5, 6], where preferences and endowments are allowed to vary from consumer to consumer in a general way. It turns out, however, for the special (nonstationary) case where utility functions are log-linear, the Gordian knot is easily *untied*. Log-linearity drastically simplifies the mathematical analysis and reduces it to a study of the properties of an infinite number of linear equations having an infinite number of unknowns. The structure of the overlapping-generations model then enables us to solve this system in a straightforward and elementary way by merely considering finite stretches of time.

Section 2 is devoted to the log-linear nonmonetary economy. The unique Walrasian equilibrium is shown to be a linear function (i.e., a degree-one polynomial function, with a "constant" term) of endowments, considered as the parameters of the economy. The log-linear monetary economy is studied in Section 3. There is a nonnegative scalar $\bar{p}^m$ with the property that if the price of money p^m exceeds $\bar{p}^m$ then there is no competitive equilibrium, and if $p^m \in [0, \bar{p}^m)$ there is exactly one competitive equilibrium. This equilibrium is a linear function of p^m. Note that in general the price of money is indeterminate and parameterizes the monetary equilibria. This result confirms our experience with monetary overlapping-generations models: we frequently encounter a vast multiplicity of equilibria.

The analysis of the nonmonetary and monetary economies is extended in Section 4 to the case where only consumer t and his successors possess log-linear utility functions. Here, too, the Gordian knot can be untied: equilibrium prices are determined by a counting-equations approach, providing a generic solution to the problem of the number of equilibria and their behavior with respect to the parameters defining the economy. Extension of our results beyond the "nearly log-linear" case seems to require the development of a transversality theory which can be applied to the appropriate infinite-dimensional spaces. How to untie this infinite-dimensional "Gordian knot" remains for the general case an unsolved problem, which is important and challenging.

2. The Log-Linear Nonmonetary Economy

We use the notation of Balasko and Shell [5]. Let

$$u_0(x_0) = \sum_{k=1}^{l} a_0^{1,k} \log x_0^{1,k}$$

and

$$u_t(x_t) = \sum_{k=1}^{l} a_t^{t,k} \log x_t^{t,k} + \sum_{k=1}^{l} a_t^{t+1,k} \log x_t^{t+1,k} \qquad \text{for} \quad t \geqslant 1,$$

be the utility functions of consumer 0 and consumer t, respectively. The coefficients $(a_t^{s,k})$ are assumed to satisfy the restrictions

$$a_0^{1,k} > 0, \qquad a_t^{t,k} > 0, \qquad a_t^{t+1,k} > 0,$$

$$\sum_{k=1}^{l} a_0^{1,k} = 1, \qquad \sum_{k=1}^{l} a_t^{t,k} + \sum_{k=1}^{l} a_t^{t+1,k} = 1.$$

Recall that the normalized price sequence $p \in \mathscr{P} = \{p \mid p^{1,1} = 1\}$ is a Walrasian equilibrium (cf. [5, Definition (2.3)]) associated with the endowment sequence $\omega = (\omega_0, \omega_1, ..., \omega_t, ...) \in X$ if and only if

$$\sum_t f_t(p, p \cdot \omega_t) = \sum_t \omega_t = r.$$

This equation is equivalent to the equation system

$$\sum_t f_t(p, w_t) = \sum_t \omega_t = r,$$

$$p \cdot \omega_t = w_t \qquad \text{for} \quad t \geqslant 0,$$

where the unknowns are not only prices $p \in \mathscr{P}$ but also "incomes" $w = (w_0, w_1, ..., w_t, ..) \in W$. If we substitute for the demand function f_t the particular expression resulting from the constrained maximization of the log-linear utility function $u_t(\cdot)$, we obtain the system of equations

$$a_{t-1}^{t,l} w^{t-1} + a_t^{t,l} w_t = p^{t,l} \cdot r^{t,l} \qquad \text{for} \quad t \geqslant 1 \text{ and } i = 1, ..., l,$$

$$w_t = p \cdot \omega_t \qquad \text{for} \quad t \geqslant 0,$$

with $p^{t+1} > 0$ and $w_t > 0$ for $t \geqslant 0$. A price sequence $p \in \mathscr{P}$ which solves the above system is a Walrasian equilibrium associated with ω and conversely.

2.1. PROPOSITION. *There is a unique Walrasian equilibrium $p \in \mathscr{P}$ (and corresponding allocation $x \in X$) associated with every endowment sequence $\omega \in X$.*

Proof. Consider the truncated equilibrium system

$$\begin{aligned}
&a_0^{1,1} w_0 + a_1^{1,1} w_1 = r^{1,1},\\
&\cdots\\
&a_t^{t+1,l} w_t + a_{t+1}^{t+1,l} w_{t+1} = p^{t+1,l} \cdot r^{t+1,l},\\
&w_0 = p^1 \cdot \omega_0^1,\\
&w_1 = p^1 \cdot \omega_1^1 + p^2 \cdot \omega_1^2,\\
&\cdots\\
&w_t = p^t \cdot \omega_t^t + p^{t+1} \cdot \omega_t^{t+1},\\
&p^1 > 0,\dots, p^{t+1} > 0,\\
&w_0 > 0,\dots, w_t > 0.
\end{aligned} \tag{2.1.t}$$

Clearly, if $p \in \mathscr{P}$ is a Walrasian equilibrium associated with ω, then the truncated vector $(p^1,\dots, p^{t+1}, w_0 = p \cdot \omega_0,\dots, w_{t+1} = p \cdot \omega_{t+1})$ is a solution to (2.1.t). From Balasko and Shell [5, Proposition (3.10)], there is at least one Walrasian equilibrium $p \in \mathscr{P}$; therefore, by truncation, there is at least one solution of the system (2.1.t) for each $t \geqslant 0$. We next show that the solution to (2.1.t) is unique.

Assume that there are at least two distinct solutions, $\pi = (p^1,\dots, p^{t+1}, w_0,\dots, w_{t+1})$ and $\pi' = (p'^1,\dots, p'^{t+1}, w'_0,\dots, w'_{t+1})$. We claim that there is then $s \in \{1,\dots, t+1\}$ such that $p^s \neq p'^s$ or $s \in \{0,\dots, t\}$ such that $w_s \neq w'_s$; i.e., we claim that, if π and π' are distinct solutions, they do not differ solely in the last components, w_{t+1} and w'_{t+1}. To prove the claim, assume that $p^s = p'^s$ for $s = 1,\dots, t+1$ and $w_s = w'_s$ for $s = 0,\dots, t$. Select any $i \in \{1,\dots, l\}$, then

$$a_t^{t+1,i} w_t + a_{t+1}^{t+1,i} w_{t+1} = p^{t+1,i} \cdot r^{t+1,i}$$

and

$$a_t^{t+1,i} w'_t + a_{t+1}^{t+1,i} w'_{t+1} = p'^{t+1,i} \cdot r^{t+1,i},$$

which together imply that $a_{t+1}^{t+1,i}(w_{t+1} - w'_{t+1}) = 0$, or $w_{t+1} = w'_{t+1}$, implying in turn that $\pi = \pi'$, contrary to assumption. Thus, if π and π' are distinct solutions, they cannot differ solely in their last components.

Let $\bar{\pi} = \lambda\pi + (1-\lambda)\pi'$, where $\lambda \in \mathbb{R}$. Clearly, if $\lambda \notin [0, 1]$, then $\bar{\pi}$ is a solution to the equalities of (2.1.t), which might violate the inequalities of (2.1.t). Recall that $\bar{\pi}$ is linear in $\lambda \in \mathbb{R}$ and that (from the previous result) at least one of the first $(l+1)(t+1)$ components of $\bar{\pi}$ is not constant in λ. Therefore, there is a $\lambda \in \mathbb{R}$ such that the vector $(\bar{p}^1,..., \bar{p}^{t+1}, \bar{w}_0 ..., \bar{w}_t) = \lambda(p^1,..., p^{t+1}, w_0,..., w_t) + (1-\lambda)(p'^1,..., p'^{t+1}, w'_0,..., w'_t)$ is nonnegative but equal to zero in at least one component. Therefore, either for some $k \in \{1,..., l\}$ and some $s \in \{1,..., t+1\}$, $\bar{p}^{s,k} = 0$ or for some $s \in \{0,..., t\}$, $\bar{w}_s = 0$. Then, from the equations in (2.1.t), either $\bar{w}_s = \bar{w}_{s-1} = 0$ or $\bar{p}^s = \bar{p}^{s+1} = 0$, which implies that $(\bar{p}^1,..., \bar{p}^{t+1}, \bar{w}_0,..., \bar{w}_t) = 0$. This contradicts the assumption that $a_0^{1,1} w_0 + a_1^{1,1} w_1 = r^{1,1} > 0$. The rest of the proof follows by varying t. ∎

Proposition (2.1) extends a well-known result on the uniqueness of Walrasian equilibrium from the standard finite model to the overlapping-generations model; see, e.g., [1].

3. The Log-Linear Monetary Economy

We use the notation of Balasko and Shell [6]. Let $p^m \in \mathbb{R}_+$ be the present price of money and let $\mu = (\mu_0, \mu_1,..., \mu_t,...) \in \mathscr{M}$ be the monetary policy. The equilibrium conditions are then

$$a_{t-1}^{t,i} w_{t-1} + a_t^{t,i} w_t = p^{t,i} r^{t,i} \quad \text{for } t \geqslant 1 \text{ and } i = 1,..., l,$$

$$w_t = p \cdot \omega_t + p^m \mu_t \quad \text{for } t \geqslant 0,$$

$$p^{t+1} > 0, \quad w_t > 0 \quad \text{for } t \geqslant 0.$$

3.1. Proposition. *The set $\mathscr{M}_B(\omega)$ of ω-bonafide normalized monetary policies is convex.*

Proof. The set of price-income equilibria (p, w) (cf. [6, Definition (5.4)]) is defined by a set of linear equations and linear inequalities and hence is obviously convex. The set $\mathscr{M}_B(\omega)$ is the image of the set of price-income equilibria by the linear mapping Φ_ω: $(p, w_0,..., w_t,...) \mapsto (w_0 - p \cdot \omega_0,..., w_t - p \cdot \omega_t,...)$ which maps convex sets onto convex sets (cf. [6, Section 6, especially Proposition (6.2)]). Therefore, $\mathscr{M}_B(\omega)$ is convex.

3.2. Proposition. *The set of equilibrium present prices of money $\mathscr{P}^m(\omega, \mu)$ is an interval for fixed endowments $\omega \in X$ and fixed monetary policy $\mu \in \mathscr{M}$.*

Proof. There is a linear bijection between $\mathscr{P}^m(\omega, \mu)$ and the intersection of the set $\mathscr{M}_B(\omega)$ with the ray $L(\mu)$; cf. Balasko and Shell [6, Proposition 7.3]. Since $\mathscr{M}_B(\omega)$ and $L(\mu)$ are each convex sets, their intersection, $\mathscr{M}_B(\omega) \cap L(\mu)$, is also convex. Therefore, $\mathscr{P}^m(\omega, \mu) \subset \mathbb{R}_+$ is convex and thus an interval. ∎

Hence, there exists some nonnegative scalar $\bar{p}^m$ (possibly equal to $+\infty$; see, however, Balasko and Shell [6, Proposition (7.4)]) such that $\mathscr{P}^m(\omega, \mu)$ is the interval with bounds 0 and $\bar{p}^m$. It may happen that $\mathscr{M}_\mu(\omega) \cap L(\mu) = \{0\}$, in which case $\bar{p}^m = 0$.

3.3. Proposition. *For any $p^m \in [0, \bar{p}^m)$ there is a unique monetary competitive equilibrium $q = (p, p^m) \in \mathscr{Q}(\omega, \mu)$ associated with (ω, μ).*

Proof. From Proposition (3.2) and the definition of $\bar{p}^m$, it follows that for any $p^m \in [0, \bar{p}^m)$, the following system has at least one solution

$$\begin{aligned} &a_{t-1}^{t,i} w_{t-1} + a_t^{t,i} w_t = p^{t,i} r^{t,i} && \text{for } t \geqslant 1 \text{ and } i = 1,\dots, l, \\ &w_t = p \cdot \omega_t + p^m \mu_t && \text{for } t \geqslant 0, \\ &p^{t+1} > 0, \quad w_t > 0 && \text{for } t \geqslant 0. \end{aligned} \tag{3.3.1}$$

The following truncated system admits as a solution the truncation of any solution of (3.3.1)

$$\begin{aligned} &a_0^{1,1} w_0 + a_1^{1,1} w_1 = r^{1,1}, \\ &\dots \\ &a_t^{t+1,l} w_t + a_{t+1}^{t+1,l} w_{t+1} = p^{t+1,l} r^{t+1,l}, \\ &w_0 = p^1 \cdot \omega_0 + p^m \mu_0, \\ &\dots \\ &w_t = p^t \cdot \omega_t^t + p^{t+1} \cdot \omega_t^{t+1} + p^m \mu_t, \\ &p^1 > 0, \dots, p^{t+1} > 0, \\ &w_0 > 0, \dots, w_t > 0. \end{aligned} \tag{3.3.t}$$

Notice that the equations in (3.3.t) have the same coefficients as the equations in (2.1.t); only the "constant terms" differ. Uniqueness of the solution to (2.1.t) implies that the equations in (2.1.t) define a nondegenerate system. Since nondegeneracy is independent of the values of the "constant terms," the linear equations in (3.3.t) are nondegenerate and have a unique solution. The proof is completed as in Proposition (2.1). ∎

3.4. Proposition. *The monetary competitive equilibrium $q = (p, p^m) \in \mathscr{Q}(\omega, \mu)$ is a linear function of p^m for each $p^m \in [0, \bar{p}^m)$.*

Proof. Since we merely need to establish this property for each coordinate of p, it suffices to analyze the truncated system (3.3.t). The proposition is then obvious after applying the explicit Cramer formula used in solving a nondegenerate system of linear equations. ∎

4. ECONOMIES IN WHICH ALL BUT A FINITE NUMBER OF UTILITY FUNCTIONS ARE LOG-LINEAR

Assume that for $t' \geqslant t$, the utility function $u_{t'}(\cdot)$ is log-linear, i.e., consumer t and his followers have log-linear functions defined on $\mathbb{R}^{2l}_{++}$. Given this assumption—that the economy is "nearly log-linear"—we are able in what follows to answer completely questions about the number and behavior of equilibria.

Consider first the nonmonetary case. Let $\omega \in X$ be the sequence of endowments. We hold total resources, $r = \sum_t \omega_t$, constant and study how the equilibria vary as ω varies.

Define the t-truncation $\omega(t)$ of ω as the vector

$$\omega(t) = (\omega_0^1, \omega_1^1, \omega_1^2, ..., \omega_{t-1}^{t-1}, \omega_{t-1}^t, \omega_t^t).$$

Notice that this truncated vector is not the same as the one used in Balasko and Shell [5, Lemma (3.9) and Proposition (3.10)]. Let $\mathfrak{X}_t = \mathbb{R}^{2lt}_{++}$ be the set of all truncated allocations $x(t) = (x_0^1, x_1^1, x_1^2, ..., x_{t-1}^{t-1}, x_{t-1}^t, x_t^t)$.

4.1. PROPOSITION. *There is an open dense subset $\mathfrak{R}_t$ of $\mathfrak{X}_t$ such that if $\omega(t)$, the truncation of $\omega \in X$, belongs to $\mathfrak{R}_t$, then:*

(a) *There is only a finite ($\neq 0$) number of nonmonetary equilibria $p \in \mathscr{P}$ associated with the endowment sequence ω;*

(b) *There is a neighborhood $\mathfrak{B} \subset \mathfrak{R}_t$ such that for any nonmonetary equilibrium $p \in \mathscr{P}(\omega)$ the prices $p^1, p^2, ..., p^t$ are smooth functions of the endowments $\omega(t) \in \mathfrak{B}$.*

Proof. We decompose the system of equilibrium equations in the following way:

$$\left.\begin{aligned}
&f_0^1(p^1, w_0) + f_1^1(p^1, p^2, w_1) = r^1, \\
&\quad \cdot \quad \cdot \quad \cdot \\
&f_{t-2}^{t-1}(p^{t-2}, p^{t-1}, w_{t-2}) + f_{t-1}^{t-1}(p^{t-1}, p^t, w_{t-1}) = r^{t-1}, \\
&f_{t-1}^{t,1}(p^{t-1}, p^t, w_{t-1}) + a_t^{t,1} \frac{w_t}{p^{t,1}} = r^{t,1}, \\
&\quad \cdot \quad \cdot \quad \cdot \\
&f_{t-1}^{t,l}(p^{t-1}, p^t, w_{t-1}) + a_t^{t,l} \frac{w_t}{p^{t,l}} = r^{t,l}, \\
&w_0 = p \cdot \omega_0,\ w_1 = p \cdot \omega_1, ..., w_{t-1} = p \cdot \omega_{t-1}, \\
&p^1 > 0, ..., p^t > 0,
\end{aligned}\right\} \qquad (4.1.1)$$

$$\left.\begin{array}{l} a_t^{t+1,1}\dfrac{w_t}{p^{t+1,1}} + a_{t+1}^{t+1,1}\dfrac{w_{t+1}}{p^{t+1,1}} = r^{t+1,1}, \\ \cdots \\ \cdots \\ w_t = p \cdot \omega_t,\ w_{t+1} = p \cdot \omega_{t+1},\ldots, \\ p^{t+1} > 0,\ p^{t+2} > 0,\ldots. \end{array}\right\} \quad (4.1.2)$$

In System (4.1.1), there are exactly as many unknowns $(p^1, p^2,..., p^t, w_0,..., w_{t-1}, w_t)$ as there are equations. Assuming that this implies determination of the solutions of (4.1.1), a subject which we shall reconsider in a moment, we can then turn to (4.1.2), where now w_t and p^t are given and where the unknowns are w_{t+1}, $w_{t+2},...$, p^{t+1}, $p^{t+2},...$ In fact, the equation system (4.1.2) is equivalent to a linear equation system which has basically the same structure as the linear equation system considered in Proposition (2.1) and hence its structure can be investigated by the same truncation process as in (2.1.t). Therefore, once w_t and p^t are known, the equalities in (4.1.2) have a unique solution which does or does not satisfy the sign constraints. This unique solution is a linear function of ω_t, $\omega_{t+1},...$ and of p^t and w_t.

Let us return to the counting-equations argument for System (4.1.1). In order to preserve the elementary nature of this paper, we merely sketch the way in which that argument can be employed to establish the generic finiteness of the number of solutions of (4.1.1). This argument parallels the one used in the standard general-equilibrium model by Balasko [2], an article to which the reader is referred for working out the details of the proof. One considers in the space of $(p^1,..., p^t, w_0,..., w_t)$ the subset defined by the equations

$$f_{i-1}^i(p, w_{i-1}) + f_i^i(p, w_i) = r^i \qquad \text{for} \quad i \geqslant 1.$$

Using, for example, the regular value theorem, one shows that this set is a smooth submanifold of the set of prices and incomes. We call it the section manifold. Equilibria correspond to the intersection of the section manifold with the linear manifold defined by the equations

$$w_0 = p \cdot \omega_0,..., w_{t-1} = p \cdot \omega_{t-1}.$$

Transversality of these two manifolds and their dimensions imply discreteness of the intersection set. The boundary behavior of individual demand functions implies that this intersection is compact, hence finite. (In the Appendix to Balasko [3], it is shown that the section manifold is properly embedded with respect to the set of linear manifolds defined by equations of the type $w_0 = p \cdot \omega_0, \ldots, w_{t-1} = p \cdot \omega_{t-1}$. This is exactly what is needed here.) We conclude by an application of Thom's transversality theorem; see [7, Theorem (4.9)].

The second part of Proposition (4.1) is an obvious consequence of the transversality properties. Note that the linear structure of (4.1.2) gives us a complete answer to the question of the local behavior of equilibria as a function of initial endowments. ∎

Let us now study the case of monetary equilibria. Let $\mu \in \mathscr{M}$ be a monetary policy. To ensure existence of proper monetary equilibria, we assume that $\mu \in \mathscr{M}_B(\omega)$, i.e., μ is a bonafide monetary policy. In view of the results of Section 3, where we have shown that monetary equilibria can be parameterized by the initial endowments ω, the monetary policy μ, and the price of money p^m, the next step is to study how the equilibrium set

$$\mathscr{P}(\omega, \mu, p^m) = \{p \mid (p, p^m) \in \mathscr{Q}(\omega, \mu)\} \subset \mathscr{P}$$

varies as (ω, μ, p^m) varies.

Let $\mu(t) = (\mu_0, \mu_1, \ldots, \mu_t)$ denote the t-truncation of t. Fixing $(\mu_{t+1}, \mu_{t+2}, \ldots)$ and $(\omega_t^{t+1}, \omega_{t+1}^{t+1}, \omega_{t+1}^{t+2}, \ldots)$ let $\mathfrak{E}$ be the subset of $\mathbb{R}^l_{++} \times \mathbb{R}^{2l}_{++} \cdots \times \mathbb{R}^{2l}_{++} \times \mathbb{R}^{t+1} \times \mathbb{R}_+$ generated by the $(\omega(t), \mu(t), p^m)$ that satisfy $\mu \in \mathscr{M}_B(\omega)$ and $\mathscr{P}(\omega, \mu, p^m) \neq \varnothing$.

4.2. Proposition. *There is an open dense subset $\mathfrak{E}_t$ of $\mathfrak{E}$ such that for any $(\omega(t), \mu(t), p^m)$ in $\mathfrak{E}_t$, then $\mathscr{P}(\omega, \mu, p^m)$ is a finite set.*

Proof. As in the case of Proposition (4.1), we decompose the equilibrium equations into two systems such that a solution of the t-truncated equation system determines in turn a solution of the second system which is in fact a linear equation system. Therefore, what is at stake once again is the t-truncation equation system. Proposition (4.2) merely states a transversality property of the linear space

$$w_0 = p \cdot \omega_0 + p^m \mu_0, \ldots, w_t = p \cdot \omega_t + p^m \mu_t$$

with the section manifold. This transversality property obviously holds for an open dense set of (ω, μ, p^m). ∎

Note that this proof also shows that when $(\omega(t), \mu(t), p^m)$ varies in $\mathfrak{E}_t$, then every equilibrium $p \in \mathscr{P}(\omega, \mu, p^m)$ can be considered as a smooth function coordinatewise of ω, μ, and p^m.

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