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Existence of Competitive Equilibrium in a General Overlapping-Generations Model*

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I. INTRODUCTION

In the companion paper [2], Balasko and Shell establish existence of competitive equilibrium for a pure-distribution, overlapping-generations economy without money (see, especially, Propositions (3.10) and (3.11) in Section 3). That argument is remarkably simple, but depends critically on some strong hypotheses concerning individual tastes and endowments. In particular, it is assumed that, in any given period, every living consumer has a utility function which is strictly increasing in every available commodity, as well as an endowment which is strictly positive in every available commodity.

These two assumptions are especially difficult to rationalize in an intergenerational context. How many teen-agers do you know who are eager viewers of Lawrence Welk? How many "middle-agers" who are likely users of Clearasil? So much for increasing (much less strictly increasing) utility functions! Regarding strictly positive endowments, we invite the reader to dream up his/her own counterexamples. (Another hint: How do you respond to the commonly heard lament, "What I wouldn't give to be young again!"?)

In the present paper, we do away with such unrealistically severe hypotheses. And we do more. Following the Arrow–Debreu tradition, we postulate fairly mild conditions on individual tastes and endowments. Of course, in line with the same tradition, we must then impose some further conditions on the potential economic interrelationships among consumers. This additional structure is an application to the overlapping-generations model of McKenzie's basic concept of irreducibility.

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II. A General Overlapping-Generations Model

The details of our formal model are closely patterned after those elaborated in [2] (see, especially, Section 2). The most significant differences in assumptions, terminology and notation are the following:

1. In each period t ($t = 1, 2, \ldots$) there is an arbitrary, finite number of completely perishable physical commodities $1 \leqslant l^t < \infty$.

2. Each consumer h ($h = 0, 1, \ldots$) lives only two periods.[1] However, in each period t (including $t = 0$, the period just prior to the inception of economic activity) the newly born group of consumers, or *generation* t, $G^t \subset \{h: h \geqslant 0\}$ consists of an arbitrary, finite number of two-period lived members $1 \leqslant \# G^t < \infty$.

3. Consumption sets are simply the nonnegative orthants

$$x_h = x_h^1 = (x_h^{1,1}, x_h^{1,2}, \ldots, x_h^{1,l^1}) \in \mathbb{R}_+^{l^1} \qquad \text{for} \quad h \in G^0$$

and

$$x_h = (x_h^t, x_h^{t+1}) = (x_h^{t,1}, \ldots, x_h^{t,l^t}, x_h^{t+1,1}, \ldots, x_h^{t+1,l^{t+1}}) \in \mathbb{R}_+^{l^t + l^{t+1}} \qquad \text{for} \quad h \in G^t, \quad t \geqslant 1.$$[2]

However, utility functions $u_h(\cdot)$ are only required to be continuous, quasiconcave and without local maxima, while endowment vectors ω_h are only required to be nonnegative and nontrivial.

4. Allocation sequences are denoted by $x = (x_1, x_2, \ldots, x_h, \ldots)$, and price

[1] Given our other maintained assumptions, this particular assumption is perfectly innocuous. The completely general economy with arbitrary, finite lifetimes is formally indistinguishable from the (apparently) singularly special one with (at most) two-period lifetimes—given suitable reinterpretation of both "periods" and "commodities." Because of its special relevance to our formulation, we present a detailed description of such reinterpretation in the concluding section.

This reinterpretation captures the essential feature of genuine generational overlap. More importantly, the assumption itself permits rather extensive simplification of the natural dynamical analysis associated with an overlapping-generations demographic structure.

[2] The introduction of consumption sets which are only required to be closed, convex and bounded below—and, more broadly, production sets (as part of endowments) which are only required to exhibit standard regularity, convexity and boundedness properties—is conceptually straightforward. Unfortunately, unlike our introduction of arbitrary numbers of coexisting commodities and contemporary consumers, this direction of generalization involves considerable additional notational and analytical complexity. This is reason enough to forego pursuing it. Another, perhaps more compelling justification for generalizing in one but not the other direction is that multiplicity of commodities and consumers is, in a sense, necessary for consistency with our very convenient specialization to two-period lifetimes—as the discussion in the concluding section makes plain.

sequences by $\mathfrak{p} = (p^1, p^2, \ldots, p^t, \ldots)$. Also, the set of feasible allocation sequences is denoted by

$$\mathfrak{X} = \left\{ \mathfrak{x}: \mathfrak{x} \geqslant 0 \text{ and } \sum_{h \in G^{t-1} \cup G^t} x_h^t \leqslant \sum_{h \in G^{t-1} \cup G^t} \omega_h^t \text{ for } t \geqslant 1 \right\},$$

and the set of conceivable price sequences by

$$\mathfrak{P} = \{\mathfrak{p}: \mathfrak{p} > 0\}.^3$$

5. As in [2], $x_h(\mathfrak{p})$ represents the *ordinary* lifetime consumption profiles for consumer h given the price sequence $\mathfrak{p} \in \mathfrak{P}$, that is, the optimal solutions to the budget-constrained utility maximization problem

$$\text{maximize } u_h(x_h) \text{ subject to } \mathfrak{p} \cdot x_h \leqslant \mathfrak{p} \cdot \omega_h \text{ and } x_h \geqslant 0.^4 \tag{1}$$

We will also find it useful to have notation representing the *compensated* lifetime consumption profiles for consumer h given the price sequence $\mathfrak{p} \in \mathfrak{P}$ and the utility level $u_h \in \mathbb{R}$. So let $x_h(\mathfrak{p}, u_h)$ denote the optimal solutions to the utility-constrained budget minimization problem

$$\text{minimize } \mathfrak{p} \cdot x_h \text{ subject to } u_h(x_h) \geqslant u_h \text{ and } x_h \geqslant 0. \tag{2}$$

6. We assume that the economy is *intertemporally irreducible* in the following precise sense (compare with McKenzie [4, pp. 58–59 and p. 247]): Let a *t-generation economy* $\mathscr{E}^t$ be specified by the tastes and endowments of the group of consumers born up through period t, that is, by $u_h(\cdot)$ and ω_h for $h \in H^t = \{h: h \in G^s \text{ for } 0 \leqslant s \leqslant t\}$. Then, we require that there be a subsequence of periods $\{t_v\}$ with the property that, given any t_v-generation economy $\mathscr{E}^{t_v}$ and any feasible allocation sequence $\mathfrak{x} \in \mathfrak{X}$, for every $H_1^{t_v}$, $H_2^{t_v} \subset H^{t_v}$ such that $H_1^{t_v}, H_2^{t_v} \neq \varnothing$, $H_1^{t_v} \cap H_2^{t_v} = \varnothing$ and $H_1^{t_v} \cup H_2^{t_v} = H^{t_v}$, there exist $y_h \geqslant 0$ for $h \in H_1^{t_v}$ and $x'_h \geqslant 0$ for $h \in H_2^{t_v}$ such that

$$\sum_{h \in H_1^{t_v}} y_h^{t,i} = 0 \qquad \text{whenever} \qquad \sum_{h \in H_1^{t_v}} \omega_h^{t,i} = 0$$
$$\text{for} \quad 1 \leqslant i \leqslant l^t, \quad 1 \leqslant t \leqslant t_v + 1,$$

$$\sum_{h \in H_2^{t_v}} x'_h \leqslant \sum_{h \in H^{t_v}} \omega_h + \sum_{h \in H_1^{t_v}} y_h = \sum_{h \in H_1^{t_v}} (\omega_h + y_h) + \sum_{h \in H_2^{t_v}} \omega_h$$

[3] We adopt the convention that, for any pair of vectors x and y, (or, similarly, sequences $\mathfrak{x}$ and $\mathfrak{y}$), "$x \geqslant y$" means $x_i \geqslant y_i$ for every i, "$x > y$" means $x_i \geqslant y_i$ for every i and $x_i > y_i$ for some i, and "$x >> y$" means $x_i > y_i$ for every i.

[4] It is also convenient to follow [2] in letting x_h (or, similarly, ω_h) denote either the vector defined earlier (as in the expression $u_h(x_h)$) or the particular corresponding sequence $x_h = (x_h^1, 0, \ldots, 0, \ldots)$ for $h \in G^0$ and $x_h = (0, \ldots, 0, x_h^t, x_h^{t+1}, 0, \ldots)$ for $h \in G^t$, $t \geqslant 1$ (as in the expression $\mathfrak{p} \cdot x_h \leqslant \mathfrak{p} \cdot \omega_h$).

and

$$u_h(x'_h) \geqslant u_h(x_h) \qquad \text{for every} \quad h \in H_2^{t_\nu}$$

and

$$u_h(x'_h) > u_h(x_h) \qquad \text{for some} \quad h \in H_2^{t_\nu}.$$

In short, no matter how we split up the group of consumers born up through period t_ν, we can always augment just those commodities provided by the first subgroup in sufficient amount so that we can then redistribute the augmented total endowment of the whole group to improve the welfare of the second subgroup.[5]

III. The Existence of Competitive Equilibrium

A. *Statement and Preliminary Discussion*

We take the assumption that commodities are completely perishable to entail that they are also freely disposable. So, in accord with the basic precept that scare goods are necessarily related to positive prices, we define $(\mathfrak{x}, \mathfrak{p}) \in \mathfrak{X} \times \mathfrak{P}$ to be a *competitive equilibrium* if

$$x_h \in x_h(\mathfrak{p}) \qquad \text{for} \quad h \geqslant 0$$

and

$$\sum_{h \in G^{t-1} \cup G^t} x_h^{t,i} = \sum_{h \in G^{t-1} \cup G^t} \omega_h^{t,i} \qquad \text{whenever } p^{t,i} > 0 \text{ for } 1 \leqslant i \leqslant l^t,\ t \geqslant 1.$$

Our central result is the following

Existence Theorem. *Under the hypotheses stated above, there exists a competitive equilibrium.*

[5] It is worth noting that our specific adaptation of McKenzie's ingenious condition is weaker than imposing the same restriction on every t-generation economy; it may simply be impossible to improve the welfare of some member of generation t without recourse to redistributions involving the endowment of some member of generation $t+1$. On the other hand, since such a circumstance can conceivably prevail for some member of all but a finite number of generations, it is equally clear that our condition is stronger than the requirement that the whole economy be irreducible (say, in the same precise sense we have prescribed for each t_ν-generation economy). Whether our existence result can be extended to cover the broader situation remains to be seen.

A suggestive example of possible nonexistence when the economy is not intertemporally irreducible is sketched in Cass [3].

The proof of this Existence Theorem is similar in spirit to that presented in [2], and revolves around the idea of taking limits of suitably truncated competitive equilibria. Specifically, we define $(\mathfrak{x}^t, \mathfrak{p}^t) \in \mathfrak{X} \times \mathfrak{P}$ to be a *t-period equilibrium* if

$$x_h \in x_h(\mathfrak{p}^t) \qquad \text{for} \quad h \in H^t$$

and

$$\sum_{h \in G^{s-1} \cup G^s} x_h^{s,i} = \sum_{h \in G^{s-1} \cup G^s} \omega_h^{s,i} \quad \text{whenever } p^{s,i} > 0 \text{ for } 1 \leqslant i \leqslant l^s,\ 1 \leqslant s \leqslant t.$$

In what follows we shall make frequent use of the obvious fact (and related observations) that if $t'' > t'$, then a t''-period equilibrium is also a t'-period equilibrium. Referring to the subsequence of periods $\{t_\nu\}$ singled out earlier, we shall demonstrate that (i) for every $\nu \geqslant 1$ there exists a t_ν-period equilibrium $(\mathfrak{x}^{t_\nu}, \mathfrak{p}^{t_\nu})$, and (ii) sequences of t_ν-period equilibria $\{(\mathfrak{x}^{t_\nu}, \mathfrak{p}^{t_\nu})\}$ essentially converge to some competitive equilibrium $(\mathfrak{x}^*, \mathfrak{p}^*)$ as ν approaches ∞.

The hard part of this argument involves establishing uniform bounds on the elements of the price sequence $\mathfrak{p}^{t_\nu}$. (The definition of $\mathfrak{X}$ already imposes uniform bounds on the elements of the feasible allocation sequence $\mathfrak{x}^{t_\nu}$.) The argument itself is somewhat easier to comprehend if we take it in gradual steps, considering first the polar special case where the generations $\{t_\nu\}$ always provide every commodity available in their second period of life,

$$\sum_{h \in G^{t_\nu}} \omega_h^{t_\nu+1} \gg 0 \qquad \text{for} \quad \nu \geqslant 1, \tag{3}$$

and then second the general case where they may not,

$$\sum_{h \in G^{t_\nu}} \omega_h^{t_\nu+1} > 0 \qquad \text{for} \quad \nu \geqslant 1.^6 \tag{4}$$

The underlying reinterpretation discussed in the concluding section provides strong motivation for being especially interested in the general case (4). However, since the analysis is substantially less complicated (in detail, not in concept) for the special case (3), we will concentrate on it, and content ourselves with merely outlining the extension of the proof to cover the general case (4).

B. *Proof for the Special Case* (3)

Given arbitrary $\nu \geqslant 1$, it is easily verified that the corresponding t_ν-generation economy $\mathscr{E}^{t_\nu}$ exhibits all of the standard properties utilized in

[6] Here and later on we appeal to the readily verified result that, for the economy to be intertemporally irreducible, every generation must provide some commodity available in each of its periods of life, $\sum_{h \in G^0} \omega_h^1 > 0$ and $\sum_{h \in G^t} \omega_h^s > 0$ for $s = t, t+1, t \geqslant 1$.

establishing the existence of competitive equilibrium. In particular, our assumption that the economy is intertemporally irreducible guarantees that, for such a truncated economy, a compensated equilibrium is also a competitive equilibrium, the crucial part in all of the standard proofs (see, for instance, Arrow and Hahn [1], especially Chapter 5). Because this fundamental result also plays an essential role in our own subsequent analysis, we sketch a simple variant of McKenzie's original argument (drafted for our specific purposes) in subsection D below.

Now, competitive equilibrium in the t_ν-generation economy necessarily yields t_ν-period equilibrium (by simply choosing arbitrary elements to fill out the feasible allocation sequence $\mathfrak{x}^{t_\nu}$ and related price sequence $\mathfrak{p}^{t_\nu}$). Thus, we know that, for every $\nu \geqslant 1$, there exists a t_ν-period equilibrium $(\mathfrak{x}^{t_\nu}, \mathfrak{p}^{t_\nu})$. But then, applying Cantor's diagonalization procedure, we also know that there exists a subsequence of periods $\{t_\xi\} \subset \{t_\nu\}$ with corresponding t_ξ-period equilibria $\{(\mathfrak{x}^{t_\xi}, \mathfrak{p}^{t_\xi})\}$ such that

$$\lim_{\xi \to \infty} \mathfrak{x}^{t_\xi} = \mathfrak{x}^* \in \mathfrak{X}.$$

Focus attention on this particular feasible allocation sequence $\mathfrak{x}^*$. *We claim that there exists* $\mathfrak{p}^* \in \mathfrak{P}$ *such that* $(\mathfrak{x}^*, \mathfrak{p}^*)$ *is a competitive equilibrium.*

This last assertion is established on the basis of the following pair of lemmas, whose role is to warrant applying Cantor's diagonalization procedure to derive the desired price sequence $\mathfrak{p}^*$ as well.

LEMMA 1. *For every* $\nu \geqslant 1$ *there exists* $\mathfrak{p}^{t_\nu *} \in \mathfrak{P}$ *such that* $(\mathfrak{x}^*, \mathfrak{p}^{t_\nu *})$ *is a* t_ν*-period equilibrium.*

Proof of Lemma 1. Given arbitrary $\nu \geqslant 1$, consider a subsequence of periods $\{t_\xi\} \subset \{t_\nu\}$ with corresponding t_ξ-period equilibria $\{(\mathfrak{x}^{t_\xi}, \mathfrak{p}^{t_\xi})\}$ such that $\lim_{\xi \to \infty} \mathfrak{x}^{t_\xi} = \mathfrak{x}^*$. Because every t_ξ-period equilibrium has the property that $x_h \in x_h(\mathfrak{p}^{t_\xi})$ for $h \in G^0$ and thus $p^1 > 0$, it follows that we can normalize $\mathfrak{p}^{t_\xi}$ so that

$$\|(p^1, p^2, \ldots, p^{t_\nu + 1})\| = 1.$$

Hence, without any loss of generality we can assume that the first

$$\sum_{t=1}^{t_\nu + 1} l^t$$

elements of $\mathfrak{p}^{t_\xi}$ converge to $(p^{1*}, p^{2*}, \ldots, p^{t_\nu + 1 *}) > 0$ as ξ approaches ∞. Finally, we can therefore employ a standard maneuver—which basically just involves switching back and forth between the budget-constrained utility maximization problem (1) and its utility-constrained budget minimization counterpart (2) (where now $u_h = u_h(x_h)$ for $x_h \in x_h(\mathrm{p}^{t_\ell})$)—in order to verify that every price sequence

$$\mathrm{p}^{t_\nu *} = (p^{1*}, p^{2*}, \ldots, p^{t_\nu+1*}, p^{t_\nu+2}, \ldots) \in \mathfrak{P}$$

yields suitable equilibrium prices:[7] We know that, for every $t_\ell \geqslant t_\nu$, the t_ℓ-period equilibrium $(\mathrm{x}^{t_\ell}, \mathrm{p}^{t_\ell})$ is also a t_ν-period equilibrium, and that in a t_ν-period equilibrium, for every $h \in G^t$, $0 \leqslant t \leqslant t_\nu$, ordinary lifetime consumption profiles are also compensated lifetime consumption profiles. Thus, introducing obvious terminology to emphasize the distinction, the t_ℓ-period equilibrium $(\mathrm{x}^{t_\ell}, \mathrm{p}^{t_\ell})$ is also a t_ν-period *compensated* equilibrium. Moreover, we know that compensated lifetime consumption profiles are upper hemi-continuous. Thus, going to the limit, $(\mathrm{x}^*, \mathrm{p}^{t_\nu *})$ is also a t_ν-period compensated equilibrium. So, now invoking the assumption that the economy is intertemporally irreducible, it is easily established that, at the price sequence $\mathrm{p}^{t_\nu *}$, every consumer born up through period t_ν must have positive income

$$\mathrm{p}^{t_\nu *} \cdot \omega_h > 0 \qquad \text{for} \quad h \in H^{t_\nu},$$

and hence that, by a well-known result, $(\mathrm{x}^*, \mathrm{p}^{t_\nu *})$ is a t_ν-period equilibrium as well. ∎

LEMMA 2. *There exist nonnegative, finite constants $0 \leqslant K^t < \infty$ for $t \geqslant 1$ with the property that, if $(\mathrm{x}^*, \mathrm{p}^{t_\nu})$ is a t_ν-period equilibrium, then p^{t_ν} can be normalized so that*

$$\| p^1 \| = 1$$

and

$$\| p^{t+1} \| \leqslant K^t \qquad \textit{for} \quad 1 \leqslant t \leqslant t_\nu.$$

Proof of Lemma 2. Given arbitrary $\nu \geqslant 1$, consider a t_ν-period equilibrium of the specific form $(\mathrm{x}^*, \mathrm{p}^{t_\nu})$. Since $x_h^* \in x_h(\mathrm{p}^{t_\nu})$ for $h \in G^0$, we know that $p^1 > 0$, or that $\| p^1 \| > 0$. Hence, we only need to establish that there exist bounds K^t such that, for every $1 \leqslant t \leqslant t_\nu$, the price sequence p^{t_ν} satisfies

$$\| p^{t+1} \| / \| p^1 \| \leqslant K^t.$$

Suppose that this were not true, that is, that for some $1 \leqslant t \leqslant t_\nu$, there were a

[7] As mentioned earlier, some details of this argument are more fully sketched in subsection D below.

sequence of t_ν-period equilibria $\{(\mathfrak{x}^*, \mathfrak{p}^\xi)\}$ ("$\mathfrak{p}^{t_\nu \xi}$" or "$(\mathfrak{p}^{t_\nu})^\xi$" would be more accurate, but also more cumbersome notation) such that

$$\lim_{\xi \to \infty} \| p^{t+1\xi} \| / \| p^{1\xi} \| = \infty.$$

Since $p^{1\xi} > 0$, we can again normalize $\mathfrak{p}^\xi$ so that $\|(p^{1\xi}, p^{2\xi}, \ldots, p^{t_\nu+1\xi})\| = 1$, and, without loss of generality, assume that the first

$$\sum_{t=1}^{t_\nu+1} l^t$$

elements of $\mathfrak{p}^\xi$ converge to $(p^{1\infty}, p^{2\infty}, \ldots, p^{t_\nu+1\infty}) > 0$ as ξ approaches ∞. But this leads to the following inconsistency: On the one hand, by basically the same reasoning as in the proof of Lemma 1, we find that every price sequence

$$\mathfrak{p}^\infty = (p^{1\infty}, p^{2\infty}, \ldots, p^{t_\nu+1\infty}, p^{t_\nu+2}, \ldots) \in \mathfrak{P}$$

yields a t_ν-period equilibrium $(\mathfrak{x}^*, \mathfrak{p}^\infty)$, so that $p^{1\infty} > 0$. On the other hand, given the particular character of the choice of $\{\mathfrak{p}^\xi\}$, we also find that for every $0 < K < \infty$ there exists $\xi_K < \infty$ such that

$$\| p^{1\xi} \| > 0, \| p^{t+1\xi} \| \leqslant 1 \qquad \text{and} \qquad \| p^{t+1\xi} \| / \| p^{1\xi} \| > K$$

or

$$1 > K \| p^{1\xi} \| > 0 \qquad \text{for} \quad \xi > \xi_K,$$

so that $p^{1\infty} = 0$. Thus, we can conclude that the requisite bounds must in fact exist. ∎

The remainder of the argument (which depends on the provisional assumption (3) only insofar as it was used in establishing Lemmas 1 and 2) is now almost routine. Lemma 1 guarantees that, for every $\nu \geqslant 1$, there exists a t_ν-period equilibrium of the specific form $(\mathfrak{x}^*, \mathfrak{p}^{t_\nu})$, while Lemma 2 justifies further restriction of the price sequence $\mathfrak{p}^{t_\nu}$ to the subset of conceivable price sequences

$$\mathfrak{P}^* = \{\mathfrak{p} \colon \mathfrak{p} \geqslant 0, \| p^1 \| = 1 \text{ and } \| p^{t+1} \| \leqslant K^t \text{ for } t \geqslant 1\} \subset \mathfrak{P}.$$

Hence, once again applying Cantor's diagonalization procedure, we can choose a subsequence of periods $\{t_\xi\} \subset \{t_\nu\}$ with corresponding t_ξ-period equilibria $\{(\mathfrak{x}^*, \mathfrak{p}^{t_\xi})\}$ such that

$$\lim_{\xi \to \infty} \mathfrak{p}^{t_\xi} = \mathfrak{p}^* \subset \mathfrak{P}^*.$$

Finally, employing the same standard maneuver as in the proofs of

Lemmas 1 and 2, it follows that, for every $\nu \geqslant 1$, (x^*, p^*) is a t_ν-period equilibrium, and, therefore that, by definition, (x^*, p^*) is also a competitive equilibrium. The proof of the Existence Theorem for the special case (3) is therefore complete.

C. *Extension to the General Case* (4)

In order to generalize the proof, we begin by observing that it is only necessary to consider the other polar special case where the generations $\{t_\nu\}$ *never* provide every commodity available in their second period of life,

$$\sum_{h \in G^{t_\nu}} \omega_h^{t_\nu+1} \not\gg 0 \qquad \text{for} \quad \nu \geqslant 1 \tag{5}$$

(since we can always select the original subsequence of periods $\{t_\nu\}$ so that either (3) or (5) obtains). Then, given the provisional assumption (5), the foregoing analysis requires two significant modifications.

1. *Establishing Existence of t_ν-Period Equilibrium*

Underlying our whole approach is the fundamental result that, for every $\nu \geqslant 1$, there exists a t_ν-period equilibrium (x^{t_ν}, p^{t_ν}). In the proof of this result the only potential problem occurs in attempting to establish the existence of competitive equilibrium for the t_ν-generation economy $\mathscr{E}^{t_\nu}$ (or for any similar truncation of the whole economy). Among other things, generation t_ν's tastes may depend crucially on some commodities which are only provided in generation $(t_\nu + 1)$'s endowments. In order to circumvent this sort of possibility, we essentially introduce a proxy to represent the critical connections between generations t_ν and $t_\nu + 1$.

We noted previously that, for the economy to be intertemporally irreducible, both generations must provide some commodity available during their common period of life,

$$\sum_{h \in G^t} \omega_h^{t_\nu+1} > 0 \qquad \text{for} \quad t = t_\nu, t_\nu + 1.$$

Based on this commonality, we fabricate a consumer ν who has tastes for just the commodities available in period $t_\nu + 1$—tastes which are represented by a continuous, quasiconcave and strictly increasing utility function, $u_\nu\colon \mathbb{R}_+^{l^{t_\nu+1}} \to \mathbb{R}$—and an endowment of just the commodities available in period $t_\nu + 1$—an endowment which is exactly the same as the total endowment of generation $t_\nu + 1$ during its first period of life,

$$\begin{aligned} \omega_\nu^t &= \sum_{h \in G^{t_\nu+1}} \omega_h^{t_\nu+1} > 0 \qquad && \text{for} \quad t = t_\nu + 1 \\ &= 0 && \text{otherwise.} \end{aligned}$$

Consider competitive equilibrium in the t_ν-generation economy after the addition of this fictitious consumer ν, say, the *augmented* t_ν-generation economy $\mathscr{E}^\nu$.

For the purpose of establishing existence, the sole condition which needs to be explicitly verified is that $\mathscr{E}^\nu$ also satisfies the restriction imposed by the assumption that the economy is intertemporally irreducible. But such verification follows directly from two observations. First, the feasible allocations for $\mathscr{E}^\nu$ are basically the same as the feasible allocation sequences up through period $t_\nu + 1$ for the whole economy—provided that we identify consumer ν's consumption with generation $(t_\nu + 1)$'s total consumption during its first period of life,

$$x_\nu^{t_\nu+1} = \sum_{h\in G^{t_\nu+1}} x_h^{t_\nu+1}.$$

Second, now letting $H^\nu = H^{t_\nu} \cup \{\nu\}$, the restriction itself only requires checking in the extreme cases where either $H_1^\nu = \{\nu\}$ or $H_2^\nu = \{\nu\}$—since in every other intermediate case, if it holds for the t_ν-generation economy $\mathscr{E}^{t_\nu}$, then it holds for the augmented t_ν-generation economy $\mathscr{E}^\nu$. And it is readily seen that the restriction holds in these extreme cases:

Case (i). $H_1^\nu = \{\nu\}$. In this case $G^{t_\nu} \subset H_2^\nu$. But because the only direct connection between generation t_ν and its successors is through generation $t_\nu + 1$, if the restriction doesn't hold for $\mathscr{E}^\nu$ with this particular partition, then it doesn't hold for $\mathscr{E}^{t_\nu+1}$ with the particular partition

$$H_1^{t_\nu+1} = \{h\colon h \in G^t \text{ for } t_\nu + 1 \leqslant t \leqslant t_{\nu+1}\}.$$

Case (ii). $H_2^\nu = \{\nu\}$. In this case $G^{t_\nu} \subset H_1^\nu$. Hence, fictitious consumer ν can be made better off by receiving the increases in total endowment of any amount such that

$$\sum_{h\in G^{t_\nu}} y_h^{t_\nu+1,i} > 0 \qquad \text{whenever } \sum_{h\in G^{t_\nu}} \omega_h^{t_\nu+1,i} > 0 \text{ for } 1 \leqslant i \leqslant l^{t_\nu+1}.$$

2. *Converting t_ν-Period Compensated Equilibrium into t_ν-Period Equilibrium*

In demonstrating both Lemmas 1 and 2, the only possible difficulty occurs in attempting to convert a t_ν-period compensated equilibrium into a t_ν-period equilibrium (along the lines more fully elaborated in the following subsection). For instance, in the last steps of the argument in the proof of Lemma 1, it may happen that, after going to the limit,

$$p^{t,i*} = 0 \qquad \text{whenever } \sum_{h\in H^{t_\nu}} \omega_h^{t,i} > 0 \text{ for } 1 \leqslant i \leqslant l^t,\ 1 \leqslant t \leqslant t_\nu + 1,$$

or that

$$\mathrm{p}^{t_\nu *} \cdot \omega_h = 0 \qquad \text{for} \quad h \in H^{t_\nu}.$$

To deal with this eventuality, we essentially just avoid those commodities which are neither provided nor consumed (in the particular feasible allocation sequence $\mathfrak{x}^*$) by generation t_ν during its second period of life. In particular, we reorder the commodities available in period $t_\nu + 1$ so that

$$\begin{aligned} \sum_{h \in G^{t_\nu}} (\omega_h^{t_\nu+1,i} + x_h^{t_\nu+1,i*}) &> 0 \qquad \text{for} \quad 1 \leqslant i \leqslant l^{t_\nu+1*} \leqslant l^{t_\nu+1} \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

renormalize p^{t_ι} so that

$$\|(p^{1}, p^2, \ldots, p^{t_\nu+1,1}, \ldots, p^{t_\nu+1, l^{t_\nu+1*}})\| = 1,$$

and repeat the same standard maneuver as before—except that now we find that, after going to the limit, $(\mathfrak{x}^*, \mathrm{p}^{t_\nu *})$ only yields a t_ν-period compensated equilibrium when we explicitly exclude the commodities presently labelled $t_\nu + 1, i$ for $l^{t_\nu+1*} < i \leqslant l^{t_\nu+1}$.[8] But this is enough to yield a similarly delimited t_ν-period equilibrium and hence, provided $\nu > 1$, a $t_{\nu-1}$-period equilibrium.

For Lemma 2, the same amendment to the analysis is enough to yield uniform bounds on the price sequence p^{t_ν} up through period t_ν, that is, provided $\nu > 1$, suitable restriction of a $t_{\nu-1}$-period equilibrium which is also a t_ν-period equilibrium. But that is more than sufficient for the remainder of the argument.

D. *Maneuvering Between Ordinary and Compensated Equilibria*

A substantial part of our analysis has implicitly employed the following basic notion: Define $(\mathfrak{x}^t, \mathrm{p}^t) \in \mathfrak{X} \times \mathfrak{P}$ to be a *t-period compensated equilibrium* if there exist $u_h \in \mathbb{R}$ for $h \in H^t$ such that

$$x_h \in x_h(\mathrm{p}^t, u_h) \qquad \text{and} \qquad \mathrm{p}^t \cdot x_h = \mathrm{p}^t \cdot \omega_h \qquad \text{for} \quad h \in H^t$$

and

$$\sum_{h \in G^{t-1} \cup G^t} x_h^{t,i} = \sum_{h \in G^{t-1} \cup G^t} \omega_h^{t,i} \qquad \text{whenever } p^{t,i} > 0 \text{ for } 1 \leqslant i \leqslant l^s,\ 1 \leqslant s \leqslant t.$$

[8] That is, when we explicitly require that $x_h^{t_\nu+1,i} = 0$ for $l^{t_\nu+1*} < i \leqslant l^{t_\nu+1}$. It is perhaps worth remarking that we cannot resort to an analogous trick in the preceding extension of the proof (establishing existence of t_ν-period equilibrium), basically because it runs afoul of Walras' law.

Typically, we have focused on the specific situation where such a t-period compensated equilibrium is generated from a predetermined t-period equilibrium $(\mathfrak{x}^t, \mathfrak{p}^t)$ (so that, for instance, $u_h = u_h(x_h)$ for $x_h \in x_h(\mathfrak{p}^t)$). Under such circumstances, the various results we have appealed to are, on the whole, so well known that they require no detailed elaboration or documentation (as can be found, say, in Arrow and Hahn's treatise). One claim, however, perhaps merits further justification. This is our assertion (toward the end of the proof of Lemma 1) that, when the economy is intertemporally irreducible, "it is easily established that ... $\mathfrak{p}^{t_\nu *} \cdot \omega_h > 0$ for $h \in H^{t_\nu}$."

The proof is essentially the same as McKenzie's. Given the limiting t_ν-period compensated equilibrium $(\mathfrak{x}^*, \mathfrak{p}^{t_\nu *})$, consider our specific condition applied to the t_ν-generation economy $\mathscr{E}^{t_\nu}$ vis-a-vis the particular feasible allocation sequence $\mathfrak{x}^*$. Choose

$$H_2^{t_\nu} = \{h\colon h \in H^{t_\nu} \text{ and } \mathfrak{p}^{t_\nu *} \cdot \omega_h > 0\}.$$

Since, by derivation, $(p^{1*}, p^{2*},..., p^{t_\nu+1*}) > 0$ and, by assumption (especially the provisional assumption (3)),

$$\sum_{h \in H^{t_\nu}} \omega_h^t \gg 0 \qquad \text{for} \quad 1 \leqslant t \leqslant t_\nu + 1,$$

we know that $\mathfrak{p}^{t_\nu *} \cdot \omega_h > 0$ for some $h \in H^{t_\nu}$, or that $H_2^{t_\nu} \neq \varnothing$. Now suppose that $H_2^{t_\nu} \neq H^{t_\nu}$, or that $H_1^{t_\nu} = H^{t_\nu} - H_2^{t_\nu} \neq \varnothing$. We will show that this entails the contradiction

$$\mathfrak{p}^{t_\nu *} \cdot \omega_h > 0 \qquad \text{for some} \quad h \in H_1^{t_\nu}. \tag{6}$$

Assume the postulated potential for redistribution denoted by $y_h \geqslant 0$ for $h \in H_1^{t_\nu}$ and $x'_h \geqslant 0$ for $h \in H_2^{t_\nu}$. By virtue of our particular choice of $H_2^{t_\nu}$ (as well as our particular derivation of $(\mathfrak{x}^*, \mathfrak{p}^{t_\nu *})$) we know that

$$\mathfrak{p}^{t_\nu *} \cdot x'_h \left\{ \begin{matrix} > \\ \geqslant \end{matrix} \right\} \mathfrak{p}^{t_\nu *} \cdot \omega_h \ \text{ according as } \ u_h(x'_h) \left\{ \begin{matrix} > \\ = \end{matrix} \right\} u_h(x_h^*) \ \text{ for } h \in H_2^{t_\nu}.$$

Thus, on the one hand, the property that

$$u_h(x'_h) \geqslant u_h(x_h^*) \qquad \text{for every} \quad h \in H_2^{t_\nu}$$

and

$$u_h(x'_h) > u_h(x_h^*) \qquad \text{for some} \quad h \in H_2^{t_\nu}$$

yields the inequality

$$p^{t_v*} \cdot \sum_{h \in H_2^{t_v}} (x'_h - \omega_h) > 0. \tag{7}$$

On the other hand, the property that

$$\sum_{h \in H_2^{t_v}} x'_h \leqslant \sum_{h \in H_1^{t_v}} (\omega_h + y_h) + \sum_{h \in H_2^{t_v}} \omega_h$$

yields the inequality

$$p^{t_v*} \cdot \sum_{h \in H_2^{t_v}} (x'_h - \omega_h) \leqslant p^{t_v*} \cdot \sum_{h \in H_1^{t_v}} (\omega_h + y_h). \tag{8}$$

But (7) and (8) together entail the inequality

$$p^{t_v*} \cdot \sum_{h \in H_1^{t_v}} (\omega_h + y_h) > 0,$$

which in turn entails the dichotomy that either (i) (6) obtains or (ii) (6) does not obtain but $p^{t_v*} \cdot y_h > 0$ for some $h \in H_1^{t_v}$. In the first case we have already finished. In the second case we simply appeal to the remaining property of the postulated potential for redistribution, that

$$\sum_{h \in H_1^{t_v}} y_h^{t,i} = 0 \qquad \text{whenever} \qquad \sum_{h \in H_1^{t_v}} \omega_h^{t,i} = 0$$
$$\text{for} \quad 1 \leqslant i \leqslant l', \quad 1 \leqslant t \leqslant t_v + 1.$$

It follows directly that this second case is impossible, that is, that the alleged contradiction (6) must in fact obtain.

IV. The Inherent Demographic Structure of Overlapping-Generations

Our purpose here is to outline an algorithm for converting a pure-distribution, overlapping-generations economy in which consumers have arbitrary but finite lifetimes into one in which they have at most two-period lifetimes. The essence of this algorithm consists in suitably redefining indices.

So now suppose that the time superscripts ℓ ($\ell = \ldots 0, 1, \ldots$) denote some conventional periods for describing economic behavior, say, for specificity, calendar dates. As before, market activity commences on calendar date 1. Continue to let the agent subscripts h ($h = 0, 1, \ldots$) denote various individual consumers ordered in terms of their first periods of life, or their *birthdates* $-\infty < b_h < \infty$. This means, in particular, that if $h'' > h'$, then $b_{h''} \geqslant b_{h'}$. Now, however, since we no longer assume that consumers live just two periods, we also need to keep track of their last as well as their first periods of life, or their *deathdates* $b_h \leqslant d_h < \infty$. Moreover, because we are only concerned with market activity, we can presume that $1 \leqslant d_h$ for $h \geqslant 0$. Likewise, because we are only concerned with genuine overlap, we can presume that, for every $\ell \geqslant 0$, there is some $h \geqslant 0$ such that $b_h = \ell < d_h$. (There may also be some $h \geqslant 0$ such that $b_h < 0 < d_h$.)

All other notation retains its previous meaning. Thus, in particular, $H^\ell = \{h\colon h \geqslant 0 \text{ and } b_h \leqslant \ell\}$ denotes the group of consumers born up through period ℓ, $1 \leqslant l^\ell < \infty$ denotes the number of commodities available in period ℓ, and $x_h^\ell \in \mathbb{R}_+^{l^\ell}$ denotes the quantity of commodities allocated to consumer h during period ℓ. Finally, note that the preceding specification entails that

$$\sup_{h \in H^\ell} d_h = \max_{h \in H^\ell} d_h > \ell \qquad \text{for} \quad \ell \geqslant 0,$$

a result which is central to our procedure for converting from the present conventional periods ℓ to the previous, say, *telescoped* periods t.

The basic idea behind this procedure is quite intuitive. At calendar date 0, the consumer with greatest future longevity must be at the end of the first half of his life. But this also means that his deathdate must occur just at the end of the second half of his life. Thus, we identify $t = 0$ with $\ell \leqslant 0 = \ell_0$ and $t = 1$ with $\ell_0 < \ell \leqslant \max_{h \in H^{\ell_0}} d_h = \ell_1$. Continuing in the same fashion, at calendar date ℓ_1, the consumer with the greatest future longevity—who will have been born during the interim $\ell_0 < \ell \leqslant \ell_1$—must be at the end of the first half of his life. So once again his deathdate must occur at the end of the second half of his life. Thus, we identify $t = 2$ with $\ell_1 < \ell \leqslant \max_{h \in H^{\ell_1}} d_h = \ell_2$.

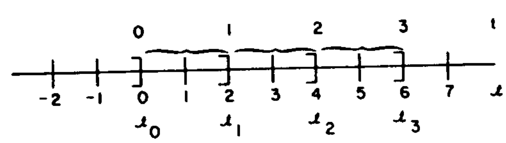

FIGURE 1

And so on. Generally, then, we identify the telescoped periods t with the interims between terms $\ell_{t-1} < \ell \leqslant \ell_t$ of the particular subsequence of conventional periods

$$\ell_0 = 0 \qquad \text{and} \qquad \ell_t = \max_{h \in H^{\ell_{t-1}}} d_h \qquad \text{for} \quad t \geqslant 1.$$

Figure 1 illustrates this correspondence for the leading special case where at least one consumer is born on each calendar date, and every consumer lives just three (conventional) periods.

All that remains is to account for generation composition and commodity classification in terms of the telescoped periods. But this amounts to a trivial exercise: Simply amalgamate indices over the corresponding interims of conventional periods. Thus, in particular, define

$$G^0 = \{h: h \geqslant 0 \text{ and } b_h \leqslant \ell_0\} \qquad \text{and}$$
$$G^t = \{h: h \geqslant 0 \text{ and } \ell_{t-1} < b_h \leqslant \ell_t\} \qquad \text{for} \quad t \geqslant 1,$$

$$l^t = \sum_{\ell_{t-1} < \ell \leqslant \ell_t} l^\ell \qquad \text{for} \quad t \geqslant 1$$

and

$$x_h^t = (x_h^{\ell_{t-1}+1}, x_h^{\ell_{t-1}+2}, \ldots, x_h^{\ell_t}) \in \mathop{\times}_{\ell_{t-1} < \ell \leqslant \ell_t} \mathbb{R}_+^{l^\ell} = \mathbb{R}_+^{l^t} \qquad \text{for} \quad t \geqslant 1.$$

It is obvious from these definitions that generation t contains just those consumers born during period t, and that, even more importantly, members of generation t have tastes for and endowments of just the commodities which are available in periods t and $t + 1$. In fact, typically— that is, when conventional lifetimes are lengthy or diverse—consumers will be directly concerned with a rather small subset of the potential goods available over their telescoped lifetimes. This observation, of course, provided an additional impetus—besides the economic motivation emphasized in the introductory section—for our undertaking this project to begin with.

References

1. K. J. Arrow and F. H. Hahn, "General Competitive Analysis," Holden-Day, San Francisco, 1971.

2. Y. BALASKO AND K. SHELL, The overlapping-generations model, I: The case of pure exchange without money, *J. Econ. Theory* **23** (1980), 281–306.
3. D. CASS, Money in consumption-loan type models: An addendum, *in* "Models of Monetary Economies" (J. Kareken and N. Wallace, Eds.), Federal Reserve Bank of Minneapolis, 1979.
4. L. W. MCKENZIE, On the existence of general equilibrium for a competitive market, *Econometrica* **27** (1959), 54–71 and Some corrections, *Econometrica* **29** (1961), 247–248.

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