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Price Level Volatility: A Simple Model of Money Taxes and Sunspots*

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We investigate sunspot equilibria in a static, one-commodity model with taxes and transfers denominated in money units. Volatility in this economy is purely monetary, since the only uncertainty is about the price level. We construct simple, robust examples of sunspot equilibria that are *not* mere randomizations over certainty equilibria. We also identify the source of these SSEs: Equilibrium in the securities market is determined *as if* there were no restricted consumers and the unrestricted consumers face *intrinsic* uncertainty. Perfect securities markets eliminate allocation uncertainty, but they exacerbate price-level volatility. *Journal of Economic Literature* Classification Numbers: D50, D52, D84, E31, E62.

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1. INTRODUCTION

We analyze sunspot equilibria (SSE) in a simple, one-period, general-equilibrium model with one commodity and a single fiat money. The fiat money is created by the government through its fiscal policy, a system of lump-sum taxes and transfers denominated in money units. Uncertainty is purely extrinsic; i.e., the fundamentals of the economy are nonstochastic.

We do three things in the present paper: (1) We make the logical first step toward integrating the theory of sunspot equilibrium[1] and the theory of money taxes and transfers.[2] (2) We present very simple, numerical examples of SSE that are not mere randomizations over the corresponding certainty equilibria (CE).[3,4] The examples are generated with the help of a new and powerful tool: the *tax-adjusted Edgeworth box*. (3) We observe an important difference between economies with government money and economies in which all securities are private.[5] Perfect securities markets eliminate volatility in the economy without any government securities.[6] In the economy with government money, perfect securities markets do eliminate allocation volatility but they can also exacerbate price-level volatility.

We introduce the certainty economy in Section 2 and the corresponding sunspots economy in Section 3. The simplicity of our model allows us to identify the basic source and nature of sunspot equilibrium. We do this in Section 4: The seed of the stochastic allocation is in the set of restricted consumers, i.e., those who are unable to hedge against the effects of price-level uncertainty. Their allocations are uncertain and this typically causes the *aggregate* allocation to the unrestricted consumers to be uncertain. The

[1] See Shell [18] and especially Cass and Shell [8]. See also Shell [19] and Shell and Smith [21].

[2] See Balasko and Shell [5] and especially Balasko and Shell [6]. See also Balasko and Shell [4] and Shell and Smith [20].

[3] Hence this paper provides a *partial* substitute to the necessarily complicated appendix to Cass and Shell [8].

[4] Goenka [12, Example 4.1] provides an example which is based on a convex, competitive economy with rationing and Shell and Wright [22] provide examples, based on the indivisible good economy, of SSE that are not randomizations over CE. For an imperfectly competitive economy, Peck and Shell [17] provide examples of SSE that are not mere randomizations over pure-strategy Nash equilibria.

[5] Since our present model is a static one, in every proper monetary equilibrium, aggregate government money must be zero. In this sense, aggregate *outside money* will be zero. The present model is, however, easily extended to a multiperiod setting in which aggregate outside money can be nonzero in each period. See Balasko and Shell [5], the references therein, and Ghiglino and Shell [11]. Wallace [24 and elsewhere] says that outside money is positive only if the government's net liability is positive. This interpretation of "outside money" requires the (infinite) overlapping-generations model; see, e.g., Balasko and Shell [4].

[6] See Cass and Shell [8, Propositions 2 and 3].

unrestricted consumers seek to hedge against the effects of price-level volatility. The equilibrium allocations of the unrestricted consumers and the state-contingent price ratio are determined in the tax-adjusted Edgeworth box. We show that the determination of equilibrium in this economy reduces to the determination of equilibrium in a smaller economy with no restrictions on market participation but in which uncertainty does affect the economic fundamentals—i.e., in which uncertainty is intrinsic. In the smaller economy, there is typically economy-wide risk. Hence allocations are usually stochastic and income is typically transferred across states of nature. The transfer of income across states of nature makes the SSE allocations different from a randomization across CE allocations.

We make some concluding remarks in Section 5. In the appendix, we establish the equivalence of contingent-commodity hedging and securities (or, contingent-money) hedging.

2. THE CERTAINTY ECONOMY

We begin with the underlying certainty economy, a simple pure-exchange, monetary economy without sunspots.[7] Consumer h has an endowment of the single commodity, $\omega_h > 0$. He must pay τ_h units of money in taxes.[8] The scalar τ_h can be positive, zero, or negative; if it is negative, the absolute value of τ_h is his money transfer. His consumption set X_h is the set of positive scalars, i.e., we have $X_h = \mathbb{R}_{++}$. The set of consumers H is finite. Consumer h chooses x_h to

$$\text{maximize } u_h(x_h)$$

subject to (2.1)

$$p x_h = p\omega_h - p^m \tau_h \qquad \text{and} \qquad x_h > 0,$$

where u_h is a strictly increasing utility function, $p > 0$ is the price of the commodity, and p^m is the price of money. The price of money must be nonnegative, i.e., we have

$$p^m \geqslant 0.$$

[7] See Balasko and Shell [6]. In the present paper, there is only one commodity, i.e., we analyze the special case of $l = 1$.

[8] One might question the realism of *nominal* taxation. Are not, for example, income taxes typically "real" taxes, i.e. automatically adjusted for price level changes? While it might be argued that 1996 income taxes are partially adjusted for the 1996 price level, the amount of 1996 tax collected in 1997 is unadjusted for the 1997 price level. Hence some parts of actual tax bills are not adjusted for inflation.

Let P^m be the *commodity* price of money. Since there is only one commodity, we can also define the general price level, P, to be the money price of the commodity.[9] Hence we have

$$P^m = p^m/p, \qquad P = p/p^m = 1/P^m,$$

and

$$0 \leqslant P^m < +\infty \qquad \text{and thus} \quad 0 < P \leqslant +\infty.$$

It will be advantageous to associate allocations in this money-taxation economy with counterparts in a no-taxation economy. To do this we define $\tilde{\omega}_h$, the *tax-adjusted endowment* of Mr. h, by

$$\tilde{\omega}_h = \omega_h - P^m \tau_h. \tag{2.2}$$

The after-tax income of consumer h, w_h, is given by

$$w_h = p\omega_h - p^m \tau_h = p[\omega_h - (p^m/p)\,\tau_h] = p\tilde{\omega}_h. \tag{2.3}$$

Thus the scalar $\tilde{\omega}_h$ is also his effective demand for the commodity and so we have

$$x_h = \tilde{\omega}_h \tag{2.4}$$

for $\tilde{\omega}_h > 0$. The price of money is not completely arbitrary. If it is too high, some taxed individual would be bankrupt. That is, his tax-adjusted income by (2.3) would be negative and hence there would be no solution to the constrained-maximization problem (2.1). From (2.3), it follows that we have

$$\text{if} \qquad \tau_h > 0 \qquad \text{then} \quad P^m < \omega_h/\tau_h. \tag{2.5}$$

(Only the consumers facing positive money taxes are involved in the restriction (2.5), because only they face the possibility of bankruptcy.)

In competitive equilibrium, aggregate demand and supply must be equal, i.e.,

$$\sum_H x_h = \sum_H \omega_h, \tag{2.6}$$

[9] In the analysis of monetary economies, it is better to use the "commodity price of money" rather than the more familiar "money price of commodity" in order to comfortably include the analytically important case of worthless money, i.e., the case of $P^m = 0$.

and the price of the commodity must be positive, i.e., $p > 0$. Substituting (2.2) and (2.4) into the left side of (2.6) yields

$$P^m \sum_H \tau_h = 0. \tag{2.7}$$

From (2.7), it follows that in a proper monetary equilibrium (i.e., one with $P^m > 0$) we have

$$\sum_H \tau_h = 0, \tag{2.8}$$

i.e., taxes are exactly offset by transfers.[10]

Note that if money is worthless (i.e., $P^m = 0$), then autarky is the competitive equilibrium allocation, since $x_h = \tilde{\omega}_h = \omega_h > 0$ for each h. On the other hand, if all taxes are zero (i.e., $\tau_h = 0$ for each h), then autarky is the unique equilibrium allocation (i.e., $x_h = \tilde{\omega}_h = \omega_h$ for each h), but then P^m is completely indeterminate.

Next we apply these ideas to a concrete pure-exchange, competitive economy with lump-sum money taxes and transfers. We focus on the non-degenerate cases, namely ones in which taxes are nontrivial (i.e., not all zero), taxes are balanced (i.e., (2.8) holds), and the price of money is strictly positive.

2.1. *Parameters for the Certainty Economy*

There are three consumers:

$$H = \{1, 2, 3\}.$$

The vector ω of before-tax endowments is given by

$$\omega = (\omega_1, \omega_2, \omega_3) = (20, 10, 5).$$

The vector of money lump-sum taxes τ is given by

$$\tau = (\tau_1, \tau_2, \tau_3) = (5, 0, -5).$$

Since $\tau_1 + \tau_2 + \tau_3 = 0$, we see that τ is *balanced* and hence we know that there will be some equilibrium in which the price of money is positive.[11]

[10] A tax vector $\tau = (..., \tau_h, ...)$ is said to be *balanced* if (2.8) holds. A tax vector τ is said to be *bonafide* if it permits an equilibrium in which money is not worthless. In finite economies, τ is bonafide if and only if it is balanced; see Balasko and Shell [6] for the proof (Corollary 3.7) and a caveat (in Section 5).

[11] See Balasko and Shell [6].

2.2. *Competitive Equilibrium in the Certainty Economy*

The set of certainty equilibrium (CE) allocations, $\mathbf{X}_{\mathrm{CE}}$, is given by

$$\{x=(x_1, x_2, x_3) \in \mathbb{R}^3_{++} \mid x_1 = 20 - 5P^m, x_2 = 10,$$
$$x_3 = 5 + 5P^m, P^m \in \mathbb{R}_+\}, \tag{2.9}$$

where P^m satisfies inequalities (2.5), which reduce in this example to the requirement $P^m \in [0, 4)$. Thus, the set of certainty equilibrium prices, $\mathbf{P}^m_{\mathrm{CE}}$, is given by

$$\mathbf{P}^m_{\mathrm{CE}} = \{P^m \mid 0 \leqslant P^m < 4\}. \tag{2.10}$$

Combining (2.9) and (2.10) yields

$$\mathbf{X}_{\mathrm{CE}} = \{(x_1, x_2, x_3) \mid x_1 = 20 - 5P^m, x_2 = 10,$$
$$x_3 = 5 + 5P^m, 0 \leqslant P^m < 4\}. \tag{2.11}$$

Hence *the set* $\mathbf{X}_{\mathrm{CE}}$ *of certainty equilibrium allocations is one-dimensional, parameterized by the commodity price of money* P^m.

2.3. *Randomizations over Certainty Equilibria*

Before analyzing the sunspots economy, it will be useful to make precise the notion of a "randomization over certainty equilibria," or equivalently a "lottery over certainty equilibria." Let s be a random variable which, for simplicity, we assume to take on one of two possible realizations, α and β, with probability $\pi(\alpha) > 0$ and $\pi(\beta) = 1 - \pi(\alpha) > 0$ respectively. Let $x_h(s)$ be the allocation to consumer h in state s and $x(s) = (x_1(s), x_2(s), x_3(s)) \in \mathbb{R}^3_{++}$.

The allocation $(x(\alpha), x(\beta)) \in \mathbb{R}^6_{++}$ is said to be a (mere) *randomization over certainty equilibria* if we have

$$x(s) \in \mathbf{X}_{\mathrm{CE}} \qquad \text{for} \quad s = \alpha, \beta.$$

Hence $\mathbf{X}_{\mathrm{RCE}}$, the set of randomizations over CE allocations, is given by

$$\mathbf{X}_{\mathrm{RCE}} = \mathbf{X}_{\mathrm{CE}} \times \mathbf{X}_{\mathrm{CE}}.$$

Let $P^m(s)$ be the goods price of money in state s. The price vector $(P^m(\alpha), P^m(\beta)) \in \mathbb{R}^2_+$ is said to be a randomization over CE prices if $P^m(s) \in \mathbf{P}^m_{CE}$ for $s = \alpha, \beta$. Therefore the set of randomizations over CE prices, $\mathbf{P}^m_{\mathrm{RCE}}$, is given by

$$\mathbf{P}^m_{\mathrm{RCE}} = \mathbf{P}^m_{\mathrm{CE}} \times \mathbf{P}^m_{\mathrm{CE}} = [0, 4) \times [0, 4). \tag{2.12}$$

Hence *the set* $\mathbf{X}_{\mathbf{RCE}}$ *of randomizations over certainty equilibria is two-dimensional, parameterized by the vector* $(P^m(\alpha), P^m(\beta))$.

3. THE SUNSPOTS ECONOMY

We extend the model of Section 2 to allow individuals to face extrinsic uncertainty about the price level. We introduce the *extrinsic random variable* s, which, as above, is assumed to satisfy $s \in \{\alpha, \beta\}$. We assume that consumers share the same beliefs;[12] hence the probabilities with which state α or β occur, denoted by $\pi(\alpha)$ and $\pi(\beta) = (1 - \pi(\alpha))$ respectively, are held in common.

By definition, extrinsic uncertainty does not affect the fundamentals of the economy. In the present example, uncertainty is extrinsic because we assume:

1. *Endowments*:

$$\omega_h(\alpha) = \omega_h(\beta) = \omega_h \tag{3.1}$$

for $h \in H$, where $\omega_h(s)$ is the endowment of consumer h in state $s = \alpha, \beta$.

2. *Taxes*:

$$\tau_h(\alpha) = \tau_h(\beta) = \tau_h \tag{3.2}$$

for $h \in H$, where $\tau_h(s)$ is the money tax on consumer h in state $s = \alpha, \beta$, and

3. *Preferences*:

$$v_h[x_h(\alpha), x_h(\beta)] = \pi(\alpha)\, u_h(x_h(\alpha)) + \pi(\beta)\, u_h(x_h(\beta)), \tag{3.3}$$

where $v_h[\cdot]$ is the *ex ante* von Neumann–Morgenstern utility function for Mr. h.

It is obvious that symmetry assumptions (3.1) and (3.2) are required if uncertainty is to be extrinsic. Symmetry assumption (3.3), along with (3.1) and (3.2), implies that uncertainty is extrinsic because $v_h[\cdot]$ is unaffected by merely permuting the labels for the states of nature.[13]

The set H of consumers is partitioned into two classes, G^0 and G^1. Every consumer has access to the spot-market for trading the commodity (and money) after state s is revealed. Consumers in G^0 (possibly "the older generation") also have access to trading on state-contingent markets. They

[12] Thus we make a strong rational-expectations hypothesis; see Shell [19]. (For a sunspots economy in which information is asymmetric, see Peck and Shell [17].)

[13] For the formalization and the generalization, see Balasko [2].

have perfect foresight about the spot-market prices that will prevail after the state s is revealed. Consumers in G^1 (possibly "the younger generation") cannot trade in contingent securities. Our time line (Fig. 3.1) can be used for an overlapping-generations interpretation of the (natural) restrictions on market participation.

Let $x_h(s)$ be consumption of the commodity by Mr. h in state s. Let the utility function u_h be strictly increasing, smooth, and strictly concave. Also assume that the behavior of indifference curves at the axes is such that free commodities are ruled out. Let $p(s) > 0$ be the price of the commodity to be delivered in state s and $p^m(s) \geqslant 0$ be the price of money delivered in state s.

Formally, consumer $h \in G^1$ chooses $x_h(s) \in \mathbb{R}_{++}$ to

$$\text{maximize } u_h(x_h(s))$$

subject to (3.4)

$$p(s)\, x_h(s) = p(s)\, \omega_h - p^m(s)\, \tau_h$$

for $s = \alpha, \beta$.

Define the money price $P^m(s)$ and the tax-adjusted endowment $\tilde{\omega}_h(s)$ by

$$P^m(s) = p^m(s)/p(s) \qquad \text{and} \qquad \tilde{\omega}_h(s) = \omega_h - P^m(s)\, \tau_h \tag{3.5}$$

for $s = \alpha, \beta$. Then the budget constraint in (3.4) can be rewritten as

$$p(s)\, x_h(s) = p(s)\, \tilde{\omega}_h(s). \tag{3.4$'$}$$

Hence for $h \in G^1$ we have

$$x_h(s) = \tilde{\omega}_h(s) \tag{3.6}$$

for $s = \alpha, \beta$; i.e., $\tilde{\omega}_h(s)$ is restricted-consumer h's demand for the commodity in state s. Therefore, aggregating over the set of restricted consumers yields

$$\sum_{G^1} x_h(s) = \sum_{G^1} \tilde{\omega}_h(s) \tag{3.7}$$

for $s = \alpha, \beta$.

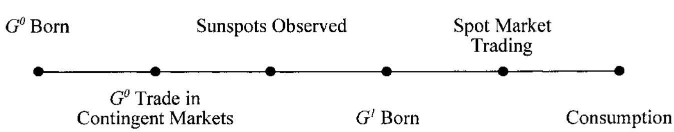

FIG. 3.1. Timeline for the OG interpretation of the restrictions on market participation.

Consumer $h \in G^0$ chooses $(x_h(\alpha), x_h(\beta)) \in \mathbb{R}^2_{++}$ to

$$\text{maximize } \pi(\alpha)\, u_h(x_h(\alpha)) + \pi(\beta)\, u_h(x_h(\beta))$$

subject to (3.8)

$$p(\alpha)\, x_h(\alpha) + p(\beta)\, x_h(\beta) = (p(\alpha) + p(\beta))\, \omega_h - (p^m(\alpha) + p^m(\beta))\, \tau_h.$$

Using (3.5), the budget constraint in (3.8) can be rewritten as

$$p(\alpha)\, x_h(\alpha) + p(\beta)\, x_h(\beta) = p(\alpha)\, \tilde{\omega}_h(\alpha) + p(\beta)\, \tilde{\omega}_h(\beta). \tag{3.8$'$}$$

The simplest (but least realistic) interpretation of (3.8) is that consumer $h \in G^0$ trades only before state s is revealed. He buys and sells contingent commodities. Taxes valued at contingent money prices are deducted from his income. We show in the appendix that (3.8) is equivalent to other, more interesting market arrangements in which consumer $h \in G^0$ trades on the spot market and hedges through the contingent commodity market and/or the contingent money market.

A competitive equilibrium for the monetary, sunspots economy is a price vector $(p(\alpha), p(\beta), p^m(\alpha), p^m(\beta))$ with $p(s) > 0$ and $p^m(s) \geqslant 0$ for $s = \alpha, \beta$ with the property that if consumers behave according to (3.4) and (3.8), then demand is equal to supply, i.e., we have

$$\sum_H x_h(s) = \sum_H \omega_h \tag{3.9}$$

for $s = \alpha, \beta$.

Summing over the budget constraints (3.8)$'$ yields

$$p(\alpha) \sum_{G^0} x_h(\alpha) + p(\beta) \sum_{G^0} x_h(\beta) = p(\alpha) \sum_{G^0} \tilde{\omega}_h(\alpha) + p(\beta) \sum_{G^0} \tilde{\omega}_h(\beta). \tag{3.10}$$

Hence the equilibrium behavior of the unrestricted consumers can be described in terms of their state-specific *tax-adjusted* endowments in a *tax-adjusted Edgeworth box* of dimensions

$$\sum_{G^0} \tilde{\omega}_h(\alpha) \times \sum_{G^0} \tilde{\omega}_h(\beta).$$

Substituting (3.7) and (3.10) into (3.9) yields

$$[p^m(\alpha) + p^m(\beta)] \sum_H \tau_h = 0. \tag{3.11}$$

Hence in a *proper* monetary equilibrium (one in which $p^m(s)>0$ for $s=\alpha, \beta)$[14] taxes must be balanced, i.e., $\sum_H \tau_h = 0$.[15, 16]

Next we specify the underlying parameters for the sunspots economy from which we build our numerical examples.

3.1. *Parameters for the Sunspots Economy*

The sunspots economy is based on the certainty economy described in Example (2.1): $H=\{1, 2, 3\}$, $\omega=(\omega_1, \omega_2, \omega_3)$, and $\tau=(\tau_1, 0, -\tau_1)$. In addition, we make the following specifications:

$$u_h(x_h(s)) = \log x_h(s) \qquad \text{for} \quad h \in H, \tag{3.12}$$

and

$$\pi = (\pi(\alpha), \pi(\beta)) = (3/4, 1/4), \tag{3.13}$$

so that we have

$$v_h = (3/4) \log x_h(\alpha) + (1/4) \log x_h(\beta) \qquad \text{for} \quad h \in G^0. \tag{3.14}$$

Four numerical examples follow.

3.2. *Totally Restricted Market Participation*

We begin with a sunspots economy example in which all consumers are restricted from participating in the state-contingent markets. Hence we have

$$G^0 = \varnothing \qquad \text{and} \qquad G^1 = H.$$

The parameters of the economy are identical to those described in example (2.1): $\omega=(\omega_1, \omega_2, \omega_3)=(20, 10, 5)$ and $\tau=(\tau_1, \tau_2, \tau_3)=(5, 0, -5)$. Since no trading across states of nature is possible in this degenerate case, the set

[14] A *seemingly* weaker requirement would be $p^m(s)>0$ for at least one s.

[15] Hence bonafidelity of taxes in the monetary sunspots economy requires balanced taxes. It is also easy to show that balancedness of taxes implies their bonafidelity in this economy.

[16] With state-specific taxation, equilibrium condition (3.11) generalizes to $p^m(\alpha) \sum_H \tau_h(\alpha) + p^m(\beta) \sum_H \tau_h(\beta) = 0$. Under this interpretation the government may have to accept for tax payment inside money issued by a consumer in G^0 and financed by his purchases of money in the other state. If $\sum_H \tau_h(\alpha)$ and $\sum_H \tau_h(\beta)$ are not zero, then they must be of opposite sign. Thus the price ratio $p^m(\alpha)/p^m(\beta)$ is (uniquely) given by the ratio $(-\sum_H \tau_h(\alpha)/\sum_H \tau_h(\beta))$. This observation is reminiscent of Fisher [10], in which it is shown that the international exchange rate is equal in absolute value to the ratio of current account deficits.

of equilibrium allocations $\mathbf{X}_{\mathbf{NP}}$, where NP denotes "no participation," is given by

$$\mathbf{X}_{\mathbf{NP}} = \{((x_1(\alpha), x_2(\alpha), x_3(\alpha)), (x_1(\beta), x_2(\beta), x_3(\beta))) \in \mathbb{R}^6_{++} \mid$$
$$x_1(s) = 20 - 5P^m(s),\ x_2(s) = 10,\ x_3(s) = 5 + 5P^m(s),$$
$$\text{and } 0 \leqslant P^m(s) < 4 \text{ for } s = \alpha, \beta\}. \tag{3.15}$$

For fixed endowments and taxes, $x_h(s)$ depends only on $P^m(s)$. Thus, the set of SSE allocations $\mathbf{X}_{\mathbf{NP}}$ and the set of sunspot equilibrium prices $\mathbf{P}^m_{\mathbf{NP}}$ satisfy, respectively,

$$\mathbf{X}_{\mathbf{NP}} = \mathbf{X}_{\mathbf{RCE}} = \mathbf{X}_{\mathbf{CE}} \times \mathbf{X}_{\mathbf{CE}}$$

and (3.16)

$$\mathbf{P}^m_{\mathbf{NP}} = \mathbf{P}^m_{\mathbf{RCE}} = \mathbf{P}^m_{\mathbf{CE}} \times \mathbf{P}^m_{\mathbf{CE}} = [0, 4) \times [0, 4).$$

Hence *the set of* SSE *allocations* is *two-dimensional, parameterized by the price vector* $(P^m(\alpha), P^m(\beta))$.

3.3. *Partially Restricted Market Participation*

In the two examples which follow, we divide the consumers into the following two groups:

$$G^0 = \{1, 2\} \qquad \text{and} \qquad G^1 = \{3\}.$$

Mr. 1 and Mr. 2 are unrestricted and can trade in securities or contingent commodities, whereas Mr. 3 cannot hedge against the effects of sunspots. Since Mr. 3 cannot trade across states of nature, he consumes (by (3.6)) his tax-adjusted endowments, i.e., we have

$$x_3(\alpha) = \tilde{\omega}_3(\alpha) \qquad \text{and} \qquad x_3(\beta) = \tilde{\omega}_3(\beta), \tag{3.17}$$

where $\tilde{\omega}_3(\alpha) = \tilde{\omega}_3(\beta)$ only if $P^m(\alpha) = P^m(\beta)$. (In the examples which follow, we always have $P^m(\alpha) \neq P^m(\beta)$.)

Any trading on the state-contingent markets must thus occur between Mr. 1 and Mr. 2. We can analyze their decisions and the resulting equilibrium allocations by means of a tax-adjusted Edgeworth box (see Figs. 3.2 and 3.3). The dimensions, $\sum_{G^0} \tilde{\omega}_h(\alpha) \times \sum_{G^0} \tilde{\omega}_h(\beta)$, represent the state-specific tax-adjusted endowments summed over Mr. 1 and Mr. 2. With the (identical, log-linear) utility functions specified in (3.14), the contract curve is the minor diagonal (with slope $\sum_{G^0} \tilde{\omega}_h(\beta)/\sum_{G^0} \tilde{\omega}_h(\alpha)$) of the tax-adjusted Edgeworth box. (For these utility functions, an allocation is

Pareto efficient for the community of unrestricted consumers if and only if the ratio of α-consumption to β-consumption is the same for each of these consumers and there are no unallocated consumption goods.)

There are two important points to be made regarding the contract curve: (1) Since the tax-adjusted Edgeworth box is now rectangular (given $P^m(\alpha) \neq P^m(\beta)$), and *not* square like the pre-tax Edgeworth box, the equilibrium allocation must be sunspot dependent for at least one consumer in G^0. (2) As long as the tax-adjusted endowments do not lie on the minor diagonal (and they do not in our examples), there must be trading across the states of nature.

Given the utility functions defined in (3.14), the demand functions are

$$x_h(s) = \frac{\pi(s)}{p(s)} w_h \qquad \text{for} \quad s = \alpha, \beta \qquad \text{and} \qquad h = 1, 2,$$

where the income of consumer h, w_h, is defined by

$$w_h = (p(\alpha) + p(\beta))\, \omega_h - (p(\alpha)\, P^m(\alpha) + p(\beta)\, P^m(\beta))\, \tau_h \qquad \text{for} \quad h = 1, 2. \tag{3.18}$$

Equating supply and demand for the state-s commodity within the tax-adjusted Edgeworth box yields the equilibrium condition

$$x_1(s) + x_2(s) = \tilde{\omega}_1(s) + \tilde{\omega}_2(s) \qquad \text{for} \quad s = \alpha, \beta. \tag{3.19}$$

Substituting the demand functions into (3.19) yields the market clearing equilibrium condition

$$\frac{p(\alpha)}{p(\beta)} = \left(\frac{\pi(\alpha)}{\pi(\beta)}\right)\left(\frac{\tilde{\omega}_1(\beta) + \tilde{\omega}_2(\beta)}{\tilde{\omega}_1(\alpha) + \tilde{\omega}_2(\alpha)}\right). \tag{3.20}$$

More generally, the equilibrium condition (3.20) is

$$\frac{p(\alpha)}{p(\beta)} = \left(\frac{\pi(\alpha)}{\pi(\beta)}\right)\left(\frac{\sum_{G^0} \tilde{\omega}_h(\beta)}{\sum_{G^0} \tilde{\omega}_h(\alpha)}\right). \tag{3.21}$$

Equation (3.21) states that the contingent-commodity price ratio must equal the after-tax scarcity ratio multiplied by the likelihood ratio.

The commodity prices of money $(P^m(\alpha), P^m(\beta))$ are restricted by the requirements that $(x_1(\alpha), x_1(\beta)) \in \mathbb{R}^2_{++}$ and $(x_2(\alpha), x_2(\beta)) \in \mathbb{R}^2_{++}$, which reduce to the requirement that

$$0 < x_1(s) < \tilde{\omega}_1(s) + \tilde{\omega}_2(s)$$

for $s=\alpha, \beta$. The set of SSE prices $\mathbf{P}^m_{\mathbf{RP}}$, where RP denotes restricted participation, is the set of $(P^m(\alpha), P^m(\beta)) \in \mathbb{R}^2_+$ that satisfy

$$P^m(\alpha) \in \left[0, \frac{\omega_1(\omega_1+\omega_2)}{\tau_1(\omega_1+\pi(\alpha)\,\omega_2)}\right) \tag{3.22}$$

and

$$P^m(\beta) \in \left[0, \frac{\omega_1(\omega_1+\omega_2)-\tau_1(\omega_1+\pi(\alpha)\,\omega_2)\,P^m(\alpha)}{\tau_1(\omega_1+\pi(\beta)\,\omega_2-\tau_1 P^m(\alpha))}\right). \tag{3.23}$$

The *set of SSE allocations* $\mathbf{X}_{\mathbf{RP}}$ is given by

$$\begin{aligned}
\mathbf{X}_{\mathbf{RP}} = \{(x_1(\alpha), x_2(\alpha), x_3(\alpha)), (x_1(\beta), x_2(\beta), x_3(\beta)) \in \mathbb{R}^6_{++} \mid \\
& x_1(\alpha) = 3/4[20-5P^m(\alpha)+(1/3\sigma)(20-5P^m(\beta))],\\
& x_2(\alpha) = 30/4[1+(1/3\sigma)],\\
& x_3(\alpha) = 5+5P^m(\alpha),\\
& x_1(\beta) = 1/4[3\sigma(20-5P^m(\alpha))+20-5P^m(\beta)],\\
& x_2(\beta) = 10/4[3\sigma+1],\\
& \text{and } x_3(\beta) = 5+5P^m(\beta)\},
\end{aligned}$$

where $\sigma=(30-5P^m(\beta))/(30-5P^m(\alpha))$ is the slope of the contract curve in the tax-adjusted Edgeworth box, and where, as in the previous section, $\omega=(\omega_1, \omega_2, \omega_3)=(20, 10, 5)$ and $\tau=(\tau_1, \tau_2, \tau_3)=(5, 0, -5)$. Given ω and τ, the allocation of the restricted consumer, $x_3(s)$ depends solely on the commodity price of money, $P^m(s)$, for $s=\alpha, \beta$. (Consequently, the tax-adjusted Edgeworth box depends on $P^m(\alpha)$ and $P^m(\beta)$ and thus in general the equilibrium allocations $x_h(s)$ for $h=1, 2$ also depend on the state contingent commodity prices of money.)

Hence *the set* $\mathbf{X}_{\mathbf{RP}}$ *is two-dimensional, parameterized the price vector* $(P^m(\alpha), P^m(\beta))$.

3.3.1. *Price Level Volatility Example*

For the remaining three examples, we continue with the economy that satifies $\omega=(\omega_1, \omega_2, \omega_3)=(20, 10, 5)$ and $\tau=(\tau_1, \tau_2, \tau_3)=(5, 0, -5)$. In addition, we set

$$P^m(\alpha)=1 \qquad \text{and} \qquad P^m(\beta)=2. \tag{3.24}$$

Relative to state α, state β is "deflationary." Given the values for τ and ω, (3.22) and (3.23) respectively yield

$$\mathbf{P}^m_{\mathrm{RP}} = \{(P^m(\alpha), P^m(\beta)) \in \mathbb{R}^2_+ \mid 0 \leqslant 11P^m(\alpha) + [9 - 2P^m(\alpha)]\, P^m(\beta) < 48 \text{ and } 0 \leqslant P^m(\alpha) < 48/11\}.$$

Hence the values chosen for the prices in the two states are consistent with equilibrium, i.e., we have $(1, 2) \in \mathbf{P}^m_{\mathrm{RP}}$. Substituting the prices from (3.24) in (3.5) yields the tax-adjusted endowments

$$(\tilde{\omega}_1(\alpha), \tilde{\omega}_1(\beta)) = (15, 10),$$
$$(\tilde{\omega}_2(\alpha), \tilde{\omega}_2(\beta)) = (10, 10),$$

and

$$(\tilde{\omega}_3(\alpha), \tilde{\omega}_3(\beta)) = (10, 15).$$

Since Mr. 3 cannot trade across states of nature, he consumes (by (3.17)) his tax adjusted endowments

$$x_3(\alpha) = \tilde{\omega}_3(\alpha) = 10 \quad \text{and} \quad x_3(\beta) = \tilde{\omega}_3(\beta) = 15. \tag{3.25}$$

Only Mr. 1 and Mr. 2 trade on the state-contingent markets. Figure 3.2 denotes the tax-adjusted Edgeworth box for this example, which is of

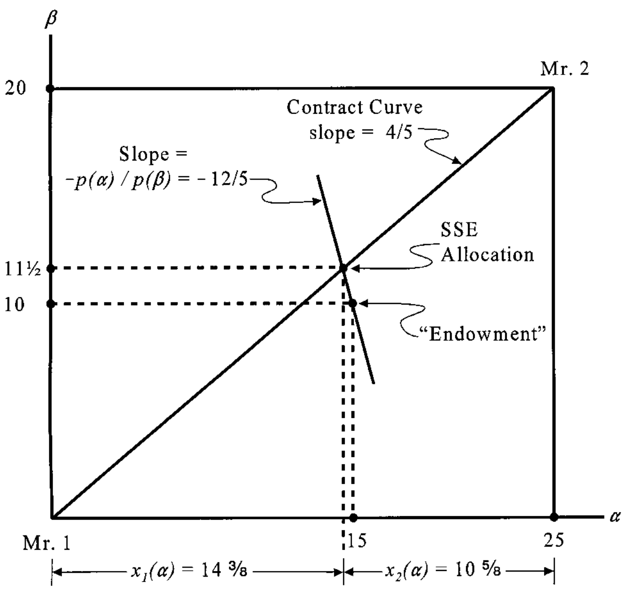

FIG. 3.2. The tax-adjusted Edgeworth box.

dimensions 25×20. The contract curve is given by the minor diagonal and has slope 20/25. From Eq. (3.20), the price ratio consistent with market clearing in this example is

$$\frac{p(\alpha)}{p(\beta)} = \left(\frac{20}{25}\right)\left(\frac{3/4}{1/4}\right) = \frac{12}{5}. \tag{3.26}$$

The resulting consumption allocations for Mr. 1 and Mr. 2 are given by

$$x_1(\alpha) = 14\tfrac{3}{8}, \quad x_1(\beta) = 11\tfrac{1}{2}, \quad x_2(\alpha) = 10\tfrac{5}{8}, \quad \text{and} \quad x_2(\beta) = 8\tfrac{1}{2}. \tag{3.27}$$

The next proposition follows directly from our example (3.3.1). It establishes that there is an SSE allocation which is not contained in the set $\mathbf{X}_{\mathbf{RCE}}$.

PROPOSITION 3.1. *The* SSE *allocation* $\{(x_1(\alpha),\ x_1(\beta)),\ (x_2(\alpha),\ x_2(\beta)),\ (x_3(\alpha),\ x_3(\beta))\}$, *described by* (3.25) *and* (3.27), *is not a randomization over certainty equilibria.*

Proof. The allocation described by (3.25) and (3.27) is a SSE allocation for the economy defined in example (3.3.1). This economy is based on the certainty economy defined in example (2.1). The set of CE allocations is given by (2.11). Because Mr. 2 is untaxed, his CE allocation must be $x_2 = 10$, and consequently, in any randomization over CE Mr. 2's allocation will also be $x_2(s) = 10$ for $s = \alpha, \beta$. But we have from (3.27) that his allocation in the corresponding sunspots economy is

$$x_2(\alpha) = 10\tfrac{5}{8} > 10 \qquad \text{and} \qquad x_2(\beta) = 8\tfrac{1}{2} < 10.$$

Hence we have shown that this SSE allocation is *not* a mere randomization over *any* CE. That is, we have found an allocation in $\mathbf{X}_{\mathbf{RP}}$ that is not in $\mathbf{X}_{\mathbf{RCE}}$. ▮

In the next example, we show that the SSE price levels are not necessarily randomizations over CE price levels.

3.3.2. *High Price-Level Volatility Example*:

We alter the prices given in (3.24) for example (3.3.1) to be

$$P^m(\alpha) = 1 \qquad \text{and} \qquad P^m(\beta) = 5, \tag{3.28}$$

and hence state β is even more "deflationary" than before. Notice that in the certainty economy, randomizations over the certainty economy, and the sunspots economy with totally restricted market participation, the respective equilibrium prices of money, P^m and $P^m(s)$ for $s = \alpha, \beta$, lie in the interval $[0, 4)$ by (2.10), (2.12) and (3.16). Hence if (3.28) is consistent with SSE, then we have shown that for one state the goods price of money is greater than could be achieved in the certainty economy or any randomization over the certainty economy.

The tax-adjusted endowments obtained by using (3.28) and (3.5) are given by

$$(\tilde{\omega}_1(\alpha), \tilde{\omega}_1(\beta)) = (15, -5),$$

$$(\tilde{\omega}_2(\alpha), \tilde{\omega}_2(\beta)) = (10, 10),$$

and (3.29)

$$(\tilde{\omega}_3(\alpha), \tilde{\omega}_3(\beta)) = (10, 30).$$

Since Mr. 3 cannot trade in the contingent markets, his allocation is identical to his tax-adjusted endowment

$$x_3(\alpha) = \tilde{\omega}_3(\alpha) = 10 \qquad \text{and} \qquad x_3(\beta) = \tilde{\omega}_3(\beta) = 30. \tag{3.30}$$

In this economy, as in example (3.3.1), only Mr. 1 and Mr. 2 trade in the commodity spot-market. The tax-adjusted Edgeworth box for this example is given in Fig. 3.3 and is of dimensions 25×5. Notice, however, that Mr. 1's tax-adjusted endowment is outside the tax-adjusted Edgeworth box, and hence it is outside his consumption set (in state β), since all allocations must lie in the strictly positive orthant.

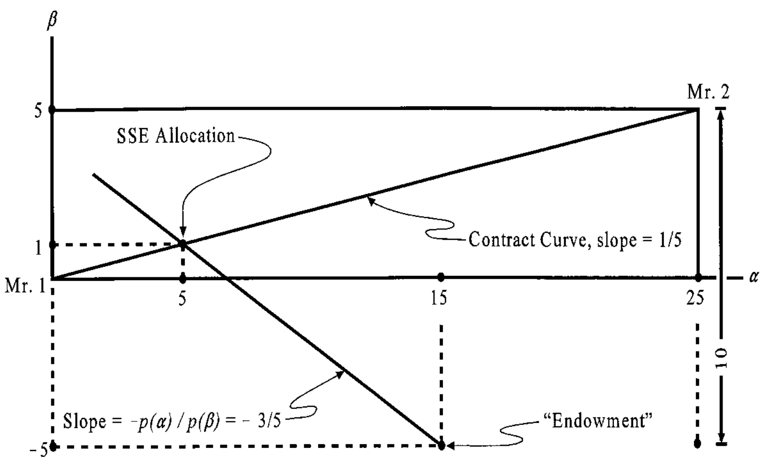

FIG. 3.3. "Endowment" outside the tax-adjusted Edgeworth box.

The contract curve is the minor diagonal (with slope 1/5); the ratio of prices consistent with market clearing in the goods market and hence with competitive equilibrium is given by

$$\frac{p(\alpha)}{p(\beta)}=\left(\frac{5}{25}\right)\left(\frac{3/4}{1/4}\right)=\frac{3}{5}, \tag{3.31}$$

from (3.20). The sunspot equilibrium allocations for Mr. 1 and Mr. 2 in this economy are given by

$$x_1(\alpha)=5, \qquad x_1(\beta)=1, \qquad x_2(\alpha)=20, \qquad \text{and} \qquad x_2(\beta)=4. \tag{3.32}$$

As in the previous example, we have constructed a SSE which is not a randomization over certainty equilibrium, as evidenced by the SSE allocations associated with Mr. 2. In addition, this economy can exhibit greater price volatility than is possible in the CE economy or the sunspots economy with totally restricted market participation since $P^m(\beta)=5\notin \mathbf{P}^m_{\mathbf{CE}}=[0,4)$.[17] As a result Mr. 1's tax-adjusted endowment is negative in state β, but because he has positive income, w_h, he can afford a strictly positive bundle. (Mr. 1 can afford a strictly positive bundle if and only if $\pi(\alpha)>5/8$. That is, Mr. 1 is viable if and only if the probability of the bad state, β for him, is less than 3/8.)

In our next example, we show that even with full market participation, the price level can be volatile in the sunspots economy and in fact, that there is room for wider price fluctuations in the economy with full hedging markets than in the corresponding economy with no scope for hedging.

3.4. *Unrestricted Market Participation*

Before we use the tax-adjusted Edgeworth box to generate our last example of a SSE, we recall that if markets are perfect then the equilibrium allocations in the economy are immune to the effects of sunspots. We also register a caveat: Even with perfect markets there is price-level volatility in the economy with money taxes and transfers. Furthermore, there is room for greater price-level volatility with perfect hedging markets than in the cases with imperfect hedging markets.

PROPOSITION 3.2. *Consider the sunspots economy with money taxes and transfers and unrestricted market participation, i.e., the economy in which*

[17] The relationship between the magnitude of the price-level volatility for the certainty economy and the sunspots economy with restricted market participants will be more fully explored in Section 3.4.1.

$G^0 = H$ (and hence $G^1 = \varnothing$). If $(..., x_h(\alpha), x_h(\beta), ...)$ is a competitive equilibrium allocation in this economy, then we have

$$x_h(\alpha) = x_h(\beta), \tag{3.33}$$

for $h \in H$. That is, the allocations are the same in each state of nature.

Proof. Consider the economy represented by the related no-taxation economy with endowments $\tilde{\omega}_h(s)$, $s = \alpha, \beta$, given by the (3.5). Since

$$\sum_H \tilde{\omega}_h(s) = \sum_H \omega_h \tag{3.34}$$

for $s = \alpha, \beta$, there is no aggregate uncertainty in the tax-adjusted economy, i.e., the tax-adjusted Edgeworth diagram is a cube.

Using (3.34) allows us to adopt the standard proof from the sunspots literature.[18] Let $(x_h(\alpha), x_h(\beta))$ be the equilibrium allocation of consumer h. Assume that for some h, $x_h(\alpha) \neq x_h(\beta)$. Let $\bar{x}_h$ be defined by $\bar{x}_h = \pi(\alpha)\, x_h(\alpha) + \pi(\beta)\, x_h(\beta)$ for $h \in H$. Then we have $\sum_H \bar{x}_h = \sum_H \omega_h$ and $v_h(\bar{x}_h, \bar{x}_h) \geqq v_h(x_h(\alpha), x_h(\beta))$ for each $h \in H$ and with strict inequality for at least one $h \in H$. Hence we have found a competitive equilibrium that is not Pareto optimal, contradicting the first theorem of welfare economics. Therefore, (nonsunspots) condition (3.33) is established. ∎

In the context of the economy without outside money, Cass and Shell use the condition in (3.33) to define the case "where sunspots do not matter." In their model, if the allocations are symmetric then so are the prices. However, in the economy with outside money, there can be price-level volatility even when the allocations are symmetric, i.e., they satisfy (3.33). Hence the Cass–Shell definition of "sunspots do not matter" is incomplete for the economy with money taxes and transfers.[19]

The set of SSE allocations $\mathbf{X}_{\mathbf{FP}}$ where FP denotes "full participation," is defined by

$$\begin{aligned} \mathbf{X}_{\mathbf{FP}} = \{ ((x_1(\alpha), x_2(\alpha), x_3(\alpha)), (x_1(\beta), x_2(\beta), x_3(\beta))) \in \mathbb{R}^6_{++} \mid \\ x_1(\alpha) = x_1(\beta) = [(3/4)(20 - 5P^m(\alpha)) + (1/4)(20 - 5P^m(\beta))], \\ x_2(\alpha) = x_2(\beta) = 10, \\ x_3(\alpha) = x_3(\beta) = [(3/4)(5 + 5P^m(\alpha)) + (1/4)(5 + 5P^m(\beta))]\}, \end{aligned}$$

[18] See Malinvaud [16]. Also see Cass and Shell [8, Proposition 3] and Goenka and Shell [13, 14].

[19] The extent of the possible price-level volatility with perfect hedging markets will be made clear in numerical Example 3.4.1.

where $\omega = (\omega_1, \omega_2, \omega_3) = (20, 10, 5)$ and $\tau = (\tau_1, \tau_2, \tau_3) = (5, 0, -5)$. Notice that $x_h(\alpha) = x_h(\beta)$ depends on $P^m(\alpha)$ and $P^m(\beta)$ solely through the sum $\pi(\alpha)\, P^m(\alpha) + \pi(\beta)\, P^m(\beta) = (3/4)\, P^m(\alpha) + (1/4)\, P^m(\beta)$.

The set of full-participation SSE money prices $\mathbf{P}^m_{\mathbf{FP}}$ is defined by

$$\mathbf{P}^m_{\mathbf{FP}} = \{(P^m(\alpha), P^m(\beta)) \in \mathbb{R}^2_+ \mid 0 \leqslant 3P^m(\alpha) + P^m(\beta) < 16\}. \tag{3.35}$$

The equilibrium allocations $x_h(s)$ for $h \in H$ and $s = \alpha, \beta$, are functions of the tax-adjusted endowments, and thus, given ω and τ depend only on the commodity prices of money, $P^m(\alpha)$ and $P^m(\beta)$, solely through the weighted average of prices $\pi(\alpha)\, P^m(\alpha) + \pi(\beta)\, P^m(\beta)$. (This results from Mr. 3 no longer being restricted from trading on the securities markets. Consequently, the ratio of commodities prices from (3.20) reduces to $p(\alpha)/p(\beta) = \pi(\alpha)/\pi(\beta) = 3$.)

Hence *the set* $\mathbf{X}_{\mathbf{FP}}$ *is one dimensional, parameterized by the scalar* $\pi(\alpha)\, P^m(\alpha) + \pi(\beta)\, P^m(\beta) = (3/4)\, P^m(\alpha) + (1/4)\, P^m(\beta)$.

This indeterminacy is the same as one finds in general-equilibrium models with incomplete markets (GEI) and with nominal financial instruments; see, e.g., Cass [7] and Werner [25]. In that literature, the financial assets are available in zero net supply. Our financial asset is money; see Appendix A.1 for the best interpretation. The nominal (coupon) return on this money is zero in each state. (Other coupon rates are easily accomodated.) In our model Mr. h's endowment of the financial asset, $-\tau_h$, is not in general zero, but the balancedness condition (2.8) corresponds to an aggregate zero supply condition. In our model, markets are *complete* but hedging on the state-contingent money market is *restricted* to consumers in G^0.

For an exposition of the monetary tax-transfer model in which $-\tau_h$ is explicitly treated as Mr. h's endowment of money, see Shell [18]. For a recasting of the monetary tax-transfer model in terms of the GEI literature, see Vilanacci [23]. It should be remarked that not all financial markets models have the same indeterminacy properties as those based on Arrow securities. For a model with different *financial* securities and different equilibrium properties, see Antinolfi and Keister [1].

3.4.1. *High Price-Level Volatility Example*

Building on example (3.3.2), we replace the restricted participation assumption with the assumption that participation on the securities market is unrestricted, i.e., we have $G^0 = H$ (and hence $G^1 = \varnothing$). The prices are the same as in (3.28) and the resulting tax-adjusted endowments are the same as in (3.29). From (3.2.1), the ratio of prices necessary for the commodity market to clear is given by

$$\frac{p(\alpha)}{p(\beta)} = \left(\frac{30}{30}\right)\left(\frac{3/4}{1/4}\right) = 3. \tag{3.36}$$

Since all individuals have access to contingent markets and thus can insure against the possibility of sunspots, allocations are invariant across states of nature; more specifically we have

$$x_1(\alpha) = x_1(\beta) = 10, \qquad x_2(\alpha) = x_2(\beta) = 10$$

and (3.37)

$$x_3(\alpha) = x_3(\beta) = 15.$$

However, as in the previous example, (3.3.2), we obtain a price level not sustainable as a CE or as a randomization over CE; in particular,

$$P^m(\beta) = 5$$

is consistent with competitive equilibrium in this example. In fact in this example, if we fix $P^m(\alpha) = 1$, then we have that equilibrium exists for every $P^m(\beta)$ satisfying

$$P^m(\beta) \in [0, 13).$$

Thus substantial price level volatility can exist. Also note that from (3.35) we have that for one state the SSE can exhibit greater deflation than is possible in the certainty economy or in the sunspots economy without securities markets.[20]

Price-Level Volatility[21]. As we have suggested in examples (3.3.2) and (3.4.1), potential price-level "volatility" has grown with each succeeding example, i.e., it increases as the restrictions on market participation are removed. We make this idea concrete by calculating the "maximum" variance in money prices for (1) the sunspots economy with totally restricted market participation (3.2), (2) the sunspots economy with partially

[20] Fix $P^m(\alpha) = 1$ and allow $\pi(\alpha) \in (0, 1)$ to vary; then for every (large) number N, there is a $\pi(\alpha)$ sufficiently close to unity, such that the interval $[0, N]$ is included in the set of equilibrium values for $P^m(\beta)$.

[21] The analysis of "general-price level volatility" suggests examination of the random variable $P(s)$. Without losing this spirit, we work instead with the random variable $P^m(s)$, the inverse of the general price level, and thereby avoid difficulties caused by the fact that $P(s)$ can have the realization $+\infty$.

restricted market participation (3.3), and (3) the sunspots economy in which all market participants are unrestricted (3.4). The variance of the price is given by

$$\operatorname{var}(P^m(s)) = \tfrac{3}{16}[P^m(\alpha) - P^m(\beta)]^2, \tag{3.38}$$

for each of the examples. Hence we have

$$\sup_{\mathbf{P}^m_{NP}} \operatorname{var}(P^m(s)) = 3 < \sup_{\mathbf{P}^m_{RP}} \operatorname{var}(P^m(s)) = \tfrac{16}{3} < \sup_{\mathbf{P}^m_{FP}} \operatorname{var}(P^m(s)) = 48$$

from (3.38). Thus, if price-level volatility is measured in terms of the "maximum" potential variance in money prices, we see that there is greater potential price-level volatility in the sunspots economy with partially restricted market participation than in the sunspots economy with totally restricted market participation. Similarly, the sunspots economy in which all market participants are unrestricted has even greater potential price-level volatility than either of the other two economies. This suggests that in an economy with outside money, nonsunspot equilibria might not be robust in the face of even small perturbations of the restrictions on market participation or other data defining the economy.

4. THE SOURCE OF SUNSPOT EQUILIBRIA

We have analyzed sunspot equilibrium in a very simple monetary model. The simplicity makes the nature and the source of the sunspot equilibria apparent. The existence of restricted consumers makes a stochastic equilibrium possible. The interaction of the restricted and unrestricted consumers is essential for producing a SSE which is not a randomization over CEs.

The restricted consumers—those in G^1, who cannot trade on sunspot contingent markets—typically face uncertain tax-adjusted endowments and hence have stochastic consumptions. Typically, the *aggregate* consumption of the consumers in G^1 is stochastic and hence the *aggregate after-tax* endowment of the consumers in G^0 will be stochastic. The unrestricted consumers—those in G^0, who can trade on sunspot contingent markets—attempt to re-allocate the sunspot risks among themselves, but they do not completely rid themselves of uncertainty: the unrestricted consumers will typically have uncertain consumptions. Because of the risk-sharing among the unrestricted consumers, the resulting SSE allocation will typically differ from a mere randomization over CE allocations. Unrestricted consumers transfer income across states of nature; their behavior is typically not maximal on a state-by-state basis.

This is also made clear in the tax-adjusted Edgeworth box, in which the contingent-claims price ratio and the final allocations to the individual unrestricted consumers are determined. The after-tax endowments of each consumer, and therefore the aggregate endowment for the community of unrestricted consumers, are typically stochastic. Consequently, as a result of the stochastic nature of aggregate endowments, the dimensions of the sides of the Edgeworth box are typically unequal. Hence the analysis is *as if* this were an economy in which all consumers are unrestricted and all uncertainty is intrinsic. Competitive-risk sharing leads to final allocations which are typically stochastic and different from the (tax-adjusted) endowments.

To summarize: the seed of the sunspot allocation is in the set of restricted consumers. Their net tax payments are uncertain because the price level is uncertain. Hence the net tax receipts of the unrestricted consumers are also uncertain. The stochastic tax-adjusted endowments of the unrestricted consumers flower into a proper sunspot equilibrium allocation through the hedging against price level uncertainty by the unrestricted consumers.

In finite, "convex," competitive economies with perfect markets for hedging against the effects of sunspots, the equilibrium allocations are not uncertain. Nonetheless in the monetary economy, hedging markets—even perfect hedging markets—increase the possible range of price-level volatility. In nonmonetary economies, small market imperfections do not induce sunspots.[22] However, for monetary economies, because of price-level volatility, it would seem that the slightest imperfection in markets would leave the door open to sunspot effects on real allocations. This remains to be investigated thoroughly.

The case in which the government possesses a (possibly unlimited) stock of commodity is probably not of much direct economic interest, but it is instructive. With a positive government stockpile, balancedness of the tax policy across private agents is not necessary for bonafidelity. The model then also allows for government aggregate real transfers to vary across states. Hence there can then be proper SSE allocations even with complete markets and unrestricted participation. This is because after-tax Edgeworth box would typically not be "square". In this case—even if markets are perfect—equilibrium allocation will typically be stochastic and not mere randomizations over CE allocations.

5. CONCLUDING REMARKS

In constructing examples, we employ a new tool: the *tax-adjusted* Edgeworth box. We hope that this tool will be useful in the systematic

[22] See Balasko, Cass, and Shell [3].

analysis of the structure of the set of competitive equilibria (and the related comparative statics) for this and more general monetary models.

We do not envision difficulty in moving to general von Neumann–Morgenstern utility functions. Moving from one commodity to several, however, will require a more sensitive use of the tax-adjusted Edgeworth box. In this case, intra-state commodity price ratios are *jointly* determined by the restricted *and* unrestricted consumers.

This is only a first attempt at integrating monetary equilibrium theory and sunspot equilibrium theory. Extension of the analysis of the present paper to *dynamic* economies is essential; static monetary analysis is clearly very incomplete. In the perfect-foresight overlapping-generations economy, it is neither necessary nor sufficient for proper monetary equilibrium that the public debt be retired.[23] Is this result substantially strengthened when we allow for the possibility of sunspot equilibria? In particular, are the existing characterizations of those fiscal policies which are consistent with positively priced money[24] significantly altered when moving from perfect-foresight equilibrium to the more general case of sunspot equilibrium?

The examples of "volatility" given in the present paper are meant to be suggestive. We give no defense of the particular volatility measure (maximum potential variance) that we use. Perhaps our calculations might prove to be provocative for the development of a theory in which government policies can be judged in terms of their implied efficiency, equity, and *stability* (the "inverse" of volatility).[25]

A. APPENDIX

Here we provide microeconomic justification for the budget constraints in (3.4) and (3.8) for the monetary, sunspots economy of Section 3. We explicitly adopt the overlapping-generations economy; the timing is given by Fig. 3.1.

Every consumer in H can trade for money and commodity in the spot-market (the market which meets after stated s is revealed). These are the only trades that the restricted consumers (those in G^1) can make (because they are born after s is revealed). The unrestricted consumers (those in G^0) can also trade in either state-contingent commodity *or* state-contingent money. Consumers in G^0 have perfect foresight about spot-market prices.

[23] See Balasko and Shell [5].

[24] See, e.g., Balasko and Shell [4, Sections 5 and 6, and especially Proposition 5.5] and Esteban, Mitra, and Ray [9].

[25] See Keister [15] for recent work in this direction based in part on the ideas in our present paper.

The notation is necessarily elaborate. We must employ a precise general-equilibrium-style approach.[26] The superscript 1 denotes prices and transactions on the spot-market. The superscript 0 denotes prices and transactions in state-contingent commodity or state-contingent money made before the state of nature is revealed.

Let $x_h^1(s)$ and $x_h^{m,1}(s)$ be respectively the amount of commodity and money purchased by Mr. h in the spot-market after state s has been revealed. Let $p^1(s) > 0$ and $p^{m,1}(s) \geqslant 0$ be the corresponding spot-market prices.

Let $x_h^0(s)$ and $x_h^{m,0}(s)$ be respectively the amount of commodity to be delivered if state s occurs and the amount of money to be delivered if state s occurs. These contingent commodities are purchased on the market that meets before the state s is revealed. Let $p^0(s)$ and $p^{m,0}(s)$ be the corresponding prices of these contingent goods.

If consumer h pays his money taxes in each state, then we have

$$x_h^{m,0}(s) + x_h^{m,1}(s) = \tau_h \tag{A.1}$$

for $s = \alpha, \beta$ and $h \in H$, but, of course, for $h \in G^1$ we have $x_h^{m,0}(s) = 0$. The consumption of consumer h in state s, $x_h(s)$ is given by

$$x_h(s) = x_h^0(s) + x_h^1(s) \tag{A.2}$$

but, of course, $x_h^0(s) = 0$ for $h \in G^1$.

We treat separately two cases: (1) Consumers in G^0 do all their hedging through purchases and sales of state-contingent money. (2) Consumers in G^0 do their hedging through purchases and sales of state-contingent commodity. We show that the model based on assumption (1) is equivalent to the model based on assumption (2) and that they are equivalent to the model of Section 3, for which it is implicitly assumed that there are contingent-money markets *and* contingent-commodity markets available to the consumers in G^0 for hedging against the effects of sunspots.

A.1. *Hedging through State-Contingent Money*

Assume, for the moment, that there is at least one consumer in G^0, i.e., we have $G^0 \neq \varnothing$. Consumer $h \in G^0$ chooses $(x_h^{m,0}(\alpha), x_h^{m,0}(\beta), x_h^{m,1}(\alpha), x_h^{m,1}(\beta), x_h^1(\alpha), x_h^1(\beta)) \in \mathbb{R}^6$ to

$$\text{maximize } \pi(\alpha)\, u_h(x_h(\alpha)) + \pi(\beta)\, u_h(x_h(\beta)) \tag{A.1.1}$$

[26] See Balasko and Shell [4, especially Section 2, Eq. 2.1].

subject to

$$p^1(s)\, x_h^1(s) + p^{m,1}(s)\, x_h^{m,1}(s) = p^1(s)\, \omega_h, \qquad \text{for} \quad s = \alpha, \beta, \tag{A.1.2}$$

$$p^{m,0}(\alpha)\, x_h^{m,0}(\alpha) + p^{m,0}(\beta)\, x_h^{m,0}(\beta) = 0, \tag{A.1.3}$$

and

$$x_h(s) = x_h^1(s) > 0 \quad \text{and} \quad x_h^{m,0}(s) + x_h^{m,1}(s) = \tau_h \quad \text{for} \quad s = \alpha, \beta. \tag{A.1.4}$$

Equation (A.1.2) says that the spot-market purchases of commodity and net accretions to money holdings are financed by the sale of the endowment of commodity. Equation (A.1.3) says that purchases of money deliverable in one state are financed by the sale of money deliverable in the other state. The first equation in (A.1.4) says that Mr. h consumes the commodities which he has purchased. The second equation in (A.1.4) says that the sum of Mr. h 's money accretions must equal his money tax obligation.

Substituting (A.1.4) and (A.1.3) in (A.1.2) gives

$$\frac{p^1(\alpha)(x_h(\alpha) - \omega_h) + p^{m,1}(\alpha)\, \tau_h}{p^{m,1}(\alpha)} = x_h^{m,0}(\alpha) \tag{A.1.5}$$

and

$$\left(\frac{p^{m,0}(\beta)}{p^{m,0}(\alpha)}\right)\left(\frac{p^1(\beta)(x_h(\beta) - \omega_h) + p^{m,1}(\beta)\, \tau_h}{p^{m,1}(\beta)}\right) = -x_h^{m,0}(\alpha). \tag{A.1.6}$$

Given prices, his endowment, and his tax-obligation, when Mr. h selects $x_h(\alpha)$ the left side of (A.1.5) is determined. Similarly, $x_h(\beta)$ determines the left side of (A.1.6). Hence (A.1.5) and (A.1.6) can be combined into a single budget constraint without altering the opportunities of Mr. h over final consumption bundles $(x_h^1(\alpha), x_h^1(\beta))$. Adding (A.1.5) and (A.1.6) results in the single budget constraint for $h \in G^0$,

$$\left(\frac{p^1(\alpha)\, p^{m,0}(\alpha)}{p^{m,1}(\alpha)}\right)(x_h(\alpha) - \omega_h) + \left(\frac{p^1(\beta)\, p^{m,0}(\beta)}{p^{m,1}(\beta)}\right)(x_h(\beta) - \omega_h)$$
$$= -(p^{m,0}(\alpha) + p^{m,0}(\beta))\, \tau_h. \tag{A.1.7}$$

The (single) budget constraint (A.1.7) reduces to the (single) budget constraint in (3.8) if we have

$$p(s) = p^1(s)\, p^{m,0}(s) / p^{m,1}(s).$$

and

$$p^m(s) = p^{m,0}(s). \tag{A.1.8}$$

Substituting the second of the two equations of (A.1.8) into the first yields

$$p^1(s)/p^{m,1}(s) = p(s)/p^m(s). \tag{A.1.9}$$

for $s = \alpha, \beta$.

We next turn to consumer $h \in G^1$ (if there is one). He chooses $(x_h^{m,1}(\alpha), x_h^{m,1}(\beta), x_h^1(\alpha), x_h^1(\beta)) \in \mathbb{R}^4$ to

$$\text{maximize } u_h(x_h(s)) \tag{A.1.10}$$

subject to

$$p^1(s)\, x_h^1(s) + p^{m,1}(s)\, x_h^{m,1}(s) = p^1(s)\, \omega_h \qquad \text{for} \quad s = \alpha, \beta \tag{A.1.11}$$

and

$$x_h(s) = x_h^1(s) > 0 \qquad \text{and} \qquad x_h^{m,1}(s) = \tau_h \qquad \text{for} \quad s = \alpha, \beta. \tag{A.1.12}$$

Substituting from (A.1.12) in (A.1.11) yields the single budget constraint for each state

$$p^1(s)\, x_h^1(s) = p^1(s)\, \omega_h - p^{m,1}(s)\, \tau_h \tag{A.1.13}$$

for $s = \alpha, \beta$. The budget constraints (A.1.13) are equivalent to the budget constraints in (3.4) if we have

$$p(s)/p^m(s) = p^1(s)/p^{m,1}(s).$$

for $s = \alpha, \beta$. But by (A.1.9) we have already chosen the prices $p(s)$ and $p^m(s)$ in a way that assures this equality. If G^0 is empty, (A.1.9) is the only restriction that needs to be imposed on $(p(s), p^m(s))$ for $s = \alpha, \beta$. We have shown that if an allocation is an equilibrium to the economy of Subsection A.1, then it is also an equilibrium for the corresponding economy in Section 3.

The budget constraint in (3.8) is equivalent to (A.1.2)–(A.1.4) if (A.1.8) holds. The budget constraint in (3.8) is equivalent to (A.1.11)–(A.1.12) if (A.1.9) holds. Hence an allocation that is an equilibrium for the economy of Section 3 is also an equilibrium for the economy of Subsection A.1. ▮

A.2. *Hedging through State-Contingent Commodity*

Consumer $h \in G^0$ chooses $(x_h^{m,0}(\alpha),\ x_h^{m,0}(\beta),\ x_h^{m,1}(\alpha),\ x_h^{m,1}(\beta),\ x_h^1(\alpha),\ x_h^1(\beta)) \in \mathbb{R}^6$ to

$$\text{maximize } \pi(\alpha)\, u_h(x_h(\alpha)) + \pi(\beta)\, u_h(x_h(\beta)) \tag{A.2.1}$$

subject to

$$p^1(s)\, x_h^1(s) + p^{m,1}(s)\, x_h^{m,1}(s) = p^1(s)\, \omega_h \qquad \text{for} \quad s = \alpha, \beta, \tag{A.2.2}$$

$$p^0(\alpha)\, x_h^0(\alpha) + p^0(\beta)\, x_h^0(\beta) = 0, \tag{A.2.3}$$

and

$$x_h(s) = x_h^0(s) + x_h^1(s) \qquad \text{and} \qquad x_h^{m,1}(s) = \tau_h \qquad \text{for} \quad s = \alpha, \beta. \tag{A.2.4}$$

Equation (A.2.2) says that in the spot-market purchases of commodity and money are financed by the sale of commodity endowment. Equation (A.2.3) says that purchases of the contingent commodity to be delivered in one of the states are financed by sales of the commodity to be delivered in the other state. The first equation in (A.2.4) says that consumption of commodity in a given state is equal to the sum of the spot-market purchases of commodity and the purchases of commodity contingent on that state. The second equation in (A.2.4) says that the money tax obligation must be met from spot-market purchases.

Substituting from (A.2.4) and (A.2.3) in (A.2.2) gives

$$\frac{p^1(\alpha)(x_h(\alpha) - \omega_h) + p^{m,1}(\alpha)\, \tau_h}{p^1(\alpha)} = x_h^0(\alpha) \tag{A.2.5}$$

and

$$\left(\frac{p^0(\beta)}{p^0(\alpha)}\right)\left(\frac{p^1(\beta)(x_h(\beta) - \omega_h) + p^{m,1}(\beta)\, \tau_h}{p^1(\beta)}\right) = -x_h^0(\alpha). \tag{A.2.6}$$

Adding (A.2.5) and (A.2.6) results in the single budget constraint for consumer $h \in G^0$,

$$\begin{aligned} &p^0(\alpha)(x_h(\alpha) - \omega_h) + p^0(\beta)(x_h(\beta) - \omega_h) \\ &\qquad = \left(\frac{p^{m,1}(\alpha)\, p^0(\alpha)}{p^1(\alpha)} + \frac{p^{m,1}(\beta)\, p^0(\beta)}{p^1(\beta)}\right) \tau_h. \end{aligned} \tag{A.2.7}$$

The budget constraint (A.2.7) reduces to the budget constraint in (3.8) if we have

$$p(s) = p^0(s)$$

and (A.2.8)

$$p^m(s) = \frac{p^{m,1}(s)\, p^0(s)}{p^1(s)}$$

for $s = \alpha, \beta$. Substituting $p(s)$ for $p^0(s)$ in the second equation in (A.2.8) yields

$$\frac{p^m(s)}{p(s)} = \frac{p^{m,1}(s)}{p^1(s)} \tag{A.2.9}$$

To complete the argument, consider consumer $h \in G^1$. He chooses $(x_h(s), x_h^m(s)) \in \mathbb{R}^2$ to

$$\text{maximize } u_h(x_h(s)) \tag{A.2.10}$$

subject to

$$p^1(s)\, x_h^1(s) + p^{m,1}(s)\, x_h^{m,1}(s) = p^1(s)\, \omega_h \qquad \text{for} \quad s = \alpha, \beta \tag{A.2.11}$$

and

$$x_h(s) = x_h'(s) > 0 \qquad \text{and} \qquad x_h^{m,1}(s) = \tau_h \qquad \text{for} \quad s = \alpha, \beta. \tag{A.2.12}$$

Substituting from (A.2.12) in (A.2.11) yields one budget constraint per state, i.e.

$$p^1(s)(x_h(s) - \omega_h) = -p^m(s)\, \tau_h \qquad \text{for} \quad s = \alpha, \beta. \tag{A.2.13}$$

Because of (A.2.9), we know that we have already chosen prices so that (A.2.13) is equivalent to the budget constraint in (3.4). Hence we have shown that an allocation that is an equilibrium for the economy of Subsection A.2 is also an equilibrium for the corresponding economy of Section 3.

The budget in (3.8) is equivalent to (A.2.2)–(A.2.4) if (A.2.8) holds. The budget constraint in (3.4) is equivalent to (A.2.11)–(A.2.12) if (A.2.9) holds. Hence an allocation that is an equilibrium for the economy of Section 3 is also an equilibrium for the economy of Subsection A.2. ▮

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