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Equilibrium Prices When the Sunspot Variable Is Continuous[1]

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We analyze sunspot-equilibrium prices in nonconvex economies with perfect markets and a continuous sunspot variable. Our primary result is that every sunspot equilibrium allocation can be supported by prices that, when adjusted for probabilities, are constant across states. This result extends to the case of a finite number of equally-probable states under a nonsatiation condition, but does not extend to general discrete state spaces. We use our primary result to establish

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the equivalence of the set of sunspot equilibrium allocations based on a continuous sunspot variable and the set of lottery equilibrium allocations. *Journal of Economic Literature* Classification Numbers: D51, D84, E32.

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1. INTRODUCTION

Cass and Shell [2] examine the role of extrinsic uncertainty in finite, convex economies with well-behaved[2] preferences and complete markets. Among other things, they show that when market participation is unrestricted, differences in the prices of state-contingent commodities across extrinsic states of nature can be attributed solely to differences in the probabilities of the states.[3] This follows from the first-order condition of any consumer's utility maximization problem and the fact that equilibrium allocations in such economies are immune from sunspot activity. If the assumption of convexity is removed, however, equilibrium allocations can depend on sunspots even when markets are perfect and hence this pricing result is lost. The literature on nonconvex economies with perfect markets contains several examples of nontrivial sunspot equilibria;[4] in some of these examples the underlying probability-adjusted prices vary across states, while in others they do not. This leaves open the question: For general (possibly nonconvex) sunspots economies, under what conditions are probability-adjusted equilibrium prices constant across states of nature?

In nonconvex economies, the purpose of the sunspot variable is to facilitate "gambles" desired by the consumers. A gamble is obtained by contracting to consume a commodity bundle in some states but not in others. One might conjecture that, in this setting, probability-adjusted equilibrium prices will always be constant. If price differences across states are not solely due to differences in the probabilities of the states, then consumption must be cheaper in some states than in others. When this happens, all consumers will want to consume in the relatively inexpensive states and not in the relatively expensive ones, resulting in disequilibrium. This simple argument is sound if the sunspot variable is rich enough and there is only one commodity. If the number of states is small or there are many commodities, however, the argument fails. When the number of states is small, consumers are limited in their choice of gambles and consequently the

[2] By this we mean preferences that satisfy the standard assumptions of smooth consumer theory.

[3] That is, state-contingent commodity prices are colinear with the probabilities of the states. See also Balasko [1].

[4] See, for example, Shell and Wright [18], Garratt [6], Goenka and Shell [7, 8], and Sorger [19].

clustering around "cheap" states that leads to disequilibrium may not occur.[5] Even if there are many states, with many commodities it is possible that consumers will buy one commodity in states where it is relatively more expensive if a complementary good is relatively cheaper in those states.

We address this issue in sunspots economies with perfect markets and arbitrary, finite numbers of consumers and commodities. We consider both continuous and discrete sunspot variables. We begin with a continuous sunspot variable that is uniformly distributed. We later show that the uniformity assumption is without loss of generality, and hence the particular distribution of the continuous sunspot variable places no restrictions on the type of sunspot allocations that are possible in equilibrium. Assuming a continuous sunspot variable might be a natural way to model extrinsic uncertainty when there are no *a priori* reasons to restrict the possibilities in advance by specifying a particular finite state space. For a given sunspot equilibrium of our model, it is typically possible to construct a finite state space that allows the same equilibrium. This state space might then be thought of as being endogenously determined.[6]

We provide an example that demonstrates that in our model there can exist sunspot equilibria in which the probability-adjusted equilibrium prices vary across states. However, the sunspot-equilibrium allocation in our example can also be supported by prices that are constant across states. As our first main result, we show that this is a general property of the model: *When the sunspot variable is continuous, any sunspot-equilibrium allocation can be supported by constant probability-adjusted prices*. Assuming (without loss of generality) that the sunspot variable is uniformly distributed, this means that no equilibrium allocations are missed if we restrict our attention to constant price functions.

In actual economies, the number and character of risk classes for insurance and securities are often restricted by law[7] and by market considerations based on transactions costs and verification costs. Even in the absence of any specific impediments to using richer randomizing devices, individuals might coordinate on some discrete randomizing device. Consequently, it is important to study in their own right sunspots economies based on restricted state spaces. We examine sunspot-equilibrium prices in economies with a finite number of extrinsic states and demonstrate that if the states are equally likely and certain nonsatiation conditions are met, any sunspot equilibrium

[5] Shell and Wright [18] present an economy with one good and two states of nature and show that probability-adjusted equilibrium prices are not constant across the two states.

[6] Previous researchers have advocated the merits of choosing the state space endogenously or have explicitly done so. See, for example, Spear [20], Peck and Shell [13], Shell and Wright [18], Garratt [6], and Goenka and Shell [8].

[7] The government may choose to manipulate risk classes as a policy tool.

allocation can again be supported by constant prices. We provide examples that demonstrate why the result fails when these conditions are not met.

Sunspot equilibrium is one theory of stochastic allocation. We show how our main result can be applied to examine the relationship between the sunspots model and an alternative theory of stochastic allocation and trading originating with Prescott and Townsend [14, 15]. In the alternative theory, each combination of the available goods is treated as a separate commodity, and the consumption set is the set of lotteries over all such commodities. A competitive equilibrium for economies in which consumers trade these lotteries is called a lottery equilibrium. The lottery model has been used successfully to study issues of private information, indivisibilities, and spatial separation, but the model is sometimes criticized because it is difficult to imagine how trades in these lotteries would actually be made. Prescott and Townsend [15] suggest that those who are uncomfortable with the lottery model can imagine lottery allocations as coming about from trades indexed to a naturally occurring random variable—that is, as sunspot allocations of the type described in Shell [17] and Cass and Shell [2]. Shell and Wright [18] support this claim by showing that Rogerson [16] equilibrium employment lotteries can be reproduced as sunspot equilibria.

Garratt [6] investigates the relationship between sunspot equilibria and lottery equilibria in an economy with indivisible goods, and shows that for any lottery-equilibrium allocation there is an associated sunspot-equilibrium allocation, but that the converse is not necessarily true. An example is provided in which a sunspot equilibrium exists when trade of contingent commodities is restricted to three equiprobable states, but the same (stochastic) allocation is not an equilibrium in the lottery model. Why can there be sunspot equilibria that are not lottery equilibria? The state space in the sunspots model is very general: it can be either restricted or continuous. When it is restricted, the set of gambles that a consumer can construct is limited. Sometimes consumers will demand compatible gambles when they are required to coordinate on a particular finite sunspot variable, even though they would not if they had the greater flexibility to construct gambles that is available with a continuous sunspot variable. In the lottery model there is no analogous way of restricting the state space. The lottery model implicitly gives the consumer a continuous random variable with which to construct gambles, and for this reason is less general than the sunspots model.

If we look only at continuous sunspot variables, then the set of gambles available to consumers is the same as in the lottery model. This alone, however, does not guarantee that the set of continuous-state-space sunspot equilibria is equivalent to the set of lottery equilibria. This is because the two models specify prices differently. In the sunspots model, prices may be state-dependent, but in the lottery model there are no commonly observed states of nature. Hence, the equivalence can be established only if it is known that equilibria in sunspots economies with a continuum of extrinsic states do not require variable supporting prices. Using our first main result, we establish our second: *The set of sunspot-equilibrium allocations based on a continuous sunspot variable coincides with the set of lottery-equilibrium allocations.*

Our second result does not apply to sunspots models with a finite number of extrinsic states of nature, but it extends to other settings with a continuous sunspot variable. In particular, Kehoe *et al.* [11] consider exchange economies with private information (about random endowments) and indivisibilities and establish the equivalence between sunspot equilibria based on a continuous sunspot variable and lottery equilibria in this environment. The lottery model they use follows Prescott and Townsend [14, 15] in having a continuum of each type of consumer. This simplifies the specification of feasible lottery allocations since it means there is no uncertainty regarding the aggregate resource requirement of the individually-demanded lotteries. We assume that the number of consumers is finite, which implies that there *is* uncertainty about the aggregate resource requirement unless the individual lotteries are coordinated. In our model sunspots provide coordination as well as randomization, while in Kehoe *et al.* [11] only the latter is needed.

Our second result should not be interpreted as meaning that there are no significant differences between sunspot equilibria and lottery equilibria. On the contrary, there are important practical differences between them. This is particularly evident in terms of the random process upon which equilibrium allocations are based. The lottery model has no explicit, public randomizing device, implying that the auctioneer not only calls out prices, he also randomizes and assigns allocations. Even if someone can be found to play the role of the auctioneer, it is not clear that he will be credible.[8] This problem does not arise in the sunspots model, as trades are based on the outcome of a publicly observed extrinsic random variable with a distribution that is determined by forces outside the control of any of the participants in the economy.[9]

[8] In Plato's *Republic* (book V) it is recommended that a lottery be set up to assign rights to procreation, and that this lottery be rigged to the benefit of the "philosopher kings."

[9] An example comes from "numbers" betting run by organized crime. In order to gain the trust of participants, winners are determined by a publicly-observed variable that is random and almost purely extrinsic. One of the principle methods used to select numbers in New York, for instance, was based on the handle (total amount wagered) at a given racetrack. Another was based on the payouts of several specified races. For further information on the details of numbers games, see Clotfelter and Cook [3].

Sunspot equilibria can also emerge without prior agreement on the sunspot variable. For instance, sunspots methods can be used to study models where a small amount of intrinsic uncertainty has large equilibrium effects. Such "high volatility" or "overreaction" equilibria are discussed by Manuelli and Peck [12] and Woodford [21]. Furthermore, it is possible for an endogenous variable (such as a price) to play the role of the sunspot variable, so that no external random variable is required. This is suggested by Spear [20] and Jackson and Peck [10].

2. CONSTANT PRICES

We consider a pure exchange economy and begin with the case of a continuous sunspot variable. Specifically, extrinsic uncertainty in the model is represented by the probability space (S, Σ, π) generated by the extrinsic random variable. We take S to be the unit interval $[0, 1)$ and Σ to be the Borel sets on S. Continuity of the sunspot variable is equivalent to the probability measure π being absolutely continuous with respect to Lebesgue measure; it also implies that the sunspot variable can be represented by a density function ϕ. We additionally assume (without loss of generality) that $\phi(s) > 0$ for every state $s \in S$. This implies that the distribution function

$$\Phi(s) = \int_0^s \phi(z)\, dz$$

is continuous and strictly increasing, and hence invertible.

There is a finite set of consumers denoted by $N = \{1, ..., h, ..., n\}$. For each state of nature, the commodity bundle c chosen by the consumer must be contained in the set C, a Borel set in $\mathbb{R}^{\ell}_{+}$. We place no restrictions on C; however, if C is bounded our results only apply to equilibria in which no consumer receives her most-preferred allocation in almost every state. We will motivate the results with examples in which C is nonconvex, because otherwise equilibrium allocations are state-independent and our results are trivial. Nonconvexities in real economies stem from many sources, including the existence of some indivisible commodities, zero-one choices in labor markets,[10] increasing returns or transactions costs.[11]

The commodity space is the set of Σ-measurable functions $x_h: S \to \mathbb{R}^{\ell}_{+}$ that are bounded in the essential supremum (or L_∞) norm. The consumption set is the subset of functions such that $x_h(s) \in C$ for every state s; we

[10] See, e.g., Hansen [9] and Rogerson [16].

[11] See Farrell [5] for an early discussion of potential sources of nonconvexities in economics.

denote this set by X. Since the uncertainty is extrinsic, the endowment of consumer h does not depend on the state of nature and is denoted by $e_h \in C$. Each consumer is assumed to be globally nonsatiated, and each consumer's preferences are represented by a von Neumann–Morgenstern utility function $u_h\colon C \to \mathbb{R}$. The consumer's problem is to

$$\max_{x_h} \int_S u_h(x_h(s))\, \phi(s)\, ds \tag{1}$$

$$\text{subject to } \int_S p(s) \cdot x_h(s)\, \phi(s)\, ds \leqslant \int_S p(s) \cdot e_h \phi(s)\, ds$$

$$x_h \in X.$$

Notice that the integration in the budget constraint involves the density function ϕ, so that our price function p directly represents probability-adjusted prices (this distinction disappears once we assume a uniform distribution). The phrase "constant prices" therefore refers to functions p that are themselves constant, i.e., take on the same values for every state s.

When the h subscript is omitted, x refers to an aggregate allocation, $x = (x_h)_{h \in N}$. An allocation is feasible if markets clear almost surely. We use F to denote the set of feasible pure (nonstochastic) allocations of the endowments. Letting $a = (a_h)_{h \in N}$ denote a pure allocation with $a_h \in C$ for every h, we therefore have

$$F = \left\{ a \in C^n : \sum_h a_h \leqslant \sum_h e_h \right\}.$$

Feasibility can then be stated as the requirement that $x(s) \in F$ for almost every s.

Two remarks about problem (1) are in order. The first is that, under the assumption that consumers are globally nonsatiated, the budget constraint in problem (1) will hold with equality at an optimum. This is true even when C is discrete because any small amount of excess income can be used to purchase a strictly preferred bundle in an appropriately small set of states. The second is that we have restricted our attention to price systems that have an inner product representation (as depicted above), that is, functions $p\colon S \to \mathbb{R}^{\ell}_{+}$ that are Σ-measurable and bounded in the L_1 norm. We denote the set of such price systems by P.

The definition of equilibrium for the sunspots economy is standard.

DEFINITION 1. A *sunspot equilibrium* consists of a price function $p^* \in P$ and an allocation $x^* \in X^n$ such that

(i) Given p^*, x_h^* solves the consumer's problem (1) for each $h \in N$,

and

(ii) x^* is feasible, i.e., we have $x^*(s) \in F$ for almost all $s \in S$.

The specification of the sunspots model permits both allocations x^* and prices p^* to vary with s. The following example shows this specification is not superfluous by providing a nondegenerate sunspot-equilibrium allocation that is supported by prices that depend on the state of nature.

EXAMPLE 1. *Sunspot equilibrium prices that vary across states of nature.* There are two consumers, so that $N = \{1, 2\}$, and the consumption set C is given by $C = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. The state space S is the unit interval, and the density ϕ is uniform. The consumers have identical utility functions satisfying $u((1, 1)) = a > 0$ and $u(c) = 0$ for $c \neq (1, 1)$. Endowments are $e_1 = (0, 1)$ and $e_2 = (1, 0)$. A sunspot equilibrium exists with prices given by

$$p^*(s) = (2 - s, 1 + s), \qquad s \in [0, 1)$$

and the allocation given by

$$x^*(s) = \begin{cases} ((1, 1), (0, 0)), & s \in [0, \frac{1}{2}) \\ ((0, 0), (1, 1)), & s \in [\frac{1}{2}, 1). \end{cases}$$

This follows from the fact that the only desirable consumption bundle, $c = (1, 1)$, costs the same amount in each state. Each consumer has enough income to purchase this bundle in exactly half of the states, and is indifferent about which states those are. It is easy to see, however, that the allocation x^* is also supported as an equilibrium by the constant prices $p^*(s) = (\frac{3}{2}, \frac{3}{2})$, $s \in [0, 1)$.

Example 1 establishes that not all equilibrium prices are constant across states, but in this example the equilibrium allocation is also supported by constant prices. In what follows, we show that the existence of constant prices supporting a sunspot equilibrium allocation is general. We begin with the case where the probability distribution over the extrinsic states is uniform. This allows us to present the results with a minimum of notation. In Subsection 2.2, we show how the same result obtains with a general continuous random variable.

2.1. *Uniform Distribution.* Suppose the probability distribution function Φ is uniform, so that its density function ϕ satisfies $\phi(s) = 1$ for $s \in [0, 1)$. We begin by defining, for any consumption plan x_h, a family of plans that are a linear "shift" of x_h across states.

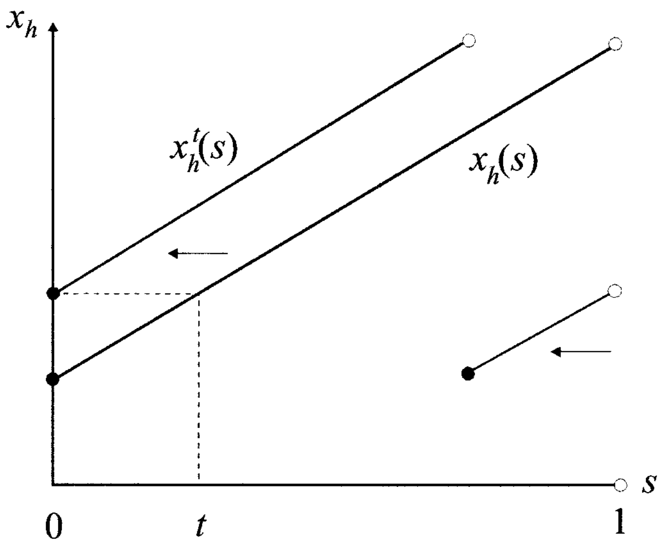

FIG. 1. An element of the shift class.

DEFINITION 2. For any $x_h \in X$, the *shift class* of x_h, denoted $T(x_h)$, is the set of functions x_h^t such that

$$x_h^t(s) = x_h(s+t) \tag{2}$$

holds for some $t \in [0, 1)$, where addition is modulo 1.

Note that every element of $T(x_h)$ is also in the consumption set X. The shift class is a family of functions that present the consumer with the same profile of consumption across states, but with a change in the actual state in which each bundle is received. Figure 1 presents a one-commodity example of x_h and a particular x_h^t.

An important property of the shift class is that a consumer will be indifferent between any two elements in the class. This statement is formalized in the following lemma.

LEMMA 1. *For any utility function u_h and any $t \in [0, 1)$, we have*

$$\int_S u_h(x_h^t(s))\, ds = \int_S u_h(x_h(s))\, ds.$$

Proof. From the definition of x^t we have

$$\int_0^1 u_h(x_h^t(s))\, ds = \int_0^{1-t} u_h(x_h(s+t))\, ds + \int_{1-t}^1 u_h(x_h(s-(1-t)))\, ds. \tag{3}$$

Using the change of variables $r = s + t$ in the first term and $r' = s - (1 - t)$ in the second term shows the right-hand side to be equal to

$$\int_t^1 u_h(x_h(r))\, dr + \int_0^t u_h(x_h(r'))\, dr'.$$

Switching the variable of integration back to s shows that we have established

$$\int_S u_h(x_h^t(s))\, ds = \int_S u_h(x_h(s))\, ds. \quad \blacksquare$$

This lemma implies that an equilibrium consumption plan must, at equilibrium prices, be a minimum-cost element of its shift class for every consumer. If this were not true, a consumer could purchase an equal-utility plan that costs less, and use the leftover income to purchase additional consumption in some set of states,[12] so that the original plan could not have been optimal. The next lemma shows that when x_h is the minimum cost element of $T(x_h)$ at prices given by p, it has an interesting property: it costs at least as much when "average" prices $Ep = \int_S p(s)\, ds$ are used in every state.

LEMMA 2. *Let $\bar{p} \in P$ denote the (degenerate) price system that satisfies $\bar{p}(s) = Ep$ for all $s \in [0, 1)$. Given $x_h \in X$, if*

$$\int_S p(s) \cdot x_h(s)\, ds \leqslant \int_S p(s) \cdot x_h^t(s)\, ds \tag{4}$$

holds for every $t \in [0, 1)$, then we have

$$\int_S p(s) \cdot x_h(s)\, ds \leqslant Ep \cdot \int_S x_h(s)\, ds. \tag{5}$$

Proof. Integrating both sides of (4) as t ranges from 0 to 1 (the left-hand side is trivial) yields

$$\int_S p(s) \cdot x_h(s)\, ds \int_0^1 dt \leqslant \int_S p(s) \cdot \left(\int_0^1 x_h(s + t)\, dt \right) ds, \tag{6}$$

where the addition is understood to be modulo 1. As in Lemma 1, applying a change of variable shows

$$\int_0^1 x_h(s + t)\, dt = \int_0^1 x_h(r)\, dr.$$

[12] Recall that the consumer is not receiving her most-preferred bundle in some states.

Substituting this into (6) yields

$$\int_S p(s) \cdot x_h(s)\, ds \leqslant \int_S p(s) \cdot \left(\int_0^1 x_h(r)\, dr \right) ds,$$

or

$$\int_S p(s) \cdot x_h(s)\, ds \leqslant \int_S p(s)\, ds \cdot \int_0^1 x_h(r)\, dr,$$

or

$$\int_S p(s) \cdot x_h(s)\, ds \leqslant Ep \cdot \int_S x_h(s)\, ds. \quad \blacksquare$$

Remark. In this model, both p and x_h are measurable functions on a probability space and are therefore random variables. For each commodity j, it is possible to examine the covariance between p^j and x_h^j, which is given by

$$\operatorname{cov}[p^j, x_h^j] = E[p^j x_h^j] - E[p^j]\, E[x_h^j],$$

where E is the expectation operator. Writing this in terms of integrals yields

$$\operatorname{cov}[p^j, x_h^j] = \int_S p^j(s)\, x_h^j(s)\, ds - Ep \cdot \int_S x_h^j(s)\, ds.$$

The result of Lemma 2 can therefore be restated as

$$\sum_{j=1}^{\ell} \operatorname{cov}[p^j, x_h^j] \leqslant 0.$$

This interpretation provides some intuition for the result. Consider, for the moment, the one-commodity case and suppose that x_h is a consumption plan in which the level of consumption varies across states. If x_h is the expenditure-minimizing element of its shift class, it will have the highest consumption levels occurring in states where consumption is least expensive, and the lowest consumption levels occurring in states where consumption is most expensive. In other words, x_h will be high when p is low and low when p is high. This is an informal way of saying that the random variables x_h and p have a covariance that is nonpositive. In the multi-commodity case, Lemma 2 says that the aggregate covariance must be nonpositive. The expenditure-minimizing consumption plan may have a positive covariance between some commodities and their prices if other commodity-price pairs have large (offsetting) negative covariances.

Lemma 2 holds for arbitrary prices p. Next, we show that at equilibrium prices, (5) must hold with equality for every consumer.

LEMMA 3. *Suppose* (p^*, x^*) *is a sunspot equilibrium and let* $Ep^* = \int_S p^*(s)\, ds$. *Then*

$$\int_S p^*(s) \cdot x_h^*(s)\, ds = Ep^* \cdot \int_S x_h^*(s)\, ds$$

holds for every $h \in N$.

Proof. Lemma 1 implies that x_h^* must be the minimal cost element of $T(x_h^*)$ under the equilibrium price function p^* for every consumer h, so that the inequality (4) in the statement of Lemma 2 is satisfied. By Lemma 2, we then have

$$\int_S p^*(s) \cdot x_h^*(s)\, ds \leqslant Ep^* \cdot \int_S x_h^*(s)\, ds$$

for every h. Suppose that the strict inequality

$$\int_S p^*(s) \cdot x_h^*(s)\, ds < Ep^* \cdot \int_S x_h^*(s)\, ds$$

held for some h. The consumer is spending all of her income,

$$\int_S p^*(s) \cdot x_h^*(s)\, ds = \int_S p^*(s) \cdot e_h\, ds,$$

and hence the strict inequality

$$\int_S p^*(s) \cdot e_h\, ds < Ep^* \cdot \int_S x_h^*(s)\, ds$$

must also hold. Adding this equation across all consumers yields

$$\int_S p^*(s) \cdot \left[\sum_{h \in N} e_h \right] ds < Ep^* \cdot \int_S \left[\sum_{h \in N} x_h^*(s) \right] ds.$$

Market clearing requires that in almost every state we have

$$\sum_{h \in N} x_h^*(s) \leqslant \sum_{h \in N} e_h.$$

It follows that the strict inequality

$$\int_s p^*(s)\, ds \cdot \sum_{h \in N} e_h < Ep^* \cdot \sum_{h \in N} e_h$$

holds, which, since $Ep^* = \int_s p^*(s)\, ds$, is a contradiction. ∎

With these lemmas in hand, we are now ready to present the first main result of the paper: When the sunspot variable is continuous, any sunspot equilibrium allocation can be supported by a price function that is constant across states.

THEOREM 1. *If* (p^*, x^*) *is a sunspot equilibrium and* $\bar{p}^*$ *is the constant price function such that* $\bar{p}^*(s) = Ep^*$ *for every* s, *then* $(\bar{p}^*, x^*)$ *is also a sunspot equilibrium.*

Proof. We prove this here for the case of a uniformly-distributed sunspot variable, and the next section shows how to generalize the proof to any continuous random variable. We only need to show that x_h^* is still an optimal choice for each consumer h when prices are given by $\bar{p}^*$. From Lemma 3 we know that x_h^* is still affordable when the price function is $\bar{p}^*$. Suppose that it is not optimal for some consumer h. Then there is another allocation $\tilde{x}_h$ that is affordable at $\bar{p}^*$ and is strictly preferred to x_h^*. Thus we would have

$$Ep^* \cdot \int_S \tilde{x}_h(s)\, ds \leqslant Ep^* \cdot e_h$$

and

$$\int_S u_h(\tilde{x}_h(s))\, ds > \int_S u_h(x_h^*(s))\, ds.$$

Let $\tilde{y}_h$ denote the minimum cost element of $T(\tilde{x}_h)$, the shift class of $\tilde{x}_h$, when prices are given by p^*. Then $\tilde{y}_h$ and $\tilde{x}_h$ cost the same amount as when prices are given by $\bar{p}^*$, and the consumer is indifferent between them. This implies that $\tilde{y}_h$ is also strictly preferred to x_h^* and

$$Ep^* \cdot \int_S \tilde{y}_h(s)\, ds \leqslant Ep^* \cdot e_h. \tag{7}$$

From Lemma 2 we know that $\tilde{y}_h$ costs no more when prices are given by p^* than when they are given by $\bar{p}^*$,

$$\int_S p^*(s) \cdot \tilde{y}_h(s)\, ds \leqslant Ep^* \cdot \int_S \tilde{y}_h(s)\, ds. \tag{8}$$

But since the endowment is constant across states, the consumer's income when prices are given by p^* is the same as when they are given by $\bar{p}^*$. Combining this with (7) and (8) then yields

$$\int_S p^*(s) \cdot \tilde{y}_h(s)\, ds \leqslant Ep^* \cdot e_h = \int_S p^*(s) \cdot e_h\, ds.$$

This shows that $\tilde{y}_h$ is also affordable when prices are given by p^*. The fact that $\tilde{y}_h$ is strictly preferred to x_h^* then contradicts the hypothesis that x_h^* is optimal for the consumer when prices are given by p^*. ∎

The converse of this theorem is not true. If $(\tilde{p}^*, x^*)$ is a sunspot equilibrium, where $\tilde{p}^*$ is a constant price function, and p^* is a price system such that $\int_S p^*(s)\, ds = \tilde{p}^*(t)$, $t \in [0, 1)$, it is *not* necessarily true that (p^*, x^*) is also a sunspot equilibrium. The problem is that, for a given consumer h, it may not be the case that x_h^* is the minimal cost element of $T(x_h^*)$ at prices p^*. An allocation x^* can only be supported by prices that satisfy this property for every consumer.

2.2. *General Distributions.* In this section, we show how to modify the argument in Subsection 2.1 to allow for a general continuous sunspot variable. A crucial element in the argument was that shifting consumption bundles across states in a linear fashion produced an equal-utility allocation. When the probability distribution is not uniform, this is clearly no longer the case. However, as we show below, an appropriate nonlinear shift does work. The necessary transformation is provided by the distribution function Φ.

Recall that we assumed that the distribution function Φ induced by the probability measure π is invertible. By applying the transformation Φ to the set S, we move to a space where the probability distribution is given by Lebesgue measure, i.e., is uniform. There a linear shift, as used in Subsection 2.1, preserves the probability of each event. Finally, the inverse transformation Φ^{-1} must be applied to return the set to the original space. Formally, note that for any set $A \in \Sigma$ (the Borel subsets of S), we have the relation $\pi(A) = \mu(\Phi(A))$, where μ is Lebesgue measure and $\Phi(A) = \{s' : s' = \Phi(s) \text{ for some } s \in A\}$. Therefore, we have

$$\pi(\Phi^{-1}(\Phi(A) + t)) = \mu(\Phi(A) + t) = \mu(\Phi(A)) = \pi(A), \tag{9}$$

so that the transformation $A \mapsto \Phi^{-1}(\Phi(A)+t)$ described above preserves probability. We use this to define the family of alternate consumption plans, which is a generalization of (2).

DEFINITION 2′. For any $x_h \in X$ and any $t \in [0, 1)$, the *shift class* of x_h, denoted $T(x_h)$, is the set of functions x_h^t such that

$$x_h^t(s) = x_h(\Phi^{-1}(\Phi(s)+t))$$

holds for some $t \in [0, 1)$, where the addition is modulo 1 and Φ is the distribution function of π.

The next step is to show the analog of Lemma 1, that a consumer must in fact be indifferent between any of the elements of this family.

LEMMA 1′. *For any utility function* u_h *and any* $t \in [0, 1)$, *we have*

$$\int_S u_h(x_h^t(s))\,\phi(s)\,ds = \int_S u_h(x_h(s))\,\phi(s)\,ds.$$

Proof. Both $x_h(s)$ and $x_h^t(s)$ induce a probability distribution over the set C. We will show that these two distributions are the same. Since a consumer with von Neumann–Morgenstern utility only cares about the distribution over C, these two allocations must provide the same utility level.

Let A be any Borel set in C. Then we have

$$\begin{aligned}
\mathrm{Prob}(x_h^t(s) \in A) &= \mathrm{Prob}(x_h(\Phi^{-1}(\Phi(s)+t)) \in A) \\
&= \mathrm{Prob}(\Phi^{-1}(\Phi(s)+t) \in x_h^{-1}(A)) \\
&= \mathrm{Prob}((\Phi(s)+t) \in \Phi(x_h^{-1}(A))) \\
&= \pi(\Phi^{-1}(\Phi(x_h^{-1}(A))-t)) \\
&= \pi(x_h^{-1}(A)) \\
&= \mathrm{Prob}(x_h(s) \in A),
\end{aligned}$$

where the penultimate equality holds by (9). Therefore x_h^t and x_h induce the same distribution over C. ▮

With this result, we can prove the generalization of Lemma 2.

LEMMA 2′. *Let* $E^\phi p = \int_S p(s)\,\phi(s)\,ds$. *If*

$$\int_S p(s)\cdot x_h(s)\,\phi(s)\,ds \leqslant \int_S p(s)\cdot x_h^t(s)\,\phi(s)\,ds \tag{10}$$

holds for every $t \in [0, 1)$, *then we have*

$$\int_S p(s) \cdot x_h(s)\, \phi(s)\, ds \leqslant E^\phi p \cdot \int_S x_h(s)\, \phi(s)\, ds.$$

Proof. The proof proceeds parallel to that of Lemma 2. Integrate both sides of (10) over t with respect to Lebesgue measure to get

$$\int_S p(s) \cdot x_h(s)\, \phi(s)\, ds \leqslant \int_S p(s) \cdot \left(\int_0^1 x_h^t(s)\, dt \right) \phi(s)\, ds. \tag{11}$$

Next, note that

$$\int_0^1 x_h^t(s)\, dt = \int_0^1 x_h(\Phi^{-1}(\Phi(s) + t))\, dt,$$

where the addition is again understood to be modulo 1. Writing the right-hand-side out explicitly yields

$$\int_0^{1-\Phi(s)} x_h(\Phi^{-1}(\Phi(s) + t))\, dt + \int_{1-\Phi(s)}^1 x_h(\Phi^{-1}[t - (1 - \Phi(s))])\, dt.$$

Through a change of variables identical to that used in (3), this becomes

$$\int_0^1 x_h(\Phi^{-1}(r))\, dr.$$

Now we use the final change of variables $z = \Phi^{-1}(r)$ (which implies $dr = \phi(z)\, dz$) so that this becomes

$$\int_0^1 x_h(z)\, \pi(dz).$$

Substituting this into (11) yields the desired result,

$$\int_S p(s) \cdot x_h(s)\, \phi(s)\, ds \leqslant E^\phi p \cdot \int_S x_h(s)\, \phi(s)\, ds. \quad \blacksquare$$

With these two lemmas, the proofs of Lemma 3 and Theorem 1 for general densities proceed exactly as in the case of uniform densities and are hence omitted. What this section has demonstrated is that assuming a uniform randomizing device is without loss of generality within the class of continuous devices.

2.3. *Finite State Spaces.* In this section, we show that the conclusion of Theorem 1 does *not* necessarily hold when there is only a finite number of states of nature. We begin by presenting an example to demonstrate this.[13] The economy in this example has two sunspot states and an equilibrium allocation that cannot be supported by any constant price function.

EXAMPLE 2. *A sunspot equilibrium allocation that cannot be supported by constant prices.* There are two consumers and the consumption set C is given by $C = \{c \in \mathbb{Z}^2_+ : c^1 \leqslant 2,\ c^2 \leqslant 2\}$, where $\mathbb{Z}$ is the set of integers. There are two extrinsic states of nature, 1 and 2, with $\pi(1) = \frac{1}{3}$ and $\pi(2) = \frac{2}{3}$. The consumers have identical utility functions $u_h \colon C \to \mathbb{R}$, satisfying $u_h(0, 0) = 0$, $u_h(1, 0) = u_h(0, 1) = 0.1$, $u_h(1, 1) = u_h(2, 0) = u_h(0, 2) = 2$, $u_h(2, 1) = u_h(1, 2) = 2.1$, and $u_h(2, 2) = 3$ for $h = 1, 2$. Endowments are $e_1 = (1, 0)$ and $e_2 = (0, 1)$. In this case, the sunspot allocation

$$(x_1^*(1), x_1^*(2)) = ((0, 0), (1, 1)), \qquad (x_2^*(1), x_2^*(2)) = ((1, 1), (0, 0))$$

is supported as an equilibrium by the prices

$$p^*(1) = (2, 1), \qquad p^*(2) = (1, 1).$$

Furthermore, this allocation cannot be supported by constant prices. To see this, note that consumer 1 prefers to consume

$$(\hat{x}_1(1), \hat{x}_1(2)) = ((1, 0), (0, 2)),$$

so that any prices supporting the allocation x^* must be such that $\hat{x}_1$ is not affordable. This implies that we must have

$$\pi(1)\, p^{1*}(1) + 2\pi(2)\, p^{2*}(2) > \pi(1)\, p^{1*}(1) + \pi(2)\, p^{1*}(2), \tag{12}$$

where the right-hand side of the equation is consumer 1's income. Prices must also be such that x_1^* is affordable, so we must have

$$\pi(2)(p^{1*}(2) + p^{2*}(2)) \leqslant \pi(1)\, p^{1*}(1) + \pi(2)\, p^{1*}(2). \tag{13}$$

Using the probability values given above, (12) reduces to $2p^{2*}(2) > p^{1*}(2)$, while (13) implies $p^{1*}(1) \geqslant 2p^{2*}(2)$. Combining these yields

$$p^{1*}(1) > p^{1*}(2),$$

so that the price of good 1 cannot be constant across the two states.

[13] Shell and Wright [18] present another such example, with a single commodity and endowments outside of the consumption set.

When there is a finite number of states, two of the key ingredients in the proof of Theorem 1 are lost. The first and most obvious problem is that unless all of the states are equally likely, the concept of the shift class of a consumption plan is no longer meaningful. This problem is evident in the above example: it would be cheaper for consumer 1 to buy her consumption bundle in the first state instead of the second, but this would also yield lower expected utility since the first state has a smaller probability of occurrence. The second problem is that, even when the states of nature are equally likely, it is not necessarily true that the equilibrium consumption plan of a consumer is the minimal cost element of the shift class or that the consumer spends all of her income. When the consumption set is discrete, a consumer can be indifferent between choosing two equal-utility plans with different costs because the income saved by purchasing the cheaper one cannot be used to raise utility in any way.

We now provide conditions under which the existence of constant supporting prices is guaranteed for the finite-state-space economy. These conditions simply amount to ruling out the two problems mentioned above. We require that the states be equally likely, so that shift classes can be meaningfully defined, and that each consumer's equilibrium consumption plan be the minimal-cost element (at equilibrium prices) of its shift class and exhaust her budget. We then briefly discuss restrictions on preferences and the consumption set that will guarantee this second property.

The notion of a shift class must be redefined for finite state spaces. This is done in a way that is exactly analogous to the continuous case. Denote the finite set of extrinsic states by $S = \{1, ..., s, ..., m\}$. For any consumption plan x_h, we define $T(x_h)$ to be the set of functions x_h^t such that

$$x_h^t(s) = x_h(s+t)$$

holds for some $t \in \{1, ..., m\}$, where the addition is modulo m. The shift class is now a finite set with m elements.

THEOREM 2. *Suppose there are m extrinsic states of nature, all of which are equally likely. Suppose* (p^*, x^*) *is a sunspot equilibrium with the property that* x_h^* *is the minimal cost element of* $T(x_h^*)$ *and*

$$\frac{1}{m} \sum_{s \in S} p^*(s) \cdot x_h^*(s) = \frac{1}{m} \sum_{s \in S} p^*(s) \cdot e_h$$

holds for every h. If $\bar{p}^*$ *is the constant price function such that*

$$\bar{p}^*(s) = \frac{1}{m} \sum_{s' \in S} p^*(s')$$

holds for all s, then $(\bar{p}^*, x^*)$ *is also a sunspot equilibrium.*

TABLE I

Equilibrium Allocation for Example 3

	Income	Consumption	Cost
Consumer 1	1.1	$(x_1^*(1), x_1^*(2)) = ((0, 0), (1, 1))$	1.1
Consumer 2	1.0	$(x_2^*(1), x_2^*(2)) = ((1, 1), (0, 0))$	1.0

The proof of this theorem follows that of Theorem 1 very closely and is thus omitted.[14] The property that x_h^* is the minimal cost element of $T(x_h^*)$ and exhausts consumer h 's budget is guaranteed if the consumption set combines with preferences to generate local nonsatiation. This will be the case, for instance, if there is at least one divisible good and utility is strictly increasing in that good.

The following two examples illustrate the importance of the assumptions in Theorem 2. Example 3 shows that the conclusion of the theorem may not hold if the equilibrium consumption plan of some consumer is not the minimal cost element of its shift class. Example 4 shows that the conclusion of the theorem may not hold if some consumer does not exhaust her budget.

EXAMPLE 3. *An equilibrium where average prices do not support the allocation because one consumer's consumption is not the minimal cost element.* The setup is exactly as in Example 2 except we now have $\pi(1) = \pi(2) = \frac{1}{2}$. Consider the price function p^* with

$$p^*(1) = (1, 1), \qquad p^*(2) = (1.2, 1).$$

These prices support the allocation given in Table I as a sunspot equilibrium.

In the specified sunspot equilibrium, each consumer spends all of her income. However, x_1^* is not the minimal cost consumption plan in $T(x_1^*)$. Consumer 1 could demand the consumption plan $((1, 1), (0, 0))$ instead of x_1^* and get the same utility with 0.1 units of income to spare. However, this extra income cannot be used to raise utility in any way, and therefore x_1^* is an optimal choice for consumer 1.

The sunspot equilibrium allocation in this example cannot be supported by the constant price function $\bar{p}^*$ which equals average prices $Ep^* = (1.1, 1)$ in both states. At these prices, the income of consumer 2 is still 1.0, but the cost of the plan $((1, 1), (0, 0))$ has increased to 1.05, so that it is no longer affordable.[15]

[14] The hypothesis of Lemma 2 holds by assumption. From that point on, the proof follows that in Subsection 2.1 line-by-line with the replacement of integrals by sums.

[15] Note that even though the allocation in Example 3 cannot be supported by $\bar{p}^*$, it can be supported by the constant price system $\hat{p}(s) \equiv (1, 1)$ each $s \in [0, 1)$.

TABLE II

Equilibrium Allocation for Example 4

	Income	Consumption	Cost
Consumer 1	1.7	$(x_1^*(1), x_1^*(2)) = ((0, 0), (0, 1))$	1.5
Consumer 2	3.6	$(x_2^*(1), x_2^*(2)) = ((1, 1), (0, 0))$	2.6

EXAMPLE 4. *An equilibrium where average prices do not support the allocation because consumers do not exhaust their budgets.* There are two consumers and the consumption set C is the lattice given by $C = \mathbb{Z}_+^2$. There are two extrinsic states of nature, 1 and 2, with $\pi(1) = \pi(2) = \frac{1}{2}$. The consumers have utility functions with $u_1(c) = c^2$, for all $c \in C$, and $u_2(c) = \min\{c^1, c^2\}$, for all $c \in C$. Endowments are $e_1 = (1, 0)$ and $e_2 = (0, 1)$. In this example, the prices

$$p^*(1) = (1, 4.2), \qquad p^*(2) = (2.4, 3)$$

support the allocation given in Table II as a sunspot equilibrium.

The allocation is clearly feasible and each consumer's consumption plan is affordable. Furthermore, neither consumer can afford a preferred consumption plan. Consumer 1 cannot afford 2 units of commodity 2 in either state. Since consumer 1 only cares about consumption of commodity 2, x_1^* is the optimal choice. Consumer 2 must consume additional units of both commodities to increase her utility, but she cannot afford the bundle $(2, 2)$ in either state. Thus x_2^* is optimal for consumer 2.

Notice that x_h^* is the minimal cost consumption plan in $T(x_h^*)$ for both consumers. However, neither consumer spends all of her income. Once again, the equilibrium allocation x^* cannot be supported by the constant price function $\bar{p}^*$ which equals average prices $Ep^* = (1.7, 3.6)$ in both states. At these prices, the consumption plan of consumer 1 costs 1.8 while his income is still 1.7. Hence, the allocation is no longer affordable for him.

Examples 3 and 4 show that the assumptions in Theorem 2 are necessary for deriving the result. The analysis here also highlights the two properties of a continuous sunspot variable that are crucial for the result in Theorem 1. First, a continuous random variable places strong restrictions on equilibrium prices since it allows consumers to generate many equal-utility allocations by shifting consumption plans across states. Second, it generates local nonsatiation, even when the consumption set is discrete. These two properties combine to guarantee that every sunspot equilibrium allocation can be supported by prices that are constant across states.

3. SUNSPOT-EQUILIBRIUM AND LOTTERY-EQUILIBRIUM ALLOCATIONS COINCIDE

In this section, we demonstrate that, for a given set of economic fundamentals, the set of sunspot equilibrium allocations based on a continuous random variable coincides with the set of lottery equilibrium allocations. This allows us to interpret the lottery model, which abstracts from issues regarding the implementation of stochastic allocations, as a reduced form of the continuous-state-space sunspots model. To analyze lottery equilibria, we let $\Delta(C)$ denote the set of probability measures over the set C. An individual lottery for consumer h is an element of $\Delta(C)$, δ_h: $\mathbb{B}(C) \to \mathbb{R}_+$, where $\mathbb{B}(C)$ denotes the Borel subsets of a given set C. Let δ denote a vector of individual lotteries, $\delta = (\delta_h)_{h \in N}$.

Let $q \in \mathbb{R}^\ell_+$ denote a vector of prices. Then consumer h's lottery-choice problem is given by

$$\max_{\delta_h} \int_C u_h(c)\, \delta_h(dc) \tag{14}$$

$$\text{subject to } q \cdot \int_C c\, \delta_h(dc) \leqslant q \cdot e_h$$

$$\delta_h \in \Delta(C).$$

Note that the distribution δ_h need not be absolutely continuous with respect to Lebesgue measure. A wide class of examples will involve solutions to the lottery choice problem that have countable support. For instance, this will be true if $C = \mathbb{R}^{\ell_1}_+ \times \mathbb{Z}^{\ell_2}_+$ and the utility function u_h is strictly concave on $\mathbb{R}^{\ell_1}_{++}$. Loosely speaking, risk averse consumers will not engage in unnecessary randomization, and hence the support of the chosen distribution tends to be small.

Feasibility of an allocation requires that each individually-demanded lottery be the marginal of some joint probability distribution over feasible pure (nonstochastic) allocations of the endowments (see Garratt [6]). Recall that $F = \{a \in C^n : \sum_h a_h \leqslant \sum_h e_h\}$ denotes the set of feasible pure allocations. Let $\Delta(F)$ denote the set of probability measures over the set F; elements of this set are called joint lotteries and are denoted by L: $\mathbb{B}(F) \to \mathbb{R}_+$.

DEFINITION 3. A *lottery equilibrium* is a price vector $q^* \in \mathbb{R}^\ell_+$ and an allocation δ^* such that

(i′) Given q^*, δ^*_h solves the consumer's lottery problem (14) for each $h \in N$, and

(ii′) There exists a joint lottery $L^* \in \Delta(F)$ such that $L^*(a\colon a_h \in A) = \delta_h^*(A)$ for any set $A \in \mathbb{B}(C)$ and for every h.

Our goal in this section is to compare the set of equilibrium allocations of the lottery economy with the set for the corresponding continuous-state-space sunspots economy. Since a lottery allocation is an element of $\Delta(C)^n$ and a sunspot allocation is an element of X^n, we must perform a translation in order to make this comparison. To do this, we first note that every sunspot allocation induces a probability distribution over F, and therefore induces a lottery allocation. At the individual level, the consumption plan x_h induces the individual lottery δ_h through the equation

$$\delta_h = \pi \circ x_h^{-1}. \tag{15}$$

This says that the probability assigned by δ_h to consumption bundles in any set $A \in \mathbb{B}(C)$ is equal to the probability assigned by π to the set of states in which the sunspot consumption bundle is in A. Note that since x_h is measurable, the inverse mapping satisfies $x_h^{-1}(A) \in \mathbb{B}(S)$, so that δ_h is well defined. It is easy to verify that δ_h is a probability measure over C, and hence an individual lottery for consumer h. A sunspot allocation x then induces a lottery allocation δ simply by inducing the individual lottery δ_h for each consumer h. The first part of our equivalence result can be stated as follows: Every sunspot equilibrium allocation induces a unique lottery equilibrium allocation.

For the reverse of this procedure, translating lottery allocations into sunspot allocations, we must show that every individual lottery δ_h is induced by some sunspot consumption plan x_h through (15). This is done in Lemma 4. The second half of our equivalence result can then be stated as: Every lottery equilibrium allocation is induced by a sunspot equilibrium allocation that is unique up to a relabeling of the states of nature.

We begin the analysis by stating and proving the lemma. We then formalize the relationship between sunspot and lottery allocations by defining a mapping between them. We discuss some of the properties of this mapping, and then state the precise equivalence result in terms of this mapping in Theorem 3.

Lemma 4. *For any individual lottery δ_h, there exists a sunspot consumption plan x_h such that $\delta_h = \pi \circ x_h^{-1}$.*

Proof. We make use of the following standard result (see, for instance, Durrett [4, p. 33]): If C is a Borel subset of a complete, separable metric space (e.g., $\mathbb{R}^{\ell}_{+}$), then there exists a one-to-one map f from C into $\mathbb{R}$ such that both f and f^{-1} are Borel measurable.

The significance of this result for us is that it implies that any probability distribution on the consumption set can be generated by some distribution on the real line, via the function f^{-1}. This removes the issue of dimension from the problem. What remains to be shown is that any such distribution on the real line can be generated by the given sunspot distribution π together with some function g on the unit interval. The sunspot consumption plan x_h will then be a composition of the functions f^{-1} and g.

For simplicity, we will assume that π is uniform over $[0, 1)$. We begin with δ_h, a probability distribution on C. This combines with the function f given by the above result to induce a distribution over $\mathbb{R}$ that is given by

$$\theta = \delta_h \circ f^{-1}. \tag{16}$$

Let Ψ denote the distribution function associated with θ, so that we have $\Psi\colon \mathbb{R} \to [0, 1)$ with

$$\Psi(z) = \theta((-\infty, z]). \tag{17}$$

We want a function $g\colon [0, 1) \to \mathbb{R}$ such that g and π combine to generate the distribution θ. This function is essentially the inverse of Ψ, where the inverse is defined carefully to deal with possible discontinuities and flat regions of Ψ,

$$g(s) = \sup\{z\colon \Psi(z) < s\}.$$

It is a standard result that g so defined combines with π to induce the distribution θ. To see this, note that for any $z \in \mathbb{R}$, we have

$$\pi(\{s\colon s \leqslant \Psi(z)\}) = \Psi(z),$$

since π is uniform over $[0, 1)$. Using (17), this implies that

$$\theta((-\infty, z]) = \pi(\{s\colon s \leqslant \Psi(z)\}).$$

Therefore, the function g induces the distribution θ if and only if we have

$$\{s\colon s \leqslant \Psi(z)\} = \{s\colon g(s) \leqslant z\}. \tag{18}$$

Suppose that $s \leqslant \Psi(z)$ and hence is in the left-hand set. Then $z \notin \{w\colon \Psi(w) < s\}$, so that $g(s) \leqslant z$ and s is in the right-hand set as well. Next suppose that $s > \Psi(z)$ and hence is not in the left-hand set. Then, since Ψ is a distribution function and thus right continuous, there exists an $\varepsilon > 0$ such that $\Psi(z + \varepsilon) < s$, and hence $g(s) \geqslant z + \varepsilon > z$, so that s is not in the right-hand set. Therefore the equality (18) holds, and $\theta = \pi \circ g^{-1}$. Another way to state this is that for any set $Z \in \mathbb{B}(\mathbb{R})$, we have

$$\theta(Z) = \pi(g^{-1}(Z)). \tag{19}$$

Now consider the function x_h: $S \to C$ defined by

$$x_h(s) = f^{-1}(g(s)). \tag{20}$$

Since f^{-1} and g are both measurable, so is x_h. This combined with the fact that the range of x_h is C implies that x_h is a sunspot consumption plan. We will now show that x_h induces δ_h, i.e.,

$$\delta_h = \pi \circ x_h^{-1}.$$

For any set $B \in \mathbb{B}(C)$, by (20) we have

$$\pi(x_h^{-1}(B)) = \pi(g^{-1} \circ f(B)).$$

Using (19), this is equal to

$$\theta(f(B)),$$

which by the definition of θ in (16) is equal to

$$\delta_h(f^{-1}(f(B))).$$

Since f is one-to-one, this reduces to

$$\delta_h(B).$$

This establishes the equality

$$\delta_h(B) = \pi(x_h^{-1}(B)),$$

and thus that x_h induces δ_h, as desired. ▮

This lemma shows that one continuous random variable (as opposed to ℓ continuous random variables) provides sufficient randomization to cover all symmetric-information lotteries. Therefore the specification of the unit interval as the state space in the sunspots model is without loss of generality. Note that the lemma states that every individual lottery is induced by *at least one* state-contingent consumption plan. There is an important sense in which any two plans that induce the same distribution over the consumption set are equivalent, since a consumer only cares about this distribution, not about the actual states in which each bundle is received. For this reason, we say that two sunspot allocations x and $\hat{x}$ are *identical up to a relabeling of the states of nature* if for every consumer they generate the same distribution over C, that is, if

$$\pi \circ x_h^{-1} = \pi \circ \hat{x}_h^{-1}, \qquad \text{for all } h. \tag{21}$$

This binary relation divides the set of sunspot allocations into equivalence classes, with all of the members of a given class inducing the same joint lottery. Since allocations from different equivalence classes induce different joint lotteries, we have a one-to-one relationship between the set of equivalence classes of sunspot allocations and the set of lottery allocations. Formally, we let R denote an equivalence class of sunspot allocations and $\mathscr{R}$ denote the set of all such classes. We then define the function $\Lambda: \mathscr{R} \to (\Delta(C))^n$ by

$$\Lambda(R) = (\pi \circ x_1^{-1}, ..., \pi \circ x_n^{-1}) \tag{22}$$

for any $x \in R$ (this is the same for all $x \in R$ by (21)). As we noted above, Λ is one-to-one by definition. From Lemma 4, we also have that Λ is onto. Therefore, Λ is a bijection between $\mathscr{R}$ and $(\Delta(C))^n$.

We now present the main result of this section. We show that if x^* is a sunspot equilibrium allocation, then so is every member of the equivalence class of x^*. We then show that the image of the set of equivalence classes of sunspot equilibrium allocations through Λ is exactly the set of lottery equilibrium allocations. This means that every sunspot equilibrium allocation induces a unique lottery equilibrium allocation, and, conversely, every lottery equilibrium allocation is induced by a sunspot equilibrium allocation that is unique up to a relabeling of the states of nature. In this sense, the set of sunspot equilibrium allocations and the set of lottery equilibrium allocations are the same.

THEOREM 3. *The mapping Λ is a bijection between the set of lottery equilibrium allocations and the set of equivalence classes of sunspot equilibrium allocations based on a continuous sunspot variable.*

Proof. We have already shown that Λ is bijective between $\mathscr{R}$ and $(\Delta(C))^n$, so we only need to show that R is an equivalence class containing a sunspot equilibrium allocation if and only if $\Lambda(R)$ is a lottery equilibrium allocation, and that in this case all the elements of R are sunspot equilibrium allocations. Let $\mathscr{A}$ denote a generic element of $\mathbb{B}(F)$. Using (22), it suffices to show that x^* is a sunspot equilibrium allocation if and only if L^* given by

$$L^*(\mathscr{A}) = \pi(\{s : x^*(s) \in \mathscr{A}\}), \qquad \text{for all} \quad \mathscr{A} \in \mathbb{B}(F) \tag{23}$$

is the joint lottery associated with a lottery equilibrium allocation. This, in turn, follows from the above lemmas and the application of the standard change of variable formula,[16]

$$\int_{\Omega} f(h(\omega))\, P(d\omega) = \int_{Y} f(y)\, P \circ h^{-1}(dy). \tag{24}$$

We proceed by showing that conditions (i) and (i′) are equivalent and conditions (ii) and (ii′) are equivalent.

Step 1. (i) $\Leftrightarrow$ (i′). Theorem 1 states that if x^* is a sunspot equilibrium allocation, then there exists $q^* \in \mathbb{R}^{\ell}_{+}$ such that x^*_h solves

$$\max_{x_h} \int_S u_h(x_h(s))\, \phi(s)\, ds \tag{25}$$

$$\text{subject to } q^* \cdot \int_S x_h(s)\, \phi(s)\, ds \leqslant q^* \cdot e_h$$

$$x_h \in X,$$

for every consumer h. Applying the change of variable formula (24), this can be rewritten as

$$\max_{x_h} \int_C u_h(c)\, \pi \circ x_h^{-1}(dc)$$

$$\text{subject to } q^* \cdot \int_C c\pi \circ x_h^{-1}(dc) \leqslant q^* \cdot e_h$$

$$x_h \in X.$$

Then using the definition $\delta^*_h = \pi \circ x^{*\,-1}_h$, this becomes

$$\max_{\delta_h} \int_C u_h(c)\, \delta_h(dc)$$

$$\text{subject to } q^* \cdot \int_C c\ \delta_h(dc) \leqslant q^* \cdot e_h$$

$$\delta_h \in \Delta(C).$$

This establishes that x^*_h solves the sunspots problem (25) if and only if $\delta^*_h = \pi \circ x^{*\,-1}_h$ solves the lottery problem (14).

[16] See, for instance, Durrett [4], p. 18.

Step 2(a). (ii) $\Rightarrow$ (ii′). From (ii) we have $x^*(s) \in F$ for almost all s in S, or

$$\pi(s: x^*(s) \in F) = 1.$$

This shows that the L^* induced by π and x^* through (23) is an element of $\Delta(F)$. Furthermore, for any h and for any $A \in \mathbb{B}(C)$, the definition of δ_h^* in (15) implies

$$\delta_h^*(A) = \pi \circ x_h^{*\,-1}(A) = \pi(s: x_h^*(s) \in A) = L^*(a: a_h \in A).$$

Hence, (δ_h^*) are the marginal distributions of L^* and (ii′) is satisfied.

Step 2(b). (ii′) $\Rightarrow$ (ii). Since $L^* \in \Delta(F)$, any x^* that induces L^* through (23) will have

$$\pi(s: x^*(s) \in F) = 1,$$

so that markets clear almost surely and (ii) holds. ∎

4. CONCLUSIONS

We examine exchange economies with a finite number of goods and establish that when the sunspot variable is continuous, any sunspot equilibrium allocation can be supported by constant probability-adjusted prices. In fact, for any nonconstant supporting price system, its mean (over states) is a constant supporting price system. This result, combined with our finding that one can without loss of generality assume a uniform density on the extrinsic state space, simplifies the specification of the consumer's problem in the sunspots model. Instead of keeping track of the ℓ price functions $p^1(s), ..., p^\ell(s)$, we need only determine relative prices in one state.

We use this result to demonstrate that introducing trade in lotteries leads to the same set of equilibrium allocations as does introducing trade in state-contingent consumption plans based on a continuous random variable. This means that the sunspots model can be viewed as an explicit implementation of the lottery model. Conversely, many (equivalent up to a relabelling of states) sunspot equilibria correspond to the same lottery equilibrium. Thus the lottery model can be viewed as a reduced form of the continuous-state-space sunspots model. This has implications for computing sunspot equilibrium. For example, when the consumption set is finite, the consumer's lottery choice problem is a linear programming problem. Much is known about the characteristics of solutions to such problems, and this information is useful in determining the characteristics of equilibrium outcomes.

In the present paper, the role for sunspots arises from nonconvexities in consumers' choice sets. Economies with convex choice sets (and perfect markets and strictly convex preferences) are not immune to sunspot activity. Sorger [19] and Goenka and Shell [7] provide examples of non-degenerate sunspot equilibria in economies with convex preferences and non-convex production technologies. Extending the present analysis to include production should be straightforward. Indeed, Goenka and Shell [7] formalize the intimate relationship between nonconvexities in consumption and nonconvexities in production, which had already been exploited in macroeconomic applications (see, e.g., Hansen [9] and Rogerson [16]).

REFERENCES

1. Y. Balasko, Extrinsic uncertainty revisited, *J. Econ. Theory* **31** (1983), 203–210.
2. D. Cass and K. Shell, Do sunspots matter? *J. Polit. Econ.* **91** (1983), 193–227.
3. C. Clotfelter and P. Cook, "Selling Hope: State Lotteries in America," Harvard Univ. Press, Cambridge, MA, 1989.
4. R. Durrett, "Probability: Theory and Examples," 2nd ed., Duxbury, Belmont, CA, 1996.
5. M. Farrell, The convexity assumption in the theory of competitive markets, *J. Polit. Econ.* **67** (1959), 377–391.
6. R. Garratt, Decentralizing lottery allocations in markets with indivisible commodities, *Econ. Theory* **5** (1995), 295–313.
7. A. Goenka and K. Shell, When sunspots don't matter, *Econ. Theory* **9** (1997), 169–178.
8. A. Goenka and K. Shell, Robustness of sunspot equilibria, *Econ. Theory* **10** (1997), 79–98.
9. G. Hansen, Indivisible labor and the business cycle, *J. Monetary Econ.* **16** (1985), 309–328.
10. M. Jackson and J. Peck, Speculation and price fluctuations with private, extrinsic signals, *J. Econ. Theory* **55** (1991), 274–295.
11. T. Kehoe, D. Levine, and E. Prescott, Lotteries, sunspots, and incentive constraints, *J. Econ. Theory* **107** (2002), 35–65, doi:10.1006/jeth.2001.2792.
12. R. Manuelli and J. Peck, Exchange rate volatility in an equilibrium asset pricing model, *Int. Econ. Rev.* **31** (1990), 559–574.
13. J. Peck and K. Shell, Market uncertainty: Correlated and sunspot equilibria in imperfectly competitive economies, *Rev. Econ. Stud.* **58** (1991), 1011–1029.
14. E. Prescott and R. Townsend, Pareto optima and competitive equilibria with adverse selection and moral hazard, *Econometrica* **52** (1984), 21–45.
15. E. Prescott and R. Townsend, General competitive analysis in an economy with private information, *Int. Econ. Rev.* **25** (1984), 1–20.
16. R. Rogerson, Indivisible labor, lotteries and equilibrium, *J. Monet. Econ.* **21** (1988), 3–16.
17. K. Shell, Monnaie et allocation intertemporelle, CNRS Séminaire Roy-Malinvaud, Paris, November 1977. [Title and abstract in French, text in English]
18. K. Shell and R. Wright, Indivisibilities, lotteries, and sunspot equilibria, *Econ. Theory* **3** (1993), 1–17.
19. G. Sorger, Convexity and sunspots: Another remark, *J. Econ. Theory* **60** (1993), 439–442.
20. S. Spear, Are sunspots necessary? *J. Polit. Econ.* **97** (1989), 965–973.
21. M. Woodford, Stationary sunspot equilibria: The case of small fluctuations around a deterministic steady state, mimeo, University of Chicago, 1986.