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Capital gains

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We analyze a simple overlapping-generations model with two capital goods. The dynamical system is defined by savings behavior and short-run perfect-foresight asset-market clearing. Because lifetimes are finite, there is no transversality condition. If there is a bubble in asset pricing, it will burst in finite time: expectations will eventually be frustrated, but this might take several generations. This raises the question of whether (infinite) long-run perfect foresight is a reasonable assumption for overlapping-generations economies and, hence, whether bursting bubbles can occur in equilibrium.

Key words bubbles, capital gains, heterogeneous capital, irreversible investment, overlapping generations, Tobin's q

JEL classification E21, E22, E32, E44

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1 Introduction

Capital gains play an essential role in capitalist economies. Changes in asset prices signal anticipated changes in relative scarcities. Capital gains can, however, fuel self-perpetuating bubbles, some of which will eventually burst.

We need a dynamic general equilibrium model with at least two assets to analyze capital gains. We follow the two-capital growth model of Shell and Stiglitz (1967),[1] where given the initial endowment of two capital goods and labor there is one and only one assignment of initial prices that is consistent with long-run balanced growth, whenever the momentary equilibrium is not unique there is one and only one allocation of investment consistent with long-run balanced growth, and on trajectories not tending to the balanced growth path the price of one of the capital goods becomes zero in finite time.

Shell and Stiglitz make an assumption that is now old-fashioned: an aggregate consumption function ungrounded in consumer optimization. In the present paper, we update their

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[1] See also Shell *et al.* (1969), Caton and Shell (1971), Burmeister *et al.* (1973), Shell (1972) and Burmeister and Graham (1974).

model by positing instead utility-maximizing individuals in an overlapping-generations (OG) model *a la* Diamond (1965) but extended to allow for the two capital goods. We also assume that the two capital goods, once installed, cannot be directly consumed or changed into the other type of capital. Therefore, investments are irreversible allowing for the prices of used machines to fall below their reproduction cost; that is, for a Tobin's q that is less than 1.

We believe that the OG model is better suited for the analysis of capital gains and bubbles in decentralized economies than is the infinite-lifetime representative-agent (ILRA) model often used in macroeconomics. The ILRA model (and other homogenous-agent models) is essentially a planning model, in which prices and, hence, capital gains, are merely dual variables to the optimization problem. The OG structure, however, highlights how prices today depend on expectations of future beliefs, including the beliefs about capital gains by unborn generations. The ILRA model is closed (but not necessarily uniquely)[2] by a transversality condition,[3] and it can be argued that the OG model is closed (not necessarily uniquely) by boundary conditions such as the non-negativity of prices. The assumption of long-run perfect foresight seems to us to be less appropriate in the OG setting where it requires agents today to predict the *market behavior* of *all* future generations.

In the two-capital, discrete-time OG model, we show that for each initial endowment of two capital goods and labor, there is a unique competitive-equilibrium path on which expectations are fulfilled in every period. On every other path, there is a bubble in that one of the capitals is overvalued relative to the other. The bubble must burst in finite time. Hence, even though Shell and Stiglitz (1967) assume ad hoc consumption behavior, their basic results do not depend on this assumption. However, because of their consumption function, Shell and Stiglitz do not allow for cases in which gross investments are both zero. In the OG model, we show that both investments are zero whenever capital–labor ratios are large. This defines a region in which Tobin's q is less than 1. We show that, on the long-run perfect-foresight path, once the economy achieves $q = 1$ it will not switch back to the $q < 1$ regime.

We compute some trajectories for an example in which the technological parameters, the depreciation rate and the consumer time-discount rate are assigned reasonable values. We assign initial capital so that one is much scarcer (based on relative marginal products) than the other, and both are above their steady-state values, so that the economy is initially wealthy.

On the path in which expectations are always realized, gross investments are zero in the first few periods because the economy is rich in capital, which is followed by a few periods in which investment is specialized to the scarcer capital good. After these two stages, the marginal products of the two capital goods are forever equalized. Asymptotically the economy tends to the steady state just as it does in the Diamond (1965) model. This is the bubble-free path.

We also compute two bubble trajectories for the same parameters and initial endowments, but with initial prices that are slightly different from those on the bubble-free trajectory. For the first few periods, gross investments are zero as on the bubble-free path, but eventually investment is specialized to the "wrong" (lower marginal product)

[2] See, for example, Benhabib and Nishimura (1998) for examples of non-uniqueness of the equilibrium path.

[3] See, for example, Shell (1969) for cases in which the so-called transversality conditions are inappropriate even in planning and ILRA models.

capital good. In approximately 120 years, the bubble bursts and it is revealed that this is a disequilibrium path in that expectations are ultimately unfulfilled.

What do we make of this? On the competitive equilibrium path in which expectations are always fulfilled, the allocation of investment is correct and there are no bubbles. On other paths, where the allocation of investment is wrong, short-run markets clear and expectations are fulfilled for a while (100–200 years), and there is a bubble that must eventually burst. This suggests to us that the long-run perfect-foresight equilibrium concept might be too rigid. In the absence of an infinite spectrum of futures markets, what mechanism ensures that prices today will be those that will never (even in the far-distant future) lead to frustrated expectations? Bursting bubbles should not be ruled out entirely.

The model treated here is very special. The (flat) one-sector technology misses important properties of the (possibly curved) multisector technology. Separable preferences are also very special. In the present paper, there is no money. Money can be an important source on non-bursting and bursting bubbles.

The one-sector model with money is analyzed by Shell *et al.* (1969) for the case with a simple savings function and by Tirole (1985) for the case of utility-maximizing OG consumers. In these models, there is a unique path tending to the non-monetary steady state. Other paths are either hyperinflationary, tending to the non-monetary steady state on which the bubble vanishes but does not burst or they are paths on which there are bubbles that eventually burst (in finite time). In models with more complicated technologies and preferences, market imperfections, externalities or increasing returns, we can have indeterminacy, dependence of long-run growth on initial endowments, and sunspot equilibria. We do not want to suggest that indeterminacy, sunspots, non-bursting bubbles, or history-dependent growth is unimportant. Quite the contrary. Our exercise here is meant to focus the role of "bursting bubbles" in the macroeconomy. We think that our simple model might help to isolate this phenomenon and to direct attention to the role of expectations in economies with no or limited futures markets.

2 The model

In each period, there is a generation of identical, old consumers and a generation of identical, young consumers. Each young consumer inelastically supplies 1 unit of labor. The old do not work. The labor force, L, grows at the rate $n \geq 0$, so we have

$$L_{t+1} = (1+n)\,L_t, \tag{1}$$

where L_t is the number of consumers born in year $t = 0, 1, \ldots$. Consumers have identical utility functions

$$u\left(x_t^y, x_t^o\right) = \log x_t^y + \beta \log x_t^o,$$

where x_t^y is Mr t's consumption when young and x_t^o is his consumption when old.

Production is given by the one-sector, three-output, three-input model:

$$C_t + Z_t^1 + Z_t^2 = Y_t = \left(K_t^1\right)^{\alpha_1}\left(K_t^2\right)^{\alpha_2} L_t^{\alpha_3}, \tag{2}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $K_t^i > 0$ is the capital of type i, $Y_t > 0$ is undifferentiated output, $C_t \geq 0$ is consumption and $Z_t^i \geq 0$ is gross investment in Capital i, all at time t, $i = 1, 2$. Investment is irreversible and capital goods are non-malleable (i.e. machines of one type cannot be turned into machines of the other type): $Z_t^i \geq 0$. Let $\mu > 0$ be the common rate of depreciation on each type of machinery. Capital accumulation is given by

$$K_{t+1}^i = (1-\mu)K_t^i + Z_t^i \tag{3}$$

for $i = 1, 2$. Denote by lower case letters quantities normalized by L, for example, $k_t = K_t/L_t$, so we have

$$c_t + z_t^1 + z_t^2 = y_t = \left(k_t^1\right)^{\alpha_1}\left(k_t^2\right)^{\alpha_2} \tag{4}$$

and

$$(1+n)k_{t+1}^i = (1-\mu)k_t^i + z_t^i \tag{5}$$

for $i = 1, 2$. Under competition, factors are rewarded by their marginal products, so we have

$$r_t^i = \alpha_i\left(k_t^i\right)^{\alpha_i - 1}\left(k_t^j\right)^{\alpha_j} > 0, \tag{6}$$

for $i = 1, 2$, and

$$w_t = \alpha_3\left(k_t^1\right)^{\alpha_1}\left(k_t^2\right)^{\alpha_2}, \tag{7}$$

where r_t^i is the rental rate on type-i capital and w_t is the wage rate.

We assume that individuals possess perfect foresight about price changes. Hence, equilibrium in the used machinery market requires that the rate of return (including capital gains) on each type of capital must be the same, or

$$\frac{(1-\mu)p_{t+1}^1 + r_{t+1}^1}{p_t^1} = \frac{(1-\mu)p_{t+1}^2 + r_{t+1}^2}{p_t^2} = \rho_{t+1}, \tag{8}$$

where $p^i \geq 0$ is the current price of machine i in terms of the consumption good and ρ is the (common) rate of return. Equation (8) is the perfect-foresight asset-market-clearing equation.

Mr t chooses consumption (x_t^y, x_t^o) and savings $s_t \geq 0$ to maximize

$$u\left(x_t^y, x_t^o\right) = \log x_t^y + \beta \log x_t^o$$

subject to

$$x_t^y = w_t - s_t$$

and

$$x_t^o = \rho_{t+1} s_t,$$

where $0 < \beta < 1$ is the discount factor, "log" denotes the natural logarithm, and s_t is savings. The consumer's problem can be stated more succinctly as

$$\max_{s_t} \log(w_t - s_t) + \beta \log(\rho_{t+1} s_t) \tag{9}$$

subject to $0 \leq s_t \leq w_t$. The solution s_t to this problem is interior and given by

$$s_t = \frac{\beta}{1+\beta} w_t. \tag{10}$$

3 Equilibrium

Young consumers use their savings to buy capital that they will rent in period t and sell in period $t+1$. In equilibrium, the value of the supply of machinery must equal the value of savings, or

$$(1+n)\left(p_t^1 k_{t+1}^1 + p_t^2 k_{t+1}^2\right) = \frac{\beta}{1+\beta} w_t = \frac{\beta}{1+\beta} \alpha_3 \left(k_t^1\right)^{\alpha_1} \left(k_t^2\right)^{\alpha_2}. \tag{11}$$

Consumption is always positive, so we can normalize prices by the price of current consumption. Under competition, firms will only produce goods with the highest market price. Hence, we have

$$\max\left(p_t^1, p_t^2\right) \leq 1.$$

If $\max(p_t^1, p_t^2) < 1$, then $z_t^1 = z_t^2 = 0$. If $\max(p_t^1, p_t^2) = 1$, then the machine with the lower price will not be produced. If $p_t^1 = p_t^2 = 1$, then the composition of investment is indeterminate. Define aggregate gross investment per worker z by

$$z_t = z_t^1 + z_t^2$$

and the allocation-of-investment fraction σ by

$$\sigma_t = z_t^1 / z_t.$$

Then σ is the upper hemi-continuous correspondence given by

$$\sigma_t \begin{cases} = 1 & \text{if} \quad p_t^1 > p_t^2 \quad \text{and} \quad z_t > 0 \\ \in [0, 1] & \text{if} \quad p_t^1 = p_t^2 \quad \text{and} \quad z_t > 0 \\ = 0 & \text{if} \quad p_t^1 < p_t^2 \quad \text{and} \quad z_t > 0 \\ \text{undefined} & \text{if} \quad \quad \quad \quad \quad \quad \quad \quad z_t = 0 \end{cases}. \tag{12}$$

Definition 1 *Given initial per capita capital stocks (k_0^1, k_0^2), a long-run perfect-foresight equilibrium is given by the sequence of allocations $\{k_{t+1}^1, k_{t+1}^2, s_t, x_t^y, x_t^o\}_{t=0}^{\infty}$ and the sequence* *of non-negative prices $\{r_t^1, r_t^2, p_t^1, p_t^2\}_{t=0}^{\infty}$ such that equations (7), (6) and (10), and the market-clearing conditions (8) and (11) are satisfied.*

4 Steady-state growth

In the steady state, both capitals are produced,

$$z^i = \lambda k^i \quad \textit{for } i = 1, 2 \textit{ where } \quad \lambda = n + \mu, \tag{13}$$

prices must be the same,

$$p^1 = p^2 = 1, \tag{14}$$

and

$$y = (k_1)^{\alpha_1} (k_2)^{\alpha_2}. \tag{15}$$

To have $p^1 = p^2 = 1$, we must have $r^1 = r^2$ and, hence, $k^1/k^2 = \alpha_1/\alpha_2$. This, together with equation (11), yields

$$k^1 = \left(\frac{\beta}{1+\beta}\frac{\alpha_3}{\alpha_1+\alpha_2}\frac{1}{1+n}\right)^{\frac{1}{\alpha_3}} \alpha_1^{\frac{1-\alpha_2}{\alpha_3}} \alpha_2^{\frac{\alpha_2}{\alpha_3}}, \tag{16}$$

$$k^2 = \left(\frac{\beta}{1+\beta}\frac{\alpha_3}{\alpha_1+\alpha_2}\frac{1}{1+n}\right)^{\frac{1}{\alpha_3}} \alpha_1^{\frac{\alpha_1}{\alpha_3}} \alpha_2^{\frac{1-\alpha_1}{\alpha_3}}, \tag{17}$$

and

$$\sigma = \frac{\alpha_1}{\alpha_1+\alpha_2}. \tag{18}$$

The following proposition summarizes the results of this section.

Proposition 1 *In the steady state, the capital to labor ratios k^1 and k^2, output per worker y, and the fraction σ of gross investment directed to machinery of type 1 are uniquely determined.*

5 Dynamic analysis

We assumed that once capital is installed it cannot be consumed. At the end of each period t, the value of the capital stock per worker is $p_t^1(1-\mu)k_t^1 + p_t^2(1-\mu)k_t^2$. The savings of young workers, $(\beta/(1+\beta))\alpha_3(k_t^1)^{\alpha_1}(k_t^2)^{\alpha_2}$, must be sufficient to buy the existing capital stock. For $z_t \geq 0$ to hold, we must have

$$p_t^1(1-\mu)k_t^1 + p_t^2(1-\mu)k_t^2 \leq \frac{\beta}{1+\beta}\alpha_3\left(k_t^1\right)^{\alpha_1}\left(k_t^2\right)^{\alpha_2}. \tag{19}$$

For the time being, we will assume that this constraint is not binding. If $\max(p_t^1, p_t^2) \leq 1$, a sufficient condition for (19) to hold is

$$(1-\mu)\left(k_t^1 + k_t^2\right) \leq \frac{\beta}{1+\beta}\alpha_3 \left(k_t^1\right)^{\alpha_1} \left(k_t^2\right)^{\alpha_2}. \tag{20}$$

We will use this condition for now, but we relax it later.

Given our temporary assumption, there are three different regimes in which we can find the economy:

Regime 1 $1 = p_t^1 > p_t^2$.
Regime 2 $1 = p_t^2 > p_t^1$.
Regime 3 $1 = p_t^1 = p_t^2$.

It is redundant to analyze both Regime 1 and Regime 2. We focus on Regime 1 and Regime 3.

Regime 1 $1 = p_t^1 > p_t^2$. Only capital of type 1 is produced, so we have $z_t^2 = 0$. Using the motion equations and the arbitrage condition, we have

$$k_{t+1}^2 = \frac{(1-\mu)}{(1+n)} k_t^2, \tag{21}$$

$$k_{t+1}^1 = \frac{\beta}{1+\beta}\frac{\alpha_3}{1+n}\left(k_t^1\right)^{\alpha_1}\left(k_t^2\right)^{\alpha_2} - p_t^2 \frac{1-\mu}{1+n} k_t^2, \tag{22}$$

and

$$p_{t+1}^2 = p_{t+1}^1 p_t^2 + \frac{r_{t+1}^1 p_t^2 - r_{t+1}^2}{1-\mu}. \tag{23}$$

From (23), we know that if $r_{t+1}^1 p_t^2 < r_{t+1}^2$, then the price of Capital 2 will decrease (and, hence, we must have $p_{1,t+1} = 1$). In period $t+1$, the price of Capital 2 will decrease at a faster absolute rate, because only Capital 1 is produced, and the marginal productivity of Capital 1 relative to Capital 2 will have decreased. With the decrease in the price of Capital 2, the value of $r_{t+1}^1 p_t^2 - r_{t+1}^2$ will remain negative. It is easy to check that in finite time the price of Capital 2 will become negative. So this trajectory cannot be a long-run equilibrium path, one on which expectations are realized at every date. Therefore, we can easily conclude that

Proposition 2 *If we have $r_{t+1}^1 < r_{t+1}^2$ there is no pair of prices (p_1^t, p_2^t) satisfying $1 = p_t^1 > p_t^2$ that can support a long-run competitive equilibrium in which expectations are always fulfilled.*

If $r_t^1 \leq r_t^2$ and $1 = p_t^1 > p_t^2$, all new investment is directed towards k^1 and, hence, we have again, $r_{t+1}^1 \leq r_{t+1}^2$. This simple observation leads to the next corollary.

Corollary 1 *If $r_t^1 \leq r_t^2$ there is no pair of prices (p_1^t, p_2^t) satisfying $1 = p_t^1 > p_t^2$ that can support a long-run competitive equilibrium in which expectations are always fulfilled.*

These results tell us that to be on a long-run equilibrium path it must be the case that the price of the relatively scarce type of machines cannot be lower than the price of the relatively abundant type of machines.[4]

Regime 3 $1 = p_t^1 = p_t^2$. In this case we have:

$$k_{t+1}^1 = \frac{1-\mu}{1+n} k_t^1 + \frac{1}{1+n} \sigma_t z_t, \tag{24}$$

$$k_{t+1}^2 = \frac{1-\mu}{1+n} k_1^2 + \frac{1}{1+n} (1 - \sigma_t) z_t, \tag{25}$$

and

$$(1 - \mu) p_{t+1}^1 + r_{t+1}^1 = (1 - \mu) p_{t+1}^2 + r_{t+1}^2. \tag{26}$$

If $r_{t+1}^1 = r_{t+1}^2$, we will have $p_{t+1}^1 = p_{t+1}^2 = 1$. If the economy stays in this regime, it will then converge to the steady state. If $r_{t+1}^i > r_{t+1}^j$ then $p_{t+1}^i < p_{t+1}^j = 1$, with $i, j = 1, 2$ and $i \neq j$. By the previous corollary we know that this is not compatible with long-run equilibrium in which expectations are always fulfilled. Hence, we have the following result.

Proposition 3 *If $1 = p_t^1 = p_t^2$, only $r_{t+1}^1 = r_{t+1}^2$ is compatible with a long-run competitive equilibrium trajectory.*

So far we have argued that the price of the relatively scarce type of capital must be equal to unity, so if we have $r_t^2 \le r_t^1$ we must have $1 = p_t^1 \ge p_t^2$.

We also know that $1 = p_t^1 \ge p_t^2$ and $r_{t+1}^1 < r_{t+1}^2$ are not compatible with long-run competitive equilibrium, so if the initial conditions are such that Capital 1 is scarcer, it will remain so, unless, of course, their marginal productivities become equal. So, if $r_t^2 < r_t^1$, unless $r_{t+1}^2 = r_{t+1}^1$, we must have $1 = p_t^1 > p_t^2$. If $r_{t+1}^2 = r_{t+1}^1$, then $1 = p_{t+1}^2 = p_{t+1}^1$.

Once the economy is in a situation with $1 = p_t^2 = p_t^1$ and $r_t^2 = r_t^1$ so that $k_t^1 / k_t^2 = \alpha_1/\alpha_2$, σ_t should be such that the ratio of Capital 1 to Capital 2 remains constant, $\sigma_t = \alpha_1 / (\alpha_1 + \alpha_2)$. Once the economy is in this path, with $k_t^1 = (\alpha_1/\alpha_2) k_t^2$, the analysis is basically as in the typical Diamond OG economy (Diamond 1965). Simplifying (11), one can see that the dynamics are reduced to the study of the difference equation $k_{t+1}^2 = A(k_t^2)^\alpha$,[5] a well-known difference equation. Hence, we know that the economy will converge to the unique steady state.

Suppose that in period zero we have $k_0^1/k_0^2 < \alpha_1/\alpha_2$. Finding the initial prices that are compatible with the long-run equilibrium trajectory is now reduced to the problem of finding the initial prices that guarantee that in some period $t^* = 0, 1, \ldots,$ we have $p_{t^*}^2 = p_{t^*}^1 = 1$ and that in the next period we have $r_{t^*+1}^2 = r_{t^*+1}^1$. Suppose that $t^* > 0$. We would expect that, in equilibrium, as Capital 1 becomes relatively less scarce, the price of Capital 2 increases. This is easily confirmed. If, for $t < t^*$, $p_{t+1}^2 \le p_t^2$ we know that $r_{t+2}^1 / r_{t+2}^2 \le r_{t+1}^1 / r_{t+1}^2 \le 1/p_{2,t} \le 1/p_{t+1}^2$. However, $r_{t+2}^1/r_{t+2}^2 \le 1/p_{t+1}^2$ implies that

[4] We use the term "scarce" as a synonym for higher marginal productivity.

[5] Where $A = \beta/1 + \beta\,(\alpha_3/((1+n)(1+\alpha_2/\alpha_1)))\,(\alpha_2/\alpha_1)^{\alpha_2} > 0$ and $\alpha = \alpha_1 + \alpha_2 < 1$.

$p^2_{t+2} \leq p^2_{t+1}$, so the price of Capital 2 cannot approach 1, contradicting our initial assumption. Therefore, if an equilibrium exists we will have $p^2_{t^*} > p^2_{t^*-1} > \cdots > p^2_0$.

This leads to the next proposition.

Proposition 4 *Let $k^1_0 < (\alpha_1/\alpha_2)\, k^2_0$. If there is a long-run equilibrium trajectory, it will be unique.*

PROOF: Consider two equilibrium price sequences. $\bar{p} = \{(\bar{p}^1_0, \bar{p}^2_0), (\bar{p}^1_1, \bar{p}^2_1), \ldots\}$ and $\tilde{p} = \{(\tilde{p}^1_0, \tilde{p}^2_0), (\tilde{p}^1_1, \tilde{p}^2_1), \ldots\}$.

(i) First we show that if $\bar{p}^2_0 > \tilde{p}^2_0$ and $\bar{p}^2_0 < \bar{p}^2_1 < \cdots < \bar{p}^2_{t^*-1} < \bar{p}^2_{t^*} = 1$, then we have $\bar{p}^2_t > \tilde{p}^2_t$ for $t \leq t^*$. In period 0, the relevant arbitrage condition is $p^2_{t+1} = p^2_t + (r^1_{t+1} p^2_t - r^2_{t+1})/(1-\mu)$. All the new investment is devoted to Capital 1. The motion equations for capital are

$$k^2_1 = \frac{(1-\mu)}{(1+n)} k^2_0$$
$$k_{1,1} = \frac{\beta}{1+\beta} \frac{(\alpha_3)}{(1+n)} k^{\alpha_1}_{1,0} k^{\alpha_2}_{2,0} - p_{2,t} \frac{(1-\mu)}{(1+n)} k_{2,0}.$$

If $\bar{p}^2_0 > \tilde{p}^2_0$ we have $\begin{cases} \bar{k}^2_1 = \tilde{k}^2_1 \\ \bar{k}^1_1 < \tilde{k}^1_1 \end{cases}$, which implies $\begin{cases} \bar{r}^2_1 < \tilde{r}^2_1 \\ \bar{r}^1_1 > \tilde{r}^1_1 \end{cases}$, which yields $\bar{r}^1_1 \bar{p}^2_0 - \bar{r}^2_1 > \tilde{r}^1_1 \tilde{p}^2_0 - \tilde{r}^2_1$, so we must have $\bar{p}^2_1 > \tilde{p}^2_1$. The same happens in the succeeding periods.

(ii) We have shown before that unless $k^1_1 = (\alpha_1/\alpha_2)\, k^2_1$, only $1 = p^1_0 > p^1_0$ is compatible with long-run equilibrium. If $k^1_1 = (\alpha_1/\alpha_2)\, k^2_1$, then we have $1 = p^1_0 = p^2_0$. Therefore, focus on the first case. Assume that $\bar{p}_{2,0} < \bar{p}_{2,1} < \cdots < \bar{p}_{2,t^*} = 1$ is compatible with the long-run equilibrium. We know that for this to be a part of a long-run equilibrium trajectory we must have $\bar{k}^1_{t^*+1} = (\alpha_1/\alpha_2)\, \bar{k}^2_{t^*+1}$. Also note that at time $t^* - 1$ we have $\bar{\sigma}_{t^*-1} = 1$.

(iii) Suppose that the alternative sequence, $\tilde{p}^2_0 < \tilde{p}^2_1 < \cdots < \tilde{p}^2_{t^{**}} = 1$, with $1 > \tilde{p}^2_0 > \bar{p}^2_0$, is also an equilibrium. Because of Step 1, we know that $t^{**} < t^*$. Because, as long as $p^2 < 1$, there is no new investment in Capital 2, at t^{**} we have $\bar{k}^2_{t^{**}} = \tilde{k}^2_{t^{**}}$. We also have $\tilde{k}^1_{t^{**}} < \bar{k}^1_{t^{**}}$. However, we also know that $\tilde{k}^1_{t^{**}+1} \leq (\alpha_1/\alpha_2)\, \tilde{k}^2_{t^{**}+1}$. Because $\tilde{k}^2_{t^{**}+1} \leq \bar{k}^2_{t^{**}+1}$ and $\tilde{k}^1_{t^{**}+1} < \bar{k}^1_{t^{**}+1}$, we have $\bar{k}^1_{t^{**}+1} < (\alpha_1/\alpha_2)\, \bar{k}^1_{t^{**}+1}$, implying that the price sequence with $\tilde{p}^2_0 < \tilde{p}^2_1 < \cdots \tilde{p}^2_{t^{**}} = 1$ cannot be an equilibrium sequence.

□

With this result, we know that for any initial conditions if we find a long-run equilibrium path it will be unique. Again, suppose, without loss of generality, that we have $k^1_0 \leq (\alpha_1/\alpha_2)\, k^2_0$. If only capital of type 1 is produced, it is easy to check that eventually this inequality will be reversed. Given our previous results, we know that the equilibrium prices must be such that exactly in the period before the inequality is reversed, say at t', prices are both equal to unity. Therefore, σ_t may take any value between zero and 1, and can be appropriately chosen so that $k^1_{t'+1} = (\alpha_1/\alpha_2)\, k^2_{t'+1}$.

Using (5), it is apparent that to have $k_{t'+1}^1 = (\alpha_1/\alpha_2)\,k_{t'+1}^2$ we must have at t'

$$(1-\mu)\,k_{t'}^1 = \frac{\alpha_1}{\alpha_2}\,(1-\mu)\,k_{t'}^2 + \left(\frac{\alpha_1}{\alpha_2}\,(1-\sigma_{t'}) - \sigma_{t'}\right) z_{t'}. \tag{27}$$

With $k_{t'}^1 \le (\alpha_1/\alpha_2)\,k_{t'}^2$, we would have $\sigma_{t'} \ge \alpha_1/(\alpha_1+\alpha_2)$. Therefore, $\sigma_{t'} \in [\alpha_1/(\alpha_1+\alpha_2), 1]$.

Consider an initial endowment of capital of type 2, say $k_0^2 = \bar{k}_0^2$, with $k_0^1 < (\alpha_1/\alpha_2)\,\bar{k}_0^2$. Is it possible to have $k_1^1 = (\alpha_1/\alpha_2)\,k_1^2$? Using (27), we can confirm that the lowest value that k_0^1 can take is $\underline{k_0^1} = [((\alpha_1+\alpha_2)/\alpha_2)((1+\beta)/\beta)((1-\mu)/\alpha_3)]^{1/\alpha_1}\,(\bar{k}_0^2)^{(1-\alpha_2)/\alpha_1}$. So if $k_{1,0} \in \left[\underline{k_0^1}, (\alpha_1/\alpha_2)\,k_{2,0}\right]$, $\left(p_0^1, p_0^2\right) = (1, 1)$ is an equilibrium price.

If $k_0^1 < \underline{k_0^1}$, we have to check if it is possible to have $k_2^1 = (\alpha_1/\alpha_2)\,k_2^2$. Noting that $k_1^2 = [(1-\mu)/(1+n)]\,k_0^2$, and that we need $\left(p_1^1, p_1^2\right) = (1, 1)$, we can use (27) again to conclude that $\underline{k_1^1} = [((\alpha_1+\alpha_2)/\alpha_2)((1+\beta)/\beta)((1-\mu)/\alpha_3)]^{1/\alpha_1}\,(\bar{k}_1^2)^{(1-\alpha_2)/\alpha_1}$. So for $\left(p_1^1, p_1^2\right) = (1, 1)$ to be an equilibrium $k_1^1 \in \left[\underline{k_1^1}, (\alpha_1/\alpha_2)((1-\mu)/(1+n))\,\bar{k}_0^2\right]$. To find the values of k_0^1 that are compatible with $k_1^1 \in \left[\underline{k_1^1}, (\alpha_1/\alpha_2)((1-\mu)/(1+n))\,\bar{k}_0^2\right]$, we can use the arbitrage equation $p_1^2 = p_0^2 + (r_1^1 p_0^2 - r_1^2)/(1-\mu)$ and $p_1^2 = 1$ to solve for p_0^2:

$$p_0^2 = \frac{(1-\mu) + \alpha_2\left(k_1^1\right)^{\alpha_1}\left(k_0^2(1-\mu)/(1+n)\right)^{\alpha_2-1}}{(1-\mu) + \alpha_1\left(k_1^1\right)^{\alpha_1-1}\left(k_0^2(1-\mu)/(1+n)\right)^{\alpha_2}}. \tag{28}$$

For $k_1^1 = (\alpha_1/\alpha_2)((1-\mu)/(1+n))\bar{k}_0^2$ we have $k_0^1 = \underline{k_0^1}$ and $(p_0^1, p_0^2) = (1, 1)$. It is immediate that if $k_1^1 < (\alpha_1/\alpha_2)((1-\mu)/(1+n))\,k_0^2$ we have p_0^2, and the lower is k_1^1 the lower will be p_0^2. It is a matter of algebra to check that for $k_1^1 = \underline{k_1^1}$ we have $\underline{\underline{k_0^1}} = \left(\left(\left(\underline{k_1^1} + p_0^2\frac{(1-\mu)}{(1+n)}\bar{k}_0^2\right)/\left(\bar{k}_0^2\right)^{\alpha_2}\right)((1+\beta)/\beta)((1+n)/\alpha_3)\right)^{1/\alpha_1}$ and to find the corresponding price $p_0^2 = \underline{\underline{p_0^2}}$.

Putting everything together, for $k_0^1 \in [\underline{k_0^1}, (\alpha_1/\alpha_2)\,k_0^2]$, we have $(p_0^1, p_0^2) = (1, 1)$. If $k_{1,0} \in [\underline{\underline{k_0^1}}, \underline{k_0^1})$ we have $p_0^1 = p_1^1 = 1$, $p_0^2 \in [\underline{\underline{p_0^2}}, 1)$ and $p_1^2 = 1$.

If $k_0^1 < \underline{\underline{k_0^1}}$, then using the same procedure we have to check if it is possible to have $k_3^1 = (\alpha_1/\alpha_2)\,k_3^2$, derive $\underline{\underline{\underline{k_0^1}}}$ and $\underline{\underline{\underline{p_0^2}}}$ and so on. We know that at some date, say $t'+1$, it will be possible to have the equality $k_{t'+1}^1 = (\alpha_1/\alpha_2)k_{t'+1}^2$ and $(p_{t'}^1, p_{t'}^2) = (1, 1)$. Hence, we have the following result.

Proposition 5 *For any initial vector (k_0^1, k_0^2) of capital per worker there is one initial price vector (p_0^1, p_0^2) compatible with the long-run competitive equilibrium in which expectations are always fulfilled.*

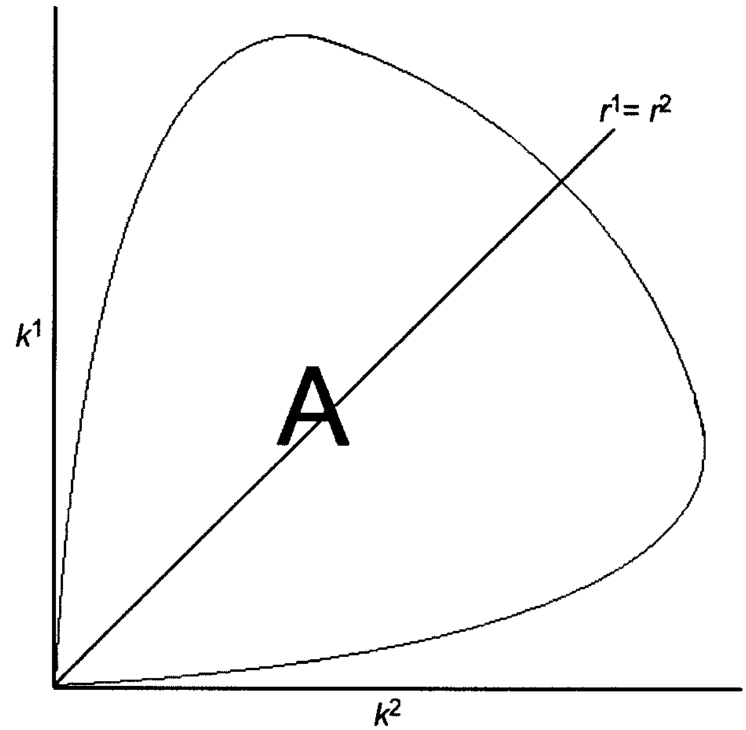

Figure 1 Region in which Tobin's $q = 1$.

6 Tobin's $q < 1$

So far we have assumed that savings are sufficient to buy the existing capital stock; namely,

$$p_t^1 (1-\mu) k_t^1 + p_t^2 (1-\mu) k_t^2 \leq \frac{\beta}{1+\beta} \alpha_3 \left(k_t^1\right)^{\alpha_1} \left(k_t^2\right)^{\alpha_2}$$

at prices satisfying $\max(p_t^1, p_t^2) = 1$.

If the above constraint is not binding, we know that $\max(p_t^1, p_t^2) = 1$. A sufficient condition for the above inequality to hold with $\max(p^1, p^2) = 1$ is

$$k_t^1 + k_t^2 \leq \frac{\beta}{1+\beta} \frac{\alpha_3}{1-\mu} \left(k_t^1\right)^{\alpha_1} \left(k_t^2\right)^{\alpha_2}, \tag{29}$$

which implicitly defines the convex Region A in Figure 1. The slope of the frontier of A, when k^1 and k^2 are close to zero, is zero or infinity, depending on whether $k^2 > k^1$ or $k^2 < k^1$.

In this section, we assume that we are outside region A. Again, without loss of generality, we assume $k_0^1 < (\alpha_1/\alpha_2) k_0^2$. If we determine p_0^2 using the algorithm described in the previous section and we get $(1-\mu) k_0^1 + p_0^2 (1-\mu) k_0^2 \leq (\beta/(1+\beta)) \alpha_3 \left(k_0^1\right)^{\alpha_1} \left(k_0^2\right)^{\alpha_2}$, then the results described before still apply. However, if instead, we conclude that

$$(1-\mu) k_0^1 + p_0^2 (1-\mu) k_0^2 > \frac{\beta}{1+\beta} \alpha_3 \left(k_0^1\right)^{\alpha_1} \left(k_0^2\right)^{\alpha_2} \tag{30}$$

Table 1 Assumed parameter values

α_1	α_2	β	μ	n	k_0^1	k_0^2
0.2	0.2	0.6	0.55	0	1	5

holds, p_0^2 cannot be an equilibrium price. If no new investment can be made in period zero, then in period 1 we will have $k_1^i = ((1-\mu)/(1+n))k_0^i$, *for* $i = 1, 2$. If the same happens again, we will have $k_2^i = ((1-\mu)/(1+n))^2 k_0^i$, and so on. Eventually the inequality will be reversed (otherwise we enter in region A, where we know for sure that the inequality will be reversed).

Suppose that in period 1 the inequality is reversed, meaning that $\bar{p}_1^2$ is an equilibrium price and

$$(1-\mu)\, k_1^1 + \bar{p}_1^2 (1-\mu)\, k_1^2 \leq \frac{\beta}{1+\beta}\alpha_3 \left(k_1^1\right)^{\alpha_1} \left(k_1^2\right)^{\alpha_2}.$$

In period zero, which prices lead to $(p_1^1, p_1^2) = (1, \bar{p}_1^2)$?

In equilibrium, the inequality (30) cannot hold, so prices will have to adjust, so that

$$\bar{p}_0^1 (1-\mu)\, k_0^1 + \bar{p}_0^2 (1-\mu)\, k_0^2 = \frac{\beta}{1+\beta}\alpha_3 \left(k_0^1\right)^{\alpha_1} \left(k_0^2\right)^{\alpha_2}, \tag{31}$$

with $\max\left(\bar{p}_0^1, \bar{p}_0^2\right) < 1$.

The arbitrage condition must hold, which implies

$$\begin{aligned} \frac{(1-\mu) + r_1^1}{\bar{p}_0^1} &= \frac{(1-\mu)\bar{p}_1^2 + r_1^2}{\bar{p}_0^2} \\ \frac{\bar{p}_0^2}{\bar{p}_0^1} &= \frac{(1-\mu)\bar{p}_1^2 + r_1^2}{(1-\mu) + r_1^1} < 1. \end{aligned} \tag{32}$$

Because we know that $k_1^i = ((1-\mu)/(1+n))k_0^i$, *for* $i = 1, 2$, we can use (31) and (32) to uniquely determine $(\bar{p}_0^1, \bar{p}_0^2)$.

This analysis can be extended to an arbitrary number of periods. For example, if only in period 2 (30) is reversed, then, using the same algorithm, we can determine the prices of period 1. Knowing these, we can determine the prices in period 0.

7 Computed examples: How long before the bubble must burst?

Our numerical exercises are inspired in part by Atkinson (1969).[6] The parameter values used in our experiments are given in Table 1.

In the two-period-lifetime OG model, we identify "youth" with the working years and "old age" with the retirement years. One period in the OG model corresponds to roughly 20 years, so $\beta = 0.6$ corresponds to an annual discount factor on the order of

[6] See pages 144–148.

Table 2 Bubble-free growth path

t	k_t^1	k_t^2	p_t^1	p_t^2	z_t^1	z_t^2
0	1	5	0.341401	0.069692	0.000000	0.000000
1	0.45	2.25	0.541100	0.114554	0.000000	0.000000
2	0.2025	1.0125	0.82821	0.194055	0.000000	0.000000
3	0.091125	0.455625	1.000000	0.322857	0.011875	0.000000
4	0.052882	0.205031	1.000000	0.543669	0.017076	0.000000
5	0.040873	0.092264	1.000000	0.902200	0.017850	0.000000
6	0.036243	0.041518	1.000000	1.000000	0.014355	0.011981
7	0.030665	0.030665	1.000000	1.000000	0.014113	0.014113
8	0.027913	0.027913	1.000000	1.000000	0.014321	0.014321
9	0.026882	0.026882	1.000000	1.000000	0.014384	0.014384
10	0.026481	0.026481	1.000000	1.000000	0.014405	0.014405
11	0.026322	0.026322	1.000000	1.000000	0.014414	0.014414
12	0.026259	0.026259	1.000000	1.000000	0.014417	0.014417
13	0.026234	0.026234	1.000000	1.000000	0.014418	0.014418

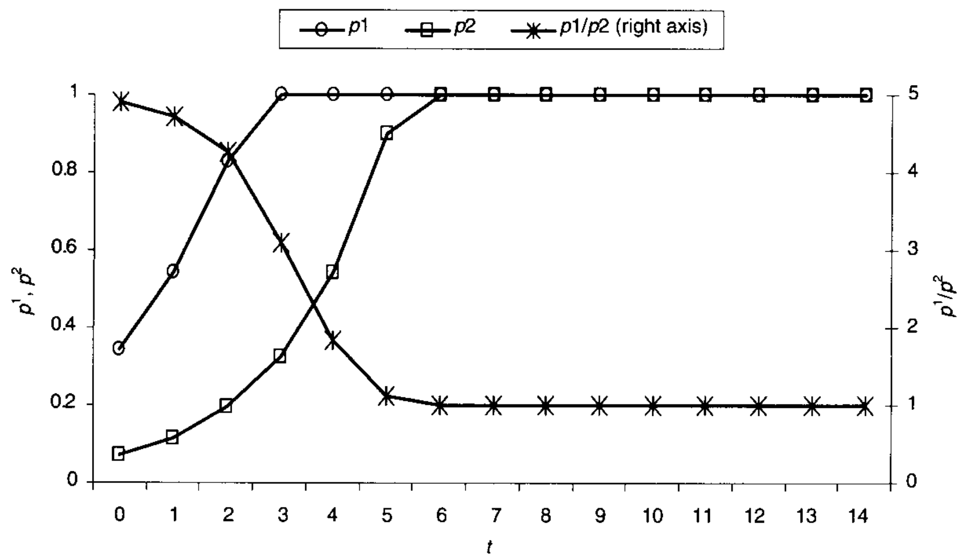

Figure 2 Prices on the bubble-free path.

97.5 percent, whereas $\mu = 0.55$ corresponds to an annual depreciation rate of approximately 4 percent.

For the economy described by Table 1, the unique bubble-free growth is displayed in Table 2 and Figures 2–4. By assumption, $k_0^2 > k_0^1$ and, hence, $r_0^1 > r_0^2$, meaning that type-1 capital is scarcer then type-2 capital. This is reflected in the capital-goods prices: $p_t^1 / p_t^2 > 1$ for $t = 0, \ldots, 5$. By assumption, $(k_0^1 + k_0^2)$ is large for this economy. This is reflected in Tobin's q: $q_t = \max(p_t^1, p_t^2) < 1$ and $z_t^1 = z_t^2 = 0$ for $t = 0, 1, 2$, $q_t = 1 = p_t^1 > p_t^2$ for $t = 3, 4, 5$ and $q_t = p_t^1 = p_t^2 = 1$ for $t = 6, 7, \ldots$. In period 6, prices are equal, $q_6 = p_6^1 = p_6^2 = 1$, but marginal products are unequal, $r_6^1 > r_6^2$ because $k_6^2 > k_6^1$, and investments are positive but unequal. After period 6, we have balanced investment:

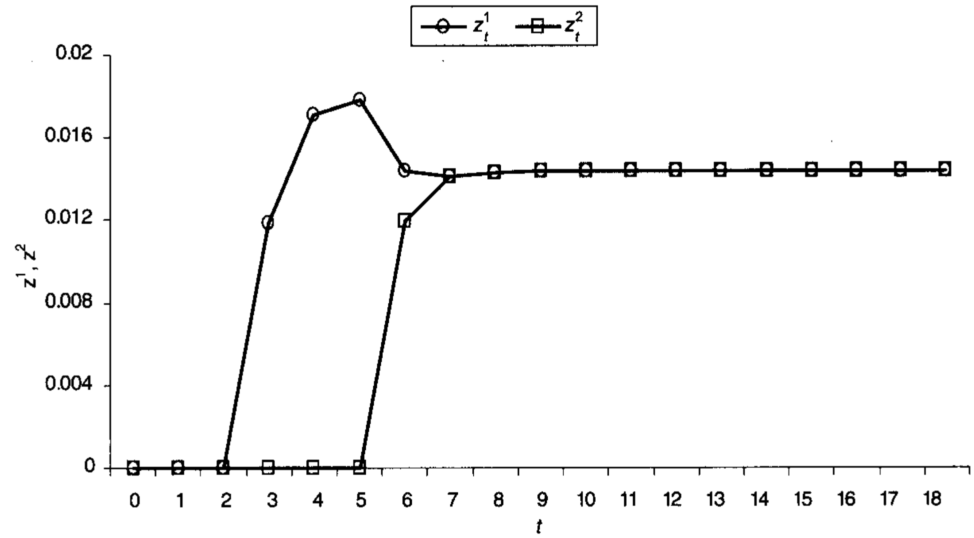

Figure 3 Savings and gross investments on the bubble-free path.

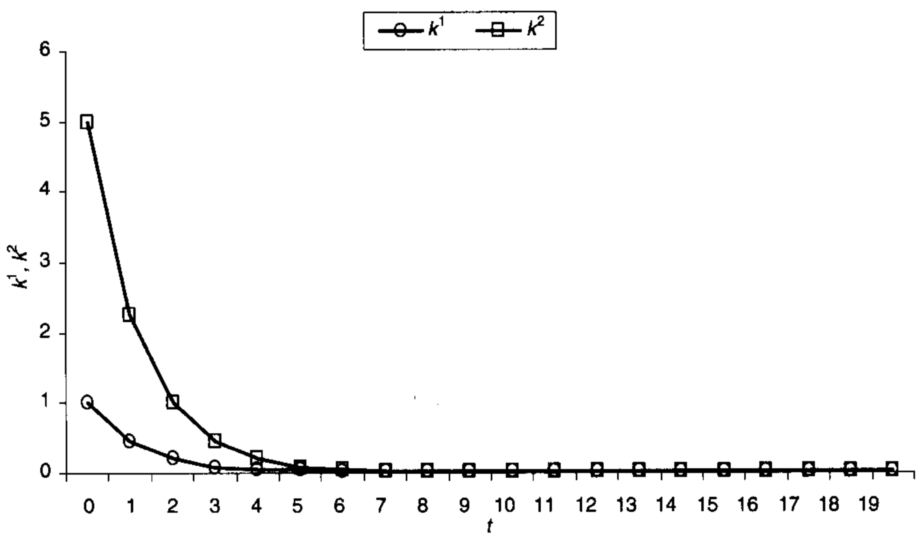

Figure 4 Capital/labor ratios on the bubble-free path.

$q_t = p^1_t = p^2_t = 1$, $z^1_t = z^2_t$, $k^1_t = k^2_t$, and $r^1_t = r^2_t$ for $t = 7, 8, \ldots$. Asymptotically the economy tends to balanced growth with $k^1 = k^2 = 0.026234$. There are no bubbles.

For Table 3 and Figure 5, we adopt the same economy as in the previous example (the one defined in Table 1), but we slightly perturb the initial prices from their bubble-free values. In particular, p^1_0 is slightly larger than its bubble-free value. In the first three periods: $1 > q_t = p^1_t > p^2_t$, $z^1_t = z^2_t = 0$ for $t = 0, 1, 2$ just as on the bubble-free path. In the next periods, Tobin's $q = 1$ and investment is specialized to type-1 capital: $1 = q_t = p^1_t > p^2_t$, $z^1_t > 0$, and $z^2_t = 0$ for $t = 3, 4, 5$. By period 6, type-2 capital is scarcer, but the economy is investing only in type-1 capital: $k^1_6 > k^2_6$, $r^2_6 > r^1_6$, $1 = q_6 = p^1_6 > p^2_6$. The growth path cannot be extended to period 7, because $p^2_7 < 0$ would be impossible because with free disposal the rate of return on type-1 capital would exceed the rate of return on type-2 capital. The bubble must burst before period 7.

Table 3 Bubble-path-I

t	k_t^1	k_t^2	p_t^1	p_t^2	z_t^1	z_t^2
0	1	5	0.341441	0.069684	0	0.000000
1	0.45	2.25	0.541281	0.114518	0	0.000000
2	0.2025	1.0125	0.829027	0.193893	0	0.000000
3	0.091125	0.455625	1.000000	0.321334	0.012188	0.000000
4	0.053194	0.205031	1.000000	0.530791	0.018232	0.000000
5	0.042169	0.092264	1.000000	0.786965	0.022513	0.000000
6	0.041489	0.041519	1.000000	0.150000	0.041538	0.000000
7	0.060208	0.018683	1.000000	<0	–	–

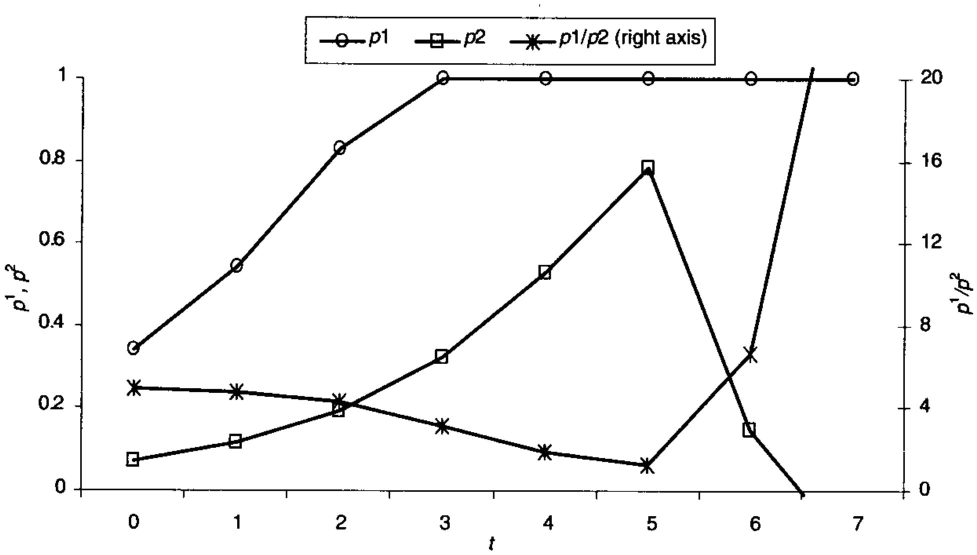

Figure 5 Prices on bubble-path-I.

Table 4 Bubble-path-II: Regime switching

t	k_t^1	k_t^2	p_t^1	p_t^2	z_t^1	z_t^2
0	1	5	0.341229	0.069727	0.000000	0.000000
1	0.45	2.25	0.540329	0.114709	0.000000	0.000000
2	0.2025	1.0125	0.824758	0.194747	0.000000	0.000000
3	0.091125	0.455625	1.000000	0.329376	0.010539403	0.000000
4	0.051545	0.205031	1.000000	0.600000	0.012016138	0.000000
5	0.035212	0.092264	0.677374	1	0.000000	0.019284287
6	0.015845	0.060803	<0	1	–	–

In Table 4 and Figures 6–7, we display bubble-path-II. This is based on the same economy as analyzed in the previous examples (and described in Table 1) except that we set p_0^1 slightly below (rather than slightly above) its value on the bubble-free path. In the first three periods, we have $1 > q_t = p_t^1 > p_t^2$ and $z_t^1 = z_t^2 = 0$ for $t = 0, 1, 2$, just as on the bubble-free path. However, here p_t^2 is increasing faster than on the bubble-free path.

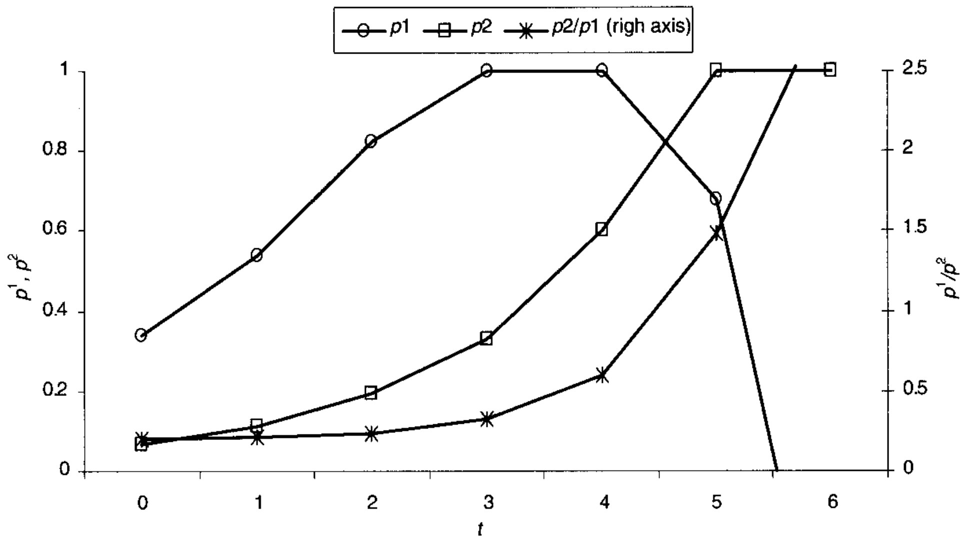

Figure 6 Prices on bubble-path-II.

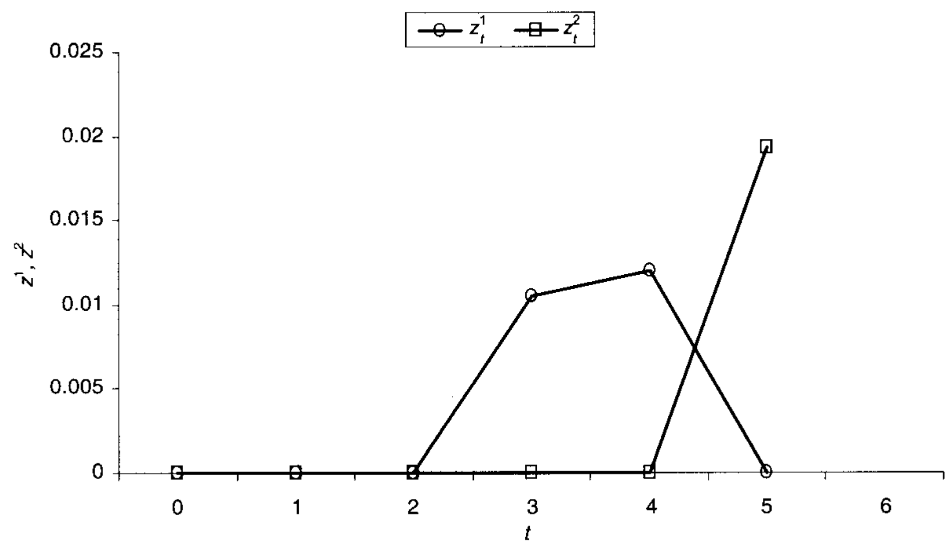

Figure 7 Investments on bubble-path-II.

By period 5 (instead of period 6), we have $p_2^5 = 1 = q_5$, $p_5^1 < 1$, $z_2^5 > 0$, $z_1^5 = 0$, $k_5^2 > k_5^1$ and, hence, $r_5^1 > r_5^2$. In period 6, there is a switch in regimes from producing the relatively scarce capital good to producing the relatively abundant capital good. On this path, p_6^1 would become negative, which is impossible if there is free disposal of capital. Hence, in period 5, the rate of return on machinery of type 1 would exceed that for machinery of type 2. Hence, the bubble must burst before period 6.

We have also investigated economies with parameter values different from those given in Table 1. Qualitatively, the results remain the same, although there are some differences. For example, the higher the depreciation rate the quicker the bubble will burst. The larger are the depreciation rates the smaller are the capital gains. Therefore, changes in the prices will have to be even greater to compensate for the differences in yields, leading to shorter-lived bubbles. If the depreciation rate were 100 percent, there would be no capital gains and, hence, there would be no perfect-foresight bubbles. Similarly, if expectations about prices were static, there would be no expected capital gains and, hence, no bubbles.

8 Summary

We have investigated asset prices and capital gains in a perfect-foresight economy. Our model is essentially a combination of the Shell and Stiglitz (1967) growth model with the Diamond (1965) OG model. We assume that investment is irreversible, allowing used machines to be sold for less than their replacement values: Tobin's q can be less than unity.[7] Just as the basic results of Shell *et al.* (1969)for the money-and-single-capital growth model carry over in the Tirole (1985) OG model, the main results of Shell and Stiglitz are unchanged in the OG environment. There is a unique competitive-equilibrium path in which expectations are always fulfilled. Complete futures markets in machinery imply that this bubble-free path is the only one that will be pursued. Even if future markets are not complete (as in the real world), the bubble trajectories will be revealed to be disequilibrium paths, but only after some decades, or centuries, or more. Bubble trajectories are not equilibrium trajectories in the usual sense, but they test the usual definition of long-run perfect-foresight in the OG environment.

Comparing the analysis of the present paper with that of Shell and Stiglitz (1967) also reveals that introducing agent optimization and discrete time allows for a richer dynamics. For example, in Shell and Stiglitz the prices of the two capital goods on the bubble-free path would be the same (and equal to unity) only when their marginal productivities are the same on the bubble-free path. In the present model, prices become the same (and equal to unity) exactly one period before the marginal productivities are equalized.

The present paper is our second attempt to analyze capital gains in an OG economy with two capital goods and perfect foresight. In Aguiar-Conraria and Shell (2005), we focus on the degenerate case in which two machines can be distinguished only by their colors (blue or red): their productivities and their replacement costs if newly produced are independent of their color, but their market prices are allowed to depend on color. We show that, on the unique bubble-free trajectory, the prices of the two capital goods are always equal, but it can take several generations before the bubble must burst.

9 Concluding remarks

Capital gains are at the heart of the capitalist economy, but they are suspected of being a source of instability. Keynes certainly mistrusted capital gains. He even went so far as to suggest that to reduce instability capital ownership be made, like marriage in his time, indissoluble except for grave cause. One interpretation of the Great Depression is that expected capital gains on holding money were very high (i.e. the general price level was falling rapidly) so that Tobin's q was driven below unity leading to drying up of investment.

[7] See Magill and Quinzii (2003).

Frank Hahn once concluded that the unstable dynamical system that results from the short-run perfect-foresight market-clearing equation is the golden nail in the coffin of capitalism.[8]

The analysis of capital gains raises fundamental questions about the formation of expectations and the nature of temporary equilibrium. These are subjects in which Jean-Michel Grandmont is the master.[9] There might also be a role for sunspots. Our formal analysis shows that in our particular (non-monetary!) model, the only infinite-horizon equilibrium path is bubble free. However, on our calculated trajectories, the bubble is revealed only after several decades. In a technical sense, bubble paths are not perfect-foresight equilibrium paths, but bubbles that burst in the far future beyond current lifetimes stretch the equilibrium concept. Perhaps the informational and strategic foundations of the expectations process should be re-examined.

Individuals perceive capital gains as part of income, as they do dividend and interest receipts. Individuals perceive capital gains as accretions to wealth and, hence, part of saving. Traditional measures of income and saving that do not include capital gains can be misleading.

Our model is special. It is non-monetary. Money allows for non-bursting bubbles. It exhibits saddlepoint dynamics. Not all multi-asset dynamics are of this type.[10] Our goal was merely to study a simple example in some detail to partially redirect the macro literature from the ILRA model to one (such as the heterogeneous agent, OG model) that might allow for destabilizing effects from capital gains.

References

Aguiar-Conraria, L., and K. Shell (2005), "Capital gains: Blue machines and red machines," *Singapore Economic Review* **50**, 437–48.

Atkinson, A. B. (1969), "The timescale of economic models: How long is the long-run?," *The Review of Economic Studies* **36**, 137–52.

Benhabib, J., and K. Nishimura (1998) "Indeterminacy and sunspots with constant returns," *Journal of Economic Theory* **81**, 58–96.

Benhabib, J., and A. Rustichini (1994), "Introduction to the symposium on growth, fluctuations, and sunspots: Confronting the data," *Journal of Economic Theory* **63**, 1–18.

Burmeister, E., C. Caton, A. Dobell, and S. Ross (1973), "The 'Saddlepoint Property' and the structure of dynamic heterogeneous capital good models," *Econometrica* **41**, 79–95.

Burmeister, E., and D. Graham (1974), "Multi-sector economic models with continuous adaptive expectations," *Review of Economic Studies* **41**, 323–36.

Cass, D., and K. Shell (1976), "The structure and stability of competitive dynamical systems," *Journal of Economic Theory* **12**, 31–70.

Caton, C., and K. Shell (1971), "An exercise in the theory of heterogeneous capital accumulation," *The Review of Economic Studies* **37**, 13–22.

[8] See Hahn (1966), the seminal paper on the "Hahn problem".

[9] See, for example, Grandmont (1974, 1977, 1983, 1985) and Grandmont and Hildenbrand (1974).

[10] See, for example, Cass and Shell (1976) for conditions under which the optimal trajectory is unique. See Benhabib and Rustichini (1994), Benhabib and Nishimura (1998) and Wen (1998) for examples of indeterminacy of the equilibrium, history dependence, and sunspots.

Diamond, P. (1965), "National debt in a neoclassical growth model," *American Economic Review* **55**, 1126–50.

Grandmont, J.-M. (1974), "On the short run equilibrium in a monetary economy," J. Drèze, ed., *Allocation under Uncertainty: Equilibrium and Optimality*, 213–28, London: Macmillan.

Grandmont, J.-M. (1977), "Temporary general equilibrium theory," *Econometrica* **45**, 535–72.

Grandmont, J.-M. (1983), *Money and Value: A Reconsideration of Classical and Neoclassical Monetary Theories*, Cambridge, UK: Cambridge University Press.

Grandmont, J.-M. (1985), "On endogenous competitive business cycles," *Econometrica* **53**, 995–1045.

Grandmont, J.-M., and W. Hildenbrand (1974), "Stochastic processes of temporary equilibria," *Journal of Mathematical Economics* **1**, 247–77.

Hahn, F. H. (1966), "Equilibrium dynamics with heterogeneous capital goods," *Quarterly Journal of Economics* **80**, 633–46.

Magill, M., and M. Quinzii (2003), "Nonshiftable capital, affine price expectations and convergence to the Golden Rule," *Journal of Mathematical Economics* **39**, 239–72.

Shell, K. (1969), "Applications of Pontryagin's Maximum Principle to Economics," H. W. Kuhn, and G. P. Szegö, eds, *Mathematical Systems Theory and Economics I*, 273–5, Berlin: Springer Verlag.

Shell, K. (1972), "On competitive dynamical systems," H. W. Kuhn, and G. P. Szegö, eds, *Differential Games and Related Topics*, 449–476, Amsterdam: North-Holland.

Shell, K., M. Sidrauski, and J. Stiglitz (1969), "Capital gains, income, and saving," *The Review of Economic Studies* **36**, 15–26.

Shell, K., and J. Stiglitz (1967), "The allocation of investment in a dynamic economy," *Quarterly Journal of Economics* **81**, 592–609.

Tirole, J. (1985), "Asset bubbles and overlapping generations," *Econometrica* **53**, 1499–528.

Wen, Y. (1998), "Capacity utilization under increasing returns to scale," *Journal of Economic Theory* **81**, 7–36.