COMPARATIVE STATICS FOR THE TWO-SECTOR MODEL

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1. INTRODUCTION.

In this note, certain simple propositions in comparative statics are developed for the two-sector model. I was led to the study of such propositions because of their relevance to the direction and stability of long-run growth in certain models; see for example [5]. It seems that these propositions may be of independent interest and therefore a separate treatment is warranted (1).

2. THE MODEL.

Following Uzawa [6], consider an economy that is composed of two sectors, labelled 1 and 2, respectively. There are two factors of production inelastically offered for employment, say, capital and labor, with respective endowments K and L. Let K_j and L_j represent the amount of the factors allocated to the jth sector. If Y_j is the amount of the homogeneous output of the jth sector, we have

$$Y_j = F_j(K_j, L_j) \qquad \text{for } j = 1, 2, \tag{1}$$

where $F_j(\cdot)$ is the neoclassical production function with the following properties:

$$\lambda Y_j = F_j(\lambda K_j, \lambda L_j) \qquad \text{for } K_j, L_j \geq 0 \text{ and } \lambda > 0 ; \tag{2}$$

$F_j(K_j, L_j)$ is continuously twice-differentiable with

$$\frac{\partial F_j}{\partial K_j} > 0, \ \frac{\partial F_j}{\partial L_j} > 0, \ \frac{\partial^2 F_j}{\partial K_j^2} < 0, \ \frac{\partial^2 F_j}{\partial L_j^2} < 0 \text{ for } 0 < K_j, L_j < \infty . \tag{3}$$

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(1) The pioneer work in two-sector comparative statics appears to be that of Rybczynski [4] whose analysis is in terms of the Samuelson-Stolper box diagram. I am indebted to Professor John Wise for the reference to Rybczynski's note.

If p is the supply price of a unit of output of the first sector in terms of a unit of output of the second sector, efficiency in production implies that

$$(4) \qquad \begin{cases} r = \dfrac{\partial F_2}{\partial K_2} = p \dfrac{\partial F_1}{K_1}, \\[2ex] w = \dfrac{\partial F_2}{\partial L_2} = p \dfrac{\partial F_1}{\partial L_1}. \end{cases}$$

If the factor markets are competitive, r and w are the respective rewards to units of the first and second factors.

Assumption (3) together with the assumption of efficiency in production implies full-employment of resources

$$(5) \qquad \begin{aligned} K_1 + K_2 &= K, \\ L_1 + L_2 &= L. \end{aligned}$$

Defining national income Y by

$$(6) \qquad Y = Y_2 + pY_1,$$

the static model is closed by the demand equation

$$(7) \qquad pY_1 = sY,$$

where s $(0 < s < 1)$ is the average propensity to save out of income. In this formulation, the average propensity to save is identical to the marginal propensity to save.

Because of the constant-returns-to-scale assumption (i), we can adopt the usual shorthand notation:

$$k = \frac{K}{L}, \qquad y = \frac{Y}{L};$$

$$k_j = \frac{K_j}{L_j}, \qquad y_j = \frac{Y_j}{L}, \qquad e_j = \frac{L_j'}{L};$$

$$\omega = \frac{w}{r}; \qquad f_j(k_j) = F_j(k_j, 1).$$

Conditions (3) imply that $f_j(k_j)$ is continuously twice differentiable with

$$(8) \qquad f_j(k_j) > 0, \; f_j'(k_j) > 0, \; f_j''(k_j) < 0 \text{ for } 0 < k_j < \infty.$$

The above system reduces to the following miniature Walrasian system of general equilibrium:

$$(9) \qquad y_1 = f_1(k_1) \frac{k_2 - k}{k_2 - k_1}, \qquad y_2 = f_2(k_2) \frac{k - k_1}{k_2 - k_1},$$

$$(10) \quad \omega = \frac{f_j(k_j)}{f_j'(k_j)} - k_j \qquad \text{for } j = 1, 2,$$

$$(11) \quad p = \frac{f_2'(k_2)}{f_1'(k_1)},$$

$$(12) \quad y = y_2 + py_1,$$

$$(13) \quad py_1 = sy.$$

3. Comparative Statics.

Differentiating (10) with respect to ω yields

$$(14) \quad \frac{dk_j}{d\omega} = \frac{-[f_j'(k_j)]^2}{f_j(k_j)\, f_j''(k_j)} > 0 \qquad \text{for } j = 1, 2.$$

Thus the implicit relations $k_j(\omega)$ are well-defined.
Combining (11)-(13) with (9) gives the basic equation

$$(15) \qquad \left(\frac{1-s}{s}\right) \frac{f_2'(k_2)}{f_1'(k_1)} = \left(\frac{k-k_1}{k_2-k}\right) \frac{f_2(k_2)}{f_1(k_1)}.$$

Logarithmically differentiating both sides of (15) with respect to k, yields the total derivative

$$(16) \quad \frac{d\omega}{dk} = \frac{\dfrac{k_2-k_1}{(k-k_1)(k_2-k)}}{\dfrac{dk_2}{d\omega}\left(\dfrac{f_2''}{f_2'} - \dfrac{f_2'}{f_2} + \dfrac{1}{k_2-k}\right) + \dfrac{dk_1}{d\omega}\left(\dfrac{f_1'}{f_1} - \dfrac{f_1''}{f_1'} + \dfrac{1}{k-k_1}\right)}$$

Because of (10) and (14), (16) can be rewritten as

$$(17) \quad \frac{d\omega}{dk} =$$

$$= \frac{\dfrac{k_2-k_1}{(k-k_1)(k_2-k)}}{\dfrac{dk_2}{d\omega}\left\{\dfrac{k+\omega}{(k_2-k)(k_2+\omega)}\right\} + \dfrac{dk_1}{d\omega}\left\{\dfrac{k+\omega}{(k-k_1)(k_1+\omega)}\right\} + \left\{\dfrac{k_2-k_1}{(k_2+\omega)(k_1+\omega)}\right\}}$$

For $k_2 \neq k \neq k_1$, the numerator and the denominator of the RHS of (17) are seen to agree in sign and therefore $\dfrac{d\omega}{dk} > 0$ for $0 < k < \infty$. This is the proposition (Uzawa) that if the demand for commodities is such that the marginal propensity to consume equals the average propensity to consume, *the higer the endowment of a factor* *of production, the lower is the equilibrium level of the relative reward to that factor.* Also given $0 < k < \infty$, the equilibrium value of ω is uniquely determined.

Next, observe the *direct* effect of differing factor endowments upon the equilibrium composition of output. From

$$\text{(9)} \qquad \frac{\partial y_1}{\partial k} = \frac{-f_1(k_1)}{k_2 - k_1},$$

$$\text{(18)} \qquad \frac{\partial y_2}{\partial k} = \frac{f_2(k_2)}{k_2 - k_1}.$$

The partial derivatives in (18) are independent of demand, and thus we have the proposition (Rybczynski): *If the rates of substitution in production are fixed,* i.e., $\frac{d\omega}{dk} = 0$, *then the higher the endowment of a factor of production, the higher (lower) is the equilibrium level of production of the commodity using relatively much (little) of that factor.*

Logarithmic differentiation of (9) yields

$$\text{(19)} \qquad \frac{1}{y_1}\frac{dy_1}{d\omega} = \frac{dk_2}{d\omega}\left(\frac{k - k_1}{(k_2 - k)(k_2 - k_1)}\right) + \frac{dk_1}{d\omega}\left(\frac{k_2 + \omega}{(k_1 + \omega)(k_2 - k_1)}\right) \gtreqless 0$$

as $k_2 \gtreqless k \gtreqless k_1$. Similarly $\frac{1}{y_2}\frac{dy_2}{d\omega} \gtreqless 0$ as $k_2 \gtreqless k \gtreqless k_1$.

Since $\frac{d\omega}{dk} > 0$, we have for the system (9)-(13) that *the direct effect* (18) *of varying factor endowment upon equilibrium levels of output is opposite in sign to the indirect effect* (19).

The *total effect* is the sum of the direct effect and the indirect effect:

$$\text{(20)} \qquad \begin{aligned} \frac{dy_1}{dk} &= \frac{\partial y_1}{\partial k} + \frac{dy_1}{d\omega}\frac{d\omega}{dk}, \\ \frac{dy_2}{dk} &= \frac{\partial y_2}{\partial k} + \frac{dy_2}{d\omega}\frac{d\omega}{dk}. \end{aligned}$$

Because of (18) and (19), the first equation in (20) can be rewritten as

$$\text{(21)} \qquad \frac{dy_1}{dk} = \frac{\partial y_1}{\partial k}\left(1 - \frac{N}{D}\right),$$

where N is defined by

$$(22)\quad N = \frac{dk_2}{d\omega}\left(\frac{k-k_1}{k_2-k_1}\right) + \frac{dk_1}{d\omega}\left(\frac{k_2-k}{k_2-k_1}\right)\left(\frac{k_2+\omega}{k_1+\omega}\right) > 0\,,$$

and D is defined by

$$(23)\qquad D = \frac{dk_2}{d\omega}\left(\frac{k-k_1}{k_2-k_1}\right) + \frac{dk_1}{d\omega}\left(\frac{k_2-k}{k_2-k_1}\right)\left(\frac{k+\omega}{k_1+\omega}\right)$$
$$+\frac{(k-k_1)(k_2-k)}{k_2-k_1}\left\{\left(\frac{f_2''}{f_2'}-\frac{f_2'}{f_2}\right)\frac{dk_2}{d\omega}-\frac{f_1''}{f_1'}\,\frac{dk_1}{d\omega}\right\}.$$

Applying (10) and (14) to (23) gives

$$(24)\qquad D = \frac{dk_2}{d\omega}\left(\frac{k-k_1}{k_2-k_1}\right) + \frac{dk_1}{d\omega}\left(\frac{k_2-k}{k_2-k_1}\right)\left(\frac{k+\omega}{k_1+\omega}\right)$$
$$+\frac{(k-k_1)(k_2-k)}{k_2-k_1}\left\{\frac{1}{k_1+\omega}-\frac{1+\dfrac{dk_2}{d\omega}}{k_2+\omega}\right\} > 0\,.$$

Consider the case where $k_1 > k > k_2$. We have from (18) that $\dfrac{\partial y_1}{\partial k} > 0$ and therefore $\dfrac{dy_1}{dk} > 0$ if and only if $D > N$. Examine the RHS of (22) and (24). The first terms are identical; for $k > k_2$ the second term in (24) is greater than the second term in (22). For $k_1 > k_2$, the third term in (24) is positive. Hence when $k_1 > k > k_2$, $D > N > 0$ or $0 < \dfrac{N}{D} < 1$.

For the two-sector economy (9)-(13), *the higher the endowment of a factor of production, the higher is the equilibrium level of output of the commodity using relatively much of that factor.*

Consider the reverse factor-intensity case, $k_2 > k > k_1$. $\dfrac{dy_1}{dk}$ is positive if and only if $(D - N) < 0$, or subtracting (22) from (23)

$$(25)\quad (k-k_1)\frac{dk_2}{d\omega}\left(\frac{f_2''}{f_2'}-\frac{f_2'}{f_2}\right)-\frac{dk_1}{d\omega}\left(\frac{(k_2-k)f_1'}{f_1}+\frac{(k-k_1)f_1''}{f_1'}\right) < 0.$$

Multiplying both sides of (25) by $\left(\dfrac{\omega}{k_1\,k_2}\right)$ and substituting from (10) and (14) yields

$$(26)\qquad \frac{(k-k_1)\,\sigma_2}{k_1\,(k_2+\omega)} + \left(\frac{\omega}{k_1\,k_2}\right)\left(\frac{k-k_1}{k_2+\omega}\right) + \\ + \frac{(k_2-k)\,\sigma_1}{k_2\,(k_1+\omega)} > \left(\frac{\omega}{k_1\,k_2}\right)\left(\frac{k-k_1}{k_1+\omega}\right),$$

where σ_j ($j = 1, 2$) is the elasticity of substitution between factors in the jth sector. This basic property of production functions was introduced by Hicks and refined by Allen ([2]). The elasticity of substitution can be written as

$$\sigma_j\,(\omega) = \frac{\omega}{k_j}\,\frac{dk_j}{d\omega} \qquad \text{for } j = 1\,,\,2\,.$$

Rearranging (26) gives

$$(27)\quad \sigma_2 > \frac{\omega}{k_1+\omega} - \frac{\omega k_1}{\omega k_2 + k_1\,k_2} - \frac{k_1\,(k_2+\omega)\,(k_2-k)\,\sigma_1}{k_2\,(k_1+\omega)\,(k-k_1)}\,.$$

From (27) a simple *sufficient* condition for $\frac{dy_1}{dk}$ to be positive is that $\sigma_2 \geq 1$. Thus *if the elasticity of substitution in the production of commodity two (one) is greater than or equal to unity, then the higher the endowment of either factor of production, the higher is the equilibrium level of output of commodity one (two).*

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([2]) Cf. pages 117, 245 in [2], pages 341-343 in [1], and [3].

APPENDIX A: *The Case of Cobb-Douglas Production Functions*

It is instructive to study the special case where the production functions [1] are linear in logarithms. For this case, we can write

$$(28) \qquad \begin{cases} f_1(k_1) = k_1^a & \text{where } 0 < a < 1 , \\ f_2(k_2) = k_2^b & \text{where } 0 < b < 1 . \end{cases}$$

Applying equation (10) to equation (28) yields

$$(29) \qquad \begin{cases} k_1 = \dfrac{a\omega}{1-a} \text{ and} \\ k_2 = \dfrac{b\omega}{1-b} . \end{cases}$$

Substituting (28) and (29) into (15) yields

$$(30) \qquad \omega = \left[\frac{s(1-a) + (1-s)(1-b)}{sa + (1-s)b} \right] k .$$

From (29) and (30)

$$k_1 = \gamma_1 k \quad \text{and} \quad k_2 = \gamma_2 k$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$ are constants fixed upon specification of the parameters s, a, b. Notice also that

$$l_1 = \frac{k_2 - k}{k_2 - k_1} = \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} \text{ and}$$
$$l_2 = \frac{k - k_1}{k_2 - k_1} = \frac{1 - \gamma_1}{\gamma_2 - \gamma_1} .$$

Therefore $0 < l_1 < 1$ and $0 < l_2 < 1$ are in the Cobb-Douglas case fixed constants. Hence if production satisfies (9)-(11), (28) and if demand satisfies (11)-(13), then $\dfrac{dy_1}{dk}$ and $\dfrac{dy_2}{dk}$ are positive and constant for all $k > 0$.

APPENDIX B: *Some Further Propositions*

In the study of the model of inventive activity and capital accumulation [5], one is interested in the sign of an expression which is equivalent to

$$(31) \qquad y_2 - k\frac{dy_2}{dk} ,$$

which can be rewritten as

$$(32) \qquad \frac{f_2(k_2)}{D(k_2 - k_1)} [kN - k_1 D]$$

where by (10), N is given by

$$N = \frac{dk_2}{d\omega}\left(\frac{k_1 + \omega}{k_2 + \omega}\right)\left(\frac{k - k_1}{k_2 - k_1}\right) + \frac{dk_1}{d\omega}\left(\frac{k_2 - k}{k_2 - k_1}\right) > 0$$

for $k_2 \neq k \neq k_1$, and where $D > 0$ is defined by equation (23).

Form the expression

$$(33)\quad kN - k_1D = \frac{k - k_1}{k_2 - k_1}\left[\frac{dk_2}{d\omega}\left\{k - k_1 - \frac{k(k_2 - k_1)}{k_2 + \omega} + \frac{k_1(k_2 - k)}{k_2 + \omega}\right\}\right.$$
$$\left. + \frac{dk_1}{d\omega}\left(\frac{k_2 - k}{k - k_1}\right)\left\{k - k_1 - \frac{k_1(k - k_1)}{k_1 + \omega}\right\} + \frac{k_1(k_2 - k)}{k_2 + \omega} - \frac{k_1(k_2 - k)}{k_1 + \omega}\right].$$

The RHS of equation (33) can be rewritten as

$$(34)\qquad \frac{(k - k_1)(k_2 - k)}{k_2 - k_1}\left[\frac{dk_2}{d\omega}\left(\frac{k - k_1}{k_2 - k}\right)\left\{1 - \frac{k_2}{k_2 + \omega}\right\} + \right.$$
$$\left. + \frac{dk_1}{d\omega}\left\{1 - \frac{k_1}{k_1 + \omega}\right\} + k_1\left\{\frac{1}{k_2 + \omega} - \frac{1}{k_1 + \omega}\right\}\right].$$

Notice that for the case $k_1 > k > k_2$, expression (34) is negative. Therefore if $k_1 > k > k_2$, then expressions (31) and (32) are positive.

For the reverse case $k_2 > k > k_1$, expression (34) tells us that $kN > k_1D$ if and only if

$$(35)\quad \frac{dk_2}{d\omega}\left(\frac{k - k_1}{k_2 - k}\right)\left(\frac{\omega}{k_2 + \omega}\right) + \frac{dk_1}{d\omega}\left(\frac{\omega}{k_1 + \omega}\right) > \frac{k_1(k_2 - k_1)}{(k_1 + \omega)(k_2 + \omega)}.$$

Diving (35) by $k_1\, k_2$ yields

$$(36)\qquad \sigma_1 > \frac{k_2}{k_2 + \omega} - \frac{k_1}{k_2 + \omega} - \left(\frac{k - k_1}{k_2 - k}\right)\left(\frac{k_1 + \omega}{k_2 + \omega}\right)\frac{k_2}{k_1}\,\sigma_2\,.$$

Therefore, if $k_2 \neq k_1$, a simple *sufficient* condition for expression (31) to be positive is for $\sigma_1 \geq 1$.