Reprinted from:

DIFFERENTIAL GAMES AND RELATED TOPICS

ON COMPETITIVE DYNAMICAL SYSTEMS*

Karl SHELL

1. Introduction to the "Sequence Economy"

Debreu's beautiful little book *Theory of Value* probably provides the best summary of the contemporary general equilibrium theory of the static, competitive economy. The interpretation of this theory is based on the notions of the Walrasian auctioneer and timeless contracting. In an atemporal setting, the auctioneer calls out a price vector and agents (consumers and producers) submit to the auctioneer their demands for goods. If markets clear for some vector of nonnegative prices, then that price system is "settled upon"; exchange and production then take place in that price environment. If markets do not clear, the auctioneer tries other price vectors until he achieves equilibrium, where for each commodity either market excess demand is zero or market excess demand is negative and the price of the commodity is zero.

The Walrasian paradigm is very powerful, but it has some shortcomings. Even in the study of short-run phenomena, for which the general equilibrium model is best suited, one must allow for the fact that in real life there is no such auctioneering process and trading can be expected to take place at "disequilibrium" prices.

The dynamic extension of the general equilibrium model is less satisfactory. Dated goods are introduced. Thus a model with m conventional goods and T periods is formally equivalent to the static model with mT goods. The usual interpretation is that the atemporal auction includes futures markets but is held only once, no recontracting is allowed. This story is difficult to relate to everyday economic life. Among other things, with births and deaths, not all agents in the dynamic economy are alive at any given period.

The implicit "equilibrium dynamics" story of many models of economic

* The preparation of these lecture notes was supported in part by National Science Foundation Grant GS-2421 to the University of Pennsylvania. Travel was supported by a grant from C.N.R., *Comitato Scienze Economiche, Sociologiche e Statistiche.*

growth is rather different from that of the Debreu model. First, there is the technical difference caused by the infinite-horizon ($T = \infty$) assumption. With an infinity of dated commodites and/or an infinity of agents, certain technical difficulties arise. For one thing, it is no longer necessarily true that in the absence of externalities competitive-equilibrium allocations are Pareto-optimal allocations. (It should be noted that many of the very recent contributions to general equilibrium theory treat the infinity problem.)

The second difference between "equilibrium dynamics" and the general-equilibrium story is more fundamental. Frank Hahn has coined the term "sequence economy" for the model often employed in growth theory. At each moment in the sequence economy all markets clear (including futures markets if they exist), but recontracting can take place at any future date.

I fully realize that my definitions of "equilibrium dynamics" and "sequence economy" are vague. This is intentional. I merely hope to set the stage for the specific models which will follow. I have been assigned the task of discussing competitive dynamical systems. I shall present Hahn's basic model of heterogeneous capital accumulation and several variations on and special instances of that model. Based on special cases, one can make a conjecture about the solution to the general "Hahn problem". The general problem cannot be stated in terms of differential equations but must be stated as a generalized dynamical system – in particular as a system of differential correspondences.

The analysis is easily extended to include paper assets (such as money and government bonds) as stores of value alternative to physical productive capital. I conclude the lectures with some reflections on the Samuelson consumption-loan paradox, transversality conditions, and other topics in the "economics of infinity".

2. The Quarter-Circle Technology*

I begin the study of the development of the sequence economy with the very simple joint-production example which my colleague Christopher Caton and I worked out. We study a one-sector, two-capital model in which a homogeneous output, Y, is produced by the cooperation of labor, L, machines of the first kind, K_1, and machines of the second kind, K_2. Assuming constant returns-to-scale and denoting quantities per unit labor by lower case letters,

* This section is based on [4].

the production relation can be written as $y = f(k_1, k_2)$. For ease of analysis we assume that the production function is linear in logarithms, so

$$(2.1) \qquad y = f(k_1, k_2) = k_1^{\alpha_1} k_2^{\alpha_2} ,$$

where $\alpha_1 > 0, \alpha_2 > 0$, and $1 - \alpha_1 - \alpha_2 > 0$. If we assume that all rentals are saved, while all wages are consumed, demand for consumption per head is given by

$$f - k_1 f_1 - k_2 f_2 = (1 - \alpha_1 - \alpha_2) k_1^{\alpha_1} k_2^{\alpha_2} ,$$

when factor markets are competitive. If the product markets are in momentary equilibrium so that demand and supply of consumption are equal,

$$(2.2) \qquad c = f - k_1 f_1 - k_2 f_2 = (1 - \alpha_1 - \alpha_2) y > 0 ,$$

when c is consumption per man. At every instant saving and gross investment per man is equal to $(\alpha_1 + \alpha_2) y \equiv z > 0$.

Following the example of Samuelson, [10], we assume a "quarter-circle" technology is available for costlessly transforming undifferentiated (gross) investment, z, into gross investment in machinery of the first kind, z_1, and gross investment in machinery of the second kind, z_2. That is,

$$(2.3) \qquad z^2 = z_1^2 + z_2^2 ,$$

with $z_1 \geqslant 0$ and $z_2 \geqslant 0$. We choose the consumption good as *numéraire*. In momentary equilibrium, the price of a unit of consumption, p_c, must equal the price of a unit of investment, p_z, since both goods are produced by the same technology, and if p_y is the price of a unit of undifferentiated output,

$$(2.4) \qquad p_y = p_z = p_c \equiv 1 ,$$

since both c and z are positive.

Under competition, profits in the conversion of undifferentiated investment into differentiated investment must be zero. Thus,

$$(2.5) \qquad p_1 z_1 + p_2 z_2 = p_z z = z ,$$

where p_1 and p_2 are the unit prices of the respective investment goods. Assume that machinery of either kind depreciates at the constant proportionate rate

μ and that the labor force is growing at the constant proportionate rate n. Then capital accumulation can be described by the equations:

$$\dot{k}_1 = z_1 - \lambda k_1 \,, \tag{2.6}$$

and

$$\dot{k}_2 = z_2 - \lambda k_2 \,, \tag{2.7}$$

where $\lambda \equiv \mu + n > 0$.

Momentary equilibrium. Momentary equilibrium in the present model is always unique. That is, given endowments and prices, the output of the consumption good and the two capital goods are uniquely determined. This is easily seen because from (2.1) and (2.2), k_1 and k_2 uniquely determine y, c, and z. Given z, p_1 and p_2, the profit-maximizing values of z_1 and z_2 are uniquely determined by (2.3) and

$$z_1/z_2 = p_1/p_2 \,. \tag{2.8}$$

Equation (2.8) states that production on the quarter circle with radius z will take place at the value of (z_1,z_2) at which the marginal rate of technical transformation is equal to the price ratio. By assumption (2.3), equation (2.8) holds for all positive but finite values of p_1 and p_2.

Squaring both sides of (2.5) and (2.3) and combining yields

$$p_1{}^2(1 - p_1{}^2) + p_2{}^2(1 - p_2{}^2) - 2p_1{}^2 p_2{}^2 = 0 \,,$$

which can be rewritten as

$$(p_1{}^2 + p_2{}^2)(p_1{}^2 + p_2{}^2 - 1) = 0 \,.$$

Ignoring the extraneous root, we have that p_1 and p_2 must lie on the unit circle, i.e.,

$$p_1{}^2 + p_2{}^2 = 1 \,. \tag{2.9}$$

Next we turn to the equilibrium conditions in the used-machinery market. *Rentiers* hold machines of the first and second kind. Expected rates of return on the two types of machinery must be equal, else all wealth holders will attempt to specialize to the capital good with the higher rate of return. If $\dot{p}_1^e$ and $\dot{p}_2^e$ are the expected rates of price change for the respective capital goods, then the expected consumption rates of return must be equalized, or

$$\dot{p}_1^e/p_1 + r_1/p_1 = \dot{p}_2^e/p_2 + r_2/p_2 \,, \tag{2.10}$$

where r_1 and r_2 are the respective rental rates on machinery. If expectations are always ratified, $\dot{p}_1^e = \dot{p}_1$, and $\dot{p}_2^e = \dot{p}_2$, then from (2.10),

$$\dot{p}_1/p_1 + f_1/p_1 = \dot{p}_2/p_2 + f_2/p_2 \,, \tag{2.11}$$

since under competition rental rates are equal to marginal products, i.e., $r_1 = f_1$ and $r_2 = f_2$.

Balanced growth. Combining (2.3), (2.5), and (2.8) yields

$$z_1 = p_1 z \quad \text{and} \quad z_2 = p_2 z \,. \tag{2.12}$$

In light of (2.9), it is legitimate to cast the analysis in terms of the price ratio $p \equiv p_2/p_1$. From (2.9)

$$p_1 = (1 + p^2)^{-1/2} \quad \text{and} \quad p_2 = p(1 + p^2)^{-1/2} \,. \tag{2.13}$$

Substituting (2.12) and (2.13) in (2.6) and (2.7) and using (2.2) gives the two capital accumulation equations

$$\dot{k}_1 = (\alpha_1 + \alpha_2)\, y(1 + p^2)^{-1/2} - \lambda k_1 \tag{2.14}$$

and

$$\dot{k}_2 = p(\alpha_1 + \alpha_2)\, y(1 + p^2)^{-1/2} - \lambda k_2 \,. \tag{2.15}$$

Since in our model $f_1 = \alpha_1 y/k_1$ and $f_2 = \alpha_2 y/k_2$, asset-market-clearing equation (2.11) can be rewritten as

$$\dot{p} = y\sqrt{1 + p^2}\, (p\alpha_1/k_1 - \alpha_2/k_2) \,, \tag{2.16}$$

using (2.13).

A balanced-growth equilibrium is then defined by $\dot{k}_1 = \dot{k}_2 = \dot{p} = 0$. If (k_1^*, k_2^*, p^*) is the value of (k_1, k_2, p) in balanced growth, then from (2.14)-(2.16) it must solve the system

$$(2.17)\qquad k_2 = p(\alpha_1 + \alpha_2)y(1+p)^{-1/2}/\lambda \,,$$

$$(2.18)\qquad p = \alpha_2 k_1/\alpha_1 k_2 = \sqrt{\alpha_2/\alpha_1} \,.$$

Equation (2.18) shows that p^* and the ratio $(k_1{}^*/k_2{}^*)$ are uniquely determined. Substituting (2.18) in (2.17) yields

$$(2.19)\qquad k_2 = p(\alpha_1 + \alpha_2)(\sqrt{\alpha_1/\alpha_2}\, k_2)^{\alpha_1}\, k_2{}^{\alpha_2}/\lambda \,.$$

Since $0 < \alpha_1 + \alpha_2 < 1$, the right-hand side of (2.19) is an increasing concave function of k_2 with a first derivative that is infinite when k_2 is zero and is zero when k_2 is infinite. Therefore, the balanced-growth equilibrium $(k_1{}^*, k_2{}^*, p^*)$ is uniquely determined.

Notice from (2.17) and (2.18) that if $\alpha_1 = \alpha_2$, then $p^* = 1$ and $k_1{}^* = k_2{}^*$. In this case the marginal product $f_1(k_1{}^*, k_2{}^*)$ equals the marginal product $f_2(k_1{}^*, k_2{}^*)$ in balanced growth. If $\alpha_1 > \alpha_2$, then $p^* < 1$ and $k_1{}^* > k_2{}^*$. But from (2.1) and (2.17)

$$f_1(k_1{}^*, k_2{}^*)/f_2(k_1{}^*, k_2{}^*) = \sqrt{\frac{\alpha_1}{\alpha_2}} > 1 \,,$$

by hypothesis. Thus, in balanced growth where $\alpha_1 > \alpha_2$, more of the first kind of machinery is produced than the second, but the stock of machines of type one is not deepened sufficiently to equate marginal products.

It will be useful to note that, across steady-states ($\dot{k}_1 = 0 = \dot{k}_2$), consumption per man, c, is maximized when $(k_1, k_2) = (k_1{}^*, k_2{}^*)$. From (2.6) and (2.7), $z_1 = \lambda k_1$ and $z_2 = \lambda k_2$ in steady states. If θ is an undetermined Lagrange multiplier, the appropriate Lagrangean form is

$$k_1{}^{\alpha_1} k_2{}^{\alpha_2} - z + \theta\,[z^2 - \lambda^2(k_1{}^2 + k_2{}^2)]$$

in light of (2.3). The first-order conditions are:

$$\alpha_1 y/k_1 = 2\lambda^2\,\theta k_1 \,,$$

$$\alpha_2 y/k_2 = 2\lambda^2\,\theta k_2 \,,$$

$$2z\theta = 1 \,,$$

which together yield the solution

$$k_1/k_2 = \sqrt{\alpha_1/\alpha_2}\,, \tag{2.20}$$

which reduces to the real equation in (2.18). Combining (2.20) with the condition that $\dot{k}_1 = 0 = \dot{k}_2$, yields the same result as is achieved when p is eliminated from (2.17) and (2.19) by using (2.18). This maximal-consumption result should come as no surprise. In balanced growth, the competitive allocation of investment is efficient. Furthermore, reminiscent of the usual Golden Rule exercises, the Marxian saving hypothesis ensures that the rate of interest equals the rate of growth.

Dynamic analysis. We have found that, in (k_1, k_2, p) phase space, there is a unique rest point $(k_1{}^*, k_2{}^*, p^*)$, which we call balanced-growth equilibrium. We turn now to a full dynamic analysis of the system of differential equations (2.14)–(2.16).

We can approximate (2.14)–(2.16) by its linear Taylor expansion at the rest point, so that

$$\begin{bmatrix} \dot{k}_1 \\ \dot{k}_2 \\ \dot{p} \end{bmatrix} \doteq \begin{bmatrix} \lambda(\alpha_1 - 1) & \lambda p \alpha_1 & \dfrac{-p\alpha_1 y}{\sqrt{1+p^2}} \\ \lambda p \alpha_1 & \lambda(p\alpha_1 - 1) & \dfrac{\alpha_1 y}{\sqrt{1+p^2}} \\ \dfrac{-p\lambda^2}{\alpha_1 y \sqrt{1+p^2}} & \dfrac{\lambda^2}{\alpha_1 y\sqrt{1+p^2}} & \lambda \end{bmatrix} \begin{bmatrix} (k_1 - k_1{}^*) \\ (k_2 - k_2{}^*) \\ (p - p^*) \end{bmatrix} \tag{2.21}$$

where the 3 X 3 matrix above is evaluated at $(k_1{}^*, k_2{}^*, p^*)$. The characteristic equation for the linear system (2.21) is then

$$\begin{vmatrix} \lambda(\alpha_1 - 1) - x & \lambda p \alpha_1 & \dfrac{-p\alpha_1 y}{\sqrt{1+p^2}} \\ \lambda p \alpha_1 & \lambda(p\alpha_1 - 1) - x & \dfrac{\alpha_1 y}{\sqrt{1+p^2}} \\ \dfrac{-p\lambda^2}{\alpha_1 y \sqrt{1+p^2}} & \dfrac{\lambda^2}{\alpha_1 y\sqrt{1+p^2}} & \lambda - x \end{vmatrix} = 0\,, \tag{2.22}$$

where x is the characteristic root and terms in (2.22) are evaluated at (k_1^*, k_2^*, p^*). Solving cubic equation (2.22) for x yields the three characteristic roots

$$(2.23) \qquad x = \lambda(\alpha_1 + \alpha_2 - 1), \quad \lambda\sqrt{2}, \quad -\lambda\sqrt{2}.$$

All roots are real, two are negative and one is positive. For the system (2.14)–(2.16) in (k_1, k_2, p) phase-space, there exists a two-dimensional surface along which all trajectories tend to (k_1^*, k_2^*, p^*) as $t \to \infty$. Also, the manifold of solutions tending to (k_1^*, k_2^*, p^*) as $t \to -\infty$ is of dimensionality 1 (a curve). Therefore, in the neighborhood of (k_1^*, k_2^*, p^*) for every endowment vector, (k_1, k_2), there exists one and only one price ratio, p, such that the system tends to (k_1^*, k_2^*, p^*) as $t \to \infty$.

We now turn to the *global* analysis of the differential equation system (2.14)–(2.16). The straightforward phase portrait for (2.14)–(2.16) must be drawn in three-dimensional space. To avoid this complexity, we adopt a trick used by Atkinson [1] which reduces the qualitative analysis to two dimensions.

Setting $k \equiv k_2/k_1$, we can study the development of our model economy in the (k,p) "phase" plane of Figure 2.1. From (2.14) and (2.15), when $k > 0$, $\dot{k} = 0$ if and only if $k = p$, so k is stationary on a ray with unitary slope. From (2.16), when $y > 0$, $\dot{p} = 0$ if and only if $p = \alpha_2 k_1/\alpha_1 k_2 = \alpha_2/\alpha_1 k$. Thus, p is stationary on a unique hyperbola in (k,p)-space. The unique intersection of the hyperbola with the ray is denoted by (k^*, p^*).

Notice that $\dot{k}$ and $\dot{p}$ are not in general uniquely determined by k and p alone. From (2.14)–(2.16), we see that to determine $\dot{k}$ and $\dot{p}$ given k and p we must also know either k_1 or k_2. Nonetheless, on the ray in Figure 2.1, $\dot{k} = 0$ for all values of k_1 and k_2 such that $k_2/k_1 \equiv k = p$. Similarly on the hyperbola, $\dot{p} = 0$, independent of the particular values of k_1 and k_2. To the northwest of the $\dot{k} = 0$ ray, k is rising, to the southeast k is falling. To the northeast of the $\dot{p} = 0$ hyperbola, p is rising, to the southwest p is falling. It should be noted that while knowledge of k and p is not sufficient to determine the speeds of motion, $\dot{k}$ and $\dot{p}$, Figure 2.1 will serve well enough for *qualitative* analysis. Substituting k in (2.14)–(2.16) and rearranging yields

$$(2.24) \qquad \dot{k} = \left(\frac{y}{k_1}\right)\left[\frac{(\alpha_1 + \alpha_2)k}{\sqrt{1 + p^2}}\left(\frac{p}{k} - 1\right)\right]$$

and

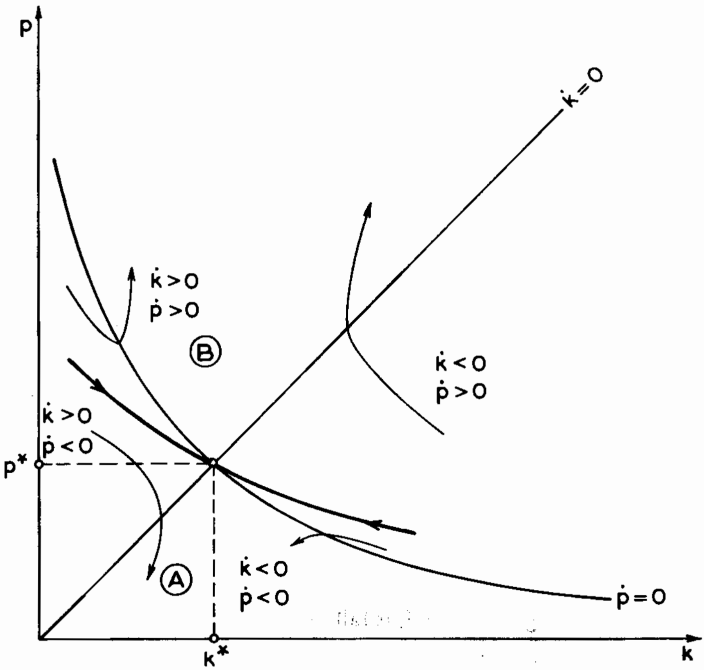

Figure 2.1.

(2.25) $$\dot{p} = \left(\frac{y}{k_1}\right)\left[\sqrt{1+p^2}\left(p\alpha_1 - \frac{\alpha_2}{k}\right)\right] .$$

In both (2.24) and (2.25) the terms in brackets are solely dependent upon the variables k and p. The term (y/k_1) depends upon k_1 and k_2 and cannot be written merely as a function of k. The right-hand size of (2.24) gives the horizontal speed of our model economy, while the right-hand side of (2.25) gives the vertical speed. Since in a "resolution-of-forces" rectangle the term (y/k_1) will "cancel", we know that from (2.24) and (2.25), k and p will uniquely determine the direction of development, although we must also know k_1 or k_2 to ascertain the speed of development. (Along any trajectory in the "phase plane" of Figure 2.1, $(dk/dp) = (\dot{k}/\dot{p}) = \psi(k,p)$.)

Some important propositions are immediate from Figure 2.1. Since for $k < k^*$, the separatrix (heavy curve) lies below the $\dot{p} = 0$ hyperbola, which in turn is asymptotic to the vertical axis, we can see that the separatrix covers the k half-line. That is, for each $k > 0$, there exists a unique $0 < p < \infty$, which will ensure that the economy tends to the rest point (k^*,p^*). For all other assignments of p, the system (2.14)–(2.16) diverges from the steady-state equilibrium.

It is essential to study further the development of the economy on trajectories not tending to the (k^*,p^*)-equilibrium. From Figure 2.1, we see that trajectories not tending to (k^*,p^*) ultimately enter either Region A, where both k and p are falling, or Region B where both k and p are rising. Consider, for example, a trajectory in Region A. From (2.16)

$$\dot{p} = \sqrt{1+p^2}\,(f_1/\alpha_1)(\alpha_1 p - \alpha_2/k)\,, \tag{2.26}$$

where f_1 is the marginal product of the first capital good. In Region A, $\alpha_1 p < \alpha_2/k$. Therefore, in this region,

$$\dot{p} < (pf_1 - f_2) < 0\,, \tag{2.27}$$

since $\sqrt{1+p^2} > 1$ for $p \neq 0$. Defining $\beta \equiv f_1/f_2$, (2.27) yields that

$$\dot{p} < (p\beta - 1)\,f_2 < 0\,. \tag{2.28}$$

In Region A, p is falling and since k is falling, $\beta = \alpha_1 k/\alpha_2$ is falling, $\beta = \alpha_1 k/\alpha_2$ is falling.

Next we show that, on a trajectory in Region A, the marginal product of the second capital good, f_2, is bounded from below. Since $f_2 = \alpha_2 y/k_2$, we can write

$$f_2 = \frac{\alpha_2}{k^{1-\alpha_2}\,k_1^{\;1-\alpha_1-\alpha_2}}\,. \tag{2.29}$$

Since in Region A, $\dot{k} < 0$, we need only demonstrate that there is an upper bound to k_1. From equation (2.14)

$$\dot{k}_1 = \frac{(\alpha_1+\alpha_2)\,k_1^{\;\alpha_1}\,k_2^{\;\alpha_2}}{\sqrt{1+p^2}} - \lambda k_1\,,$$

or

$$\dot{k}_1 = \frac{(\alpha_1+\alpha_2)\,k_1^{\;\alpha_1+\alpha_2}\,k^{\alpha_2}}{\sqrt{1+p^2}} - \lambda k_1\,,$$

which yields

(2.30) $$\dot{k}_1 < (\alpha_1 + \alpha_2)\, k_1^{\alpha_1 + \alpha_2}\, k^{\alpha_2} - \lambda k_1$$

for $p \neq 0$. In Region A, k is falling and since $\alpha_1 + \alpha_2 < 1$, the right-hand side of (2.30) becomes negative for k_1 sufficiently large. In Region A, k_1 is bounded from above, and thus f_2 is bounded from below.

Therefore, from (2.28) we know that in Region A, p falls to zero in finite time. From zero-profit condition (2.5), when $p_2 \leqslant 0$ we have that $p_1 = 1$ since $z_2 = 0$ and $z_1 = z$, i.e., investment is specialized to machinery of the first kind. If we assume *for the moment* that differential equation (2.11) also holds for non-positive prices*, then from (2.11)

$$\dot{p} = pf_1 - f_2 < -f_2$$

when $p \leqslant 0$. We have shown that in Region A marginal product f_2 is bounded from below, so that starting from Region A, $p \to -\infty$ and $k \to \infty$ as $t \to \infty$.

By symmetry, for trajectories in Region B, p becomes infinite in finite time. Again, assuming for the moment, that differential equation (2.11) holds everywhere, on trajectories starting in Region B, $p(t)$ has a pole at finite t; to the right of the pole, p is negative, and $p \to 0$ with $k \to \infty$ as $t \to \infty$.

Since all "errant" trajectories (i.e., trajectories not tending to balanced growth) enter either Region A or Region B, we know that on such "errant" trajectories in finite time $p = 0$ or $p = \infty$. The development of the price ratio, p, is summarized in Figure 2.2. For given initial relative capital intensity, k_0, there exists exactly one initial price ratio, p_0, such that the system (2.14)–(2.16) develops to (k^*, p^*). This unique price trajectory is shown by the heavy curve in Figure 2.2. For smaller initial price ratios (see, for example, the light curve in Figure 2.2), p becomes zero in finite time and tends to minus infinity as $t \to \infty$. For initial price ratios greater than the unique assignment tending to balanced growth (see, for example, the dashed curve in Figure 2.2), $p(t)$ rises to a pole at some finite date, then ultimately tends to zero as $t \to \infty$.

If, on the other hand, *capital goods are freely disposable,* the differential equation system (2.14)–(2.16) holds only for finite and positive p. Consider, for example, an "errant" trajectory in Region A. In finite time, $p = 0$. Therefore, by (2.13), in finite time $p_1 = 1$ and $p_2 = 0$. At this point asset-market-clearing equation (2.11) can be written as

* Essentially assuming non-disposability of capital goods.

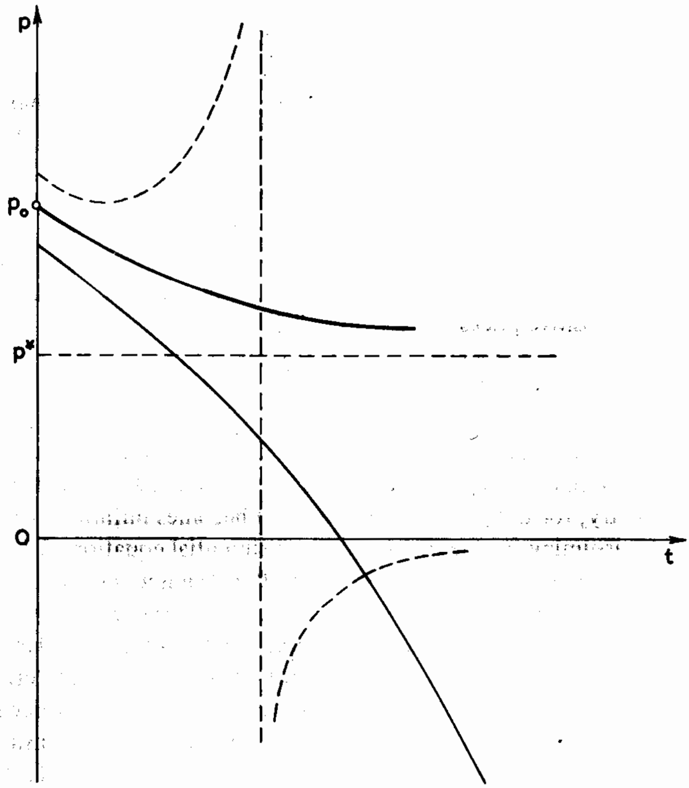

Figure 2.2.

$$(2.31) \qquad \dot{p}_1 + f_1 = \dot{p}_2/p_2 + f_2/p_2 \,.$$

Because of free disposability, $(\dot{p}_2/p_2) \geqslant 0$ when $p_2 = 0$. Since $f_2 > 0$, the right-hand side of (2.31) must be $+\infty$. But, unless p_1 is discontinuous and thus frustrates the most acute adaptive expectations (short-run perfect foresight), asset-market-clearing equation (2.31) will not hold. Then, the second capital good will have an infinite rate of return while the first capital good bears a finite rate of return. All asset-holders will desire to specialize to capital of the second kind; the capital goods market does not clear.

What forces are there in capitalism that prevent the economy from following "errant" trajectories? Walrasian (i.e., essentially atemporal) futures markets extending indefinitely into the future ensure that the economy will be stable – that is, ensure that it will develop to (k^*, p^*) independent of the initial capital intensity, k_0, since the initial competitive price ratio, p_0, will be a function of k_0, lying on the heavy curve in Figure 2.1. The Walrasian auctioneer rejects all price ratios but the unique ratio, $p_0(k_0)$, since all other price ratios imply that the spot capital goods market will not clear at some (finite) future date.

On the other hand, if there are no markets for selling and renting machinery, then the economy is stable since there are no capital gains. It is furthermore true (although I do not show it here) that if we assume that capitalists possess static expectations, i.e., $\dot{p}_1{}^e = 0 = \dot{p}_2{}^e$, then development always tends to the balanced-growth equilibrium $(k_1{}^*, k_2{}^*, p^*)$.

3. Heterogeneous Capital Accumulation when Momentary Equilibrium is not Unique†

In the previous section, we studied a model in which the production possibility frontier (PPF) in (c, z_1, z_2) space is sufficiently curved to ensure uniqueness of momentary equilibrium. Thus, our dynamical system could be written as a system of ordinary differential equations. If, however, the PPF is flat, momentary equilibrium is not unique. The analysis must then be couched in terms of a system of differential correspondences. In a particular model due to Joseph Stiglitz and me, such an analysis has been carried out. We use no new general theorems from the theory of differential correspondences, but observe that our system can be represented *almost everywhere* by a Lipschitzian system of differential equations. The analysis follows from "piecing" together the various Lipschitzian regions.

We use as before the Cobb-Douglas production function

$$y = k_1{}^{\alpha_1} k_2{}^{\alpha_2} = f(k_1, k_2), \tag{3.1}$$

but assume instead that along the PPF consumption, investment goods of the first kind, and investment goods of the second kind are perfect substitutes

$$y = c + z_1 + z_2, \tag{3.2}$$

where $c \geqslant 0$, $z_1 \geqslant 0$, and $z_2 \geqslant 0$. Again assume that all wages are consumed and all rentals are saved, so that in momentary equilibrium

† This section is based on my work with Stiglitz [12].

$$c = (1 - \alpha_1 - \alpha_2)y \,. \tag{3.3}$$

The production supply prices of the three goods are identical. Since consumption is always positive, its market price must be no less than the market price of the more expensive investment good. Since investment is always positive, the price of consumption cannot exceed the price of the more expensive good. That is, in momentary equilibrium

$$\max(p_1, p_2) = p_c \equiv 1 \,, \tag{3.4}$$

if consumption is the *numéraire*.

If both capital goods depreciate at the same proportionate rate, then our dynamical system can be written as

$$\begin{aligned} \dot{k}_1 &= z_1 - \lambda k_1 = \sigma(\alpha_1 + \alpha_2)y - \lambda k_1 \,, \\ \dot{k}_2 &= z_2 - \lambda k_2 = (1 - \sigma)(\alpha_1 + \alpha_2)y - \lambda k_2 \,, \end{aligned} \tag{3.5}$$

where σ is the upper-semicontinuous correspondence defined by

$$\sigma \begin{cases} = 1 & \text{if } p_2 < p_1 \\ \epsilon [0,1] & \text{if } p_2 = p_1 \\ = 0 & \text{if } p_2 > p_1 \,. \end{cases}$$

I now turn to the question of momentary equilibrium in the used-machinery market. If expectations about capital gains are ratified, then in equilibrium the rate-of-return on machinery types must be equalized, i.e.,

$$\frac{\dot{p}_1}{p_1} + \frac{f_1}{p_1} = \frac{\dot{p}_2}{p_2} + \frac{f_2}{p_2} \,. \tag{3.6}$$

Balanced growth. In the steady-state, $\dot{p}_1 = 0 = \dot{p}_2$; hence from (3.6), $f_1 = f_2$. Also, $\dot{k}_1 = 0 = \dot{k}_2$; hence from (3.5), $f_1 = f_2 = \lambda$ since $p_1 = p_2 = 1$. Since along $f_1 = f_2$,

$$\lim_{k_1 \to 0} f_1 = \infty \quad \text{and} \quad \lim_{k_1 \to \infty} f_1 = 0 \,,$$

the solution $(k_1{}^*, k_2{}^*)$ to the system $f_1 = \alpha_1 f/k_1 = f_2 = \alpha_2 f/k_2 = \lambda$ is uniquely determined.

Dynamic analysis. Define the three regimes by:
Regime I. $p_2 < p_1 = 1$, only capital good 1 is produced $(\sigma = 1)$.
Regime II. $1 = p_2 > p_1$, only capital good 2 is produced $(\sigma = 0)$.
Regime III. $p_1 = p_2 = 1$, σ is indeterminate.
The differential correspondences (3.5) and (3.6) reduce to (using 3.4):

Regime I	Regime II	Regime III
$\dot{k}_1 = \pi - \lambda k_1$	$\dot{k}_1 = -\lambda k_1$	$\dot{k}_1 = \sigma\pi - \lambda k_1$
$\dot{k}_2 = -\lambda k_2$	$\dot{k}_2 = \pi - \lambda k_2$	$\dot{k}_2 = (1-\sigma)\pi - \lambda k_2$
$\dot{p}_1 = 0$	$\dot{p}_1 = p_1 f_2 - f_1$	$\dot{p}_1 = \rho + \mu - f_1$
$\dot{p}_2 = p_2 f_1 - f_2$	$\dot{p}_2 = 0$	$\dot{p}_2 = \rho + \mu - f_2$

where profits per capita $\pi = (\alpha_1 + \alpha_2)y$.

Although this is a system of four differential correspondences in k_1, k_2, p_1, and p_2, our simple assumptions about production and demand for consumption allow us to make a complete dynamic analysis of the system in (k_1, k_2) phase space. In Figure 3.1 I have drawn the ray OA along which $k_1 = k_2\alpha_1/\alpha_2$ (i.e., along which $f_1 = f_2$). Above OA, $f_2 > f_1$; below OA, $f_1 > f_2$.

In Regime I, $\dot{k}_1 = 0$ along the curve denoted in Figure 3.1 by OB which is the locus of points such that

$$k_1 = \left[\frac{(\alpha_1 + \alpha_2)k_2\alpha_2}{\lambda}\right]^{\frac{1}{1-\alpha_1}} .$$

Differentiating yields

$$\left(\frac{dk_1}{dk_2}\right)_{\substack{\dot{k}_1 = 0 \\ \sigma = 1}} > 0 \quad \text{and} \quad \left(\frac{d^2k_1}{dk_2{}^2}\right)_{\substack{\dot{k}_1 = 0 \\ \sigma = 1}} < 0 .$$

Similarly for Regime II we can describe the locus of points such that $\dot{k}_2 = 0$, and this curve is denoted in Figure 3.1 by the curve OC.

Thus, Figure 3.1 is divided into six basic regions: A_1 which lies to the right

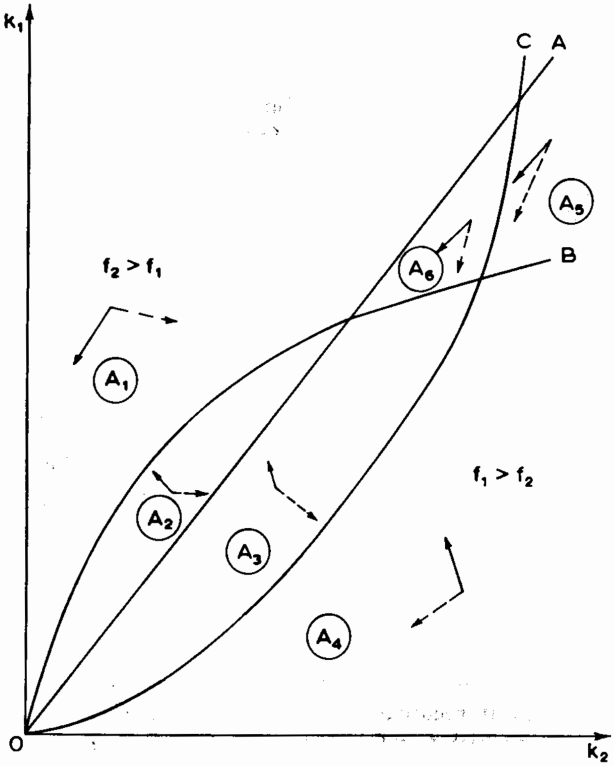

Figure 3.1.

of the k_1 (vertical axis and above OA, OB, and OC; A_2 which lies above OA and below OB; A_3 which lies above OC and below OA and OB; A_4 which lies above the k_2 (horizontal) axis and below OC and OB; A_5 which lies above OB and below OC; A_6 which lies below OA and above OB and OC. The solid arrows indicate the direction of development in the respective regions when $\sigma = 1$ (Regime I). The dashed arrows indicate the direction of development when $\sigma = 0$ (Regime II).

So far, we have ignored the behavior of prices. We recall that in Regime I, $\dot{p}_2/p_2 = f_1 - f_2/p_2$. If the economy is in Regime I and above OA (i.e., $f_2 > f_1$), we know that $f_1 < f_2/p_2$ since $p_1 > p_2$. Thus, in this case p_2 falls and as long as the economy is above OA it cannot switch to Regime II. It continues to specialize investment in the capital good with the lower marginal product – a clear instance of the Keynesian disparity between social and private returns due to capital gains. Similarly, if the economy is in Regime II and below OA, as long as it is below OA it cannot switch to Regime I.

We are now ready to put all this information to use for a full dynamic analysis. Consider, for example, an economy which begins initially in A_2. If the economy begins with $p_1 > p_2$ (i.e., in Regime I), it must remain in Regime I, so that the economy moves towards the curve OB, crosses it, and then moves towards the origin.

If initially $p_2 > p_1$ and initially the economy is in A_2, it moves towards OA, but since $\dot{p}_1/p_1 = f_2 - f_1/p_1$ and $f_1 < f_2$, if p_1 is sufficiently large ($> f_1/f_2$), p_1 is rising. It is possible then that, before the economy gets to OA, p_1 becomes equal to p_2 (= 1). But since $f_1 < f_2, p_2$ must begin to fall, and we switch to Regime I. From then on the story follows as before. Alternatively, the economy can cross the ray OA with p_2 greater than p_1. The story for the economy in Regime II in A_3 is analogous to that of the economy in Regime I in A_2. The economy moves to OC, crosses it and proceeds to the origin.

One important case remains: The economy begins in A_2 with $p_2 > p_1$, but $p_1 = p_2$ at exactly the moment that $f_1 = f_2$. The economy is then in Regime III in which momentary equilibrium is not unique. There is a unique allocation σ^* which will enable the economy to move along the ray OA to the steady-state solution; we require $\dot{k}_1/\dot{k}_2$ to equal the slope of OA

$$\frac{\alpha_1}{\alpha_2} = \frac{\sigma^*\pi - \lambda k_1}{(1-\sigma^*)\pi - \lambda k_2} ,$$

or

$$0 < \sigma^* = \frac{\alpha_1}{\alpha_1 + \alpha_2} < 1 .$$

If σ deviates from σ^* for more than an infinitesimal length of time, clearly f_1 will no longer equal f_2. If, for instance, $\sigma < \sigma^*$, k_2 becomes slightly greater than $\alpha_2 k_1/\alpha_1$, i.e., f_1 becomes greater than f_2. Our price differential equation, for Regime III, is $\dot{p}_2 - \dot{p}_1 = f_1 - f_2 > 0$. Hence p_1 decreases relative to p_2 (= 1), and the economy moves into Regime II. From then on, the story is familiar.

But in the model as presented thus far there is no mechanism with only short-run perfect foresight by which σ can be maintained at σ^*.

The dynamic behavior for the economy with initial endowments in other regions can be analyzed in a similar manner. For each initial assignment of the endowment vector (k_1, k_2), *there is one and only one assignment of initial prices* (p_1, p_2) *that allows the economy to proceed to long-run balanced growth.* We have shown that, if we assign $\sigma = \sigma^*$ in Regime III, *the unique bal-*

anced growth equilibrium is a saddlepoint in the (k_1, k_2, p_1, p_2) *phase-space.* So far, there is no mechanism endogenous to the model which ensures that initial prices will be chosen so as to allow for long-run balanced growth. Paths not tending toward balanced growth tend to the origin, and even along paths which allow for long-run balanced growth there is no mechanism to ensure that $\sigma = \sigma^*$ in Regime III.

Moreover, *along paths not tending to balanced growth, the price of one of the two capital goods goes to zero in finite time.* To see this, consider once again the economy above OA ($f_2 > f_1$) but in Regime I. Defining $\beta = f_1/f_2$ yields

$$\dot{p}_2 = (p_2\beta - 1)f_2 < 0 .$$

Observe that in this case β is declining through time. This is because β is a constant along every ray through the origin (where β decreases as the slope of the ray increases), and the path of development cuts every ray from the right. Also in this case, f_2 is increasing through time since

$$\frac{d \log f_2}{dt} = (1 - \alpha_1 - \alpha_2)\lambda + \frac{\alpha_1 \pi}{k_1} = 0 .$$

Therefore, p_2 is falling at a rate faster than a constant *absolute* rate, thus for all paths not leading to long-run balanced growth, the price of the capital good with the higher marginal product goes to zero in *finite* time.

As before, paths not tending to balanced growth cannot be competitive equilibrium paths at every instant.

Sensitivity to assumptions about PPF. The shaded triangle in Figure 3.2 represents the Shell-Stiglitz technology. The production frontier is given by $z_1 + z_2 = z$ and the associated price system obeys the law max $(p_1, p_2) = 1$. The shaded quarter circle in Figure 3.3 represents the production set employed in the analysis of Caton-Shell. The price system for the "quarter-circle" technology obeys the law $p_1{}^2 + p_2{}^2 = 1$.

We can also consider another familiar geometric form to the production set, the "square technology". The shaded area in Figure 3.4 represents the production set defined by $\max(z_1, z_2) \leqslant z$. For $0 < p < \infty$, production is nonspecialized with $z_1 = z = z_2$. The zero-profit condition, $p_1 z_1 + p_2 z_2 = z$, then yields that the associated price system must follow the law $p_1 + p_2 = 1$.

Even in the extreme case of the "square technology" the basic results carry over. Letting $p = p_2/p_1$, yields for this case,

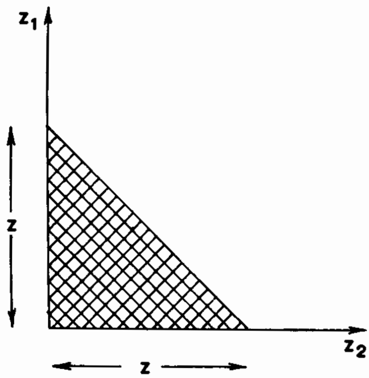

Figure 3.2: The "Triangular Technology" of Shell-Stiglitz.

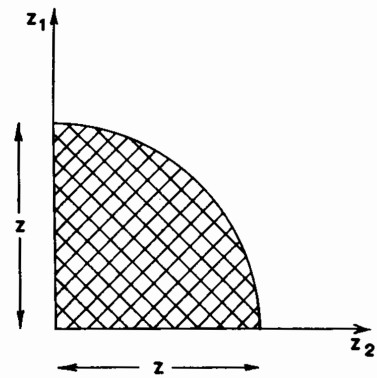

Fig. 3.3: The "Quarter-Circle" Technology of Caton-Shell.

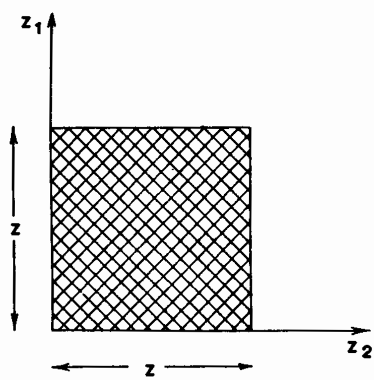

Figure 3.4: The "Square Technology".

$$p_1 = \frac{1}{1+p} \quad \text{and} \quad p_2 = \frac{p}{1+p} \ .$$

Therefore, market-clearing equation (2.11) can be written as

$$\dot{p} = (1+p)(pf_1 - f_2) \ .$$

Assume that the two capital-labor ratios are equal to their unique long-run equilibrium values, $k_1 = k_1{}^*$ and $k_2 = k_2{}^*$, but that $p < p^*$, i.e., $p < f_2/f_1$. In this case,

$$\dot{p} < pf_1 - f_2 < 0 \,,$$

where f_1 and f_2 remain at their equilibrium values. Therefore, p is falling at a rate faster than a constant absolute rate and must become zero in finite time. Thus, for three intrinsically different assumptions about technology, the two main qualitative properties of the competitive dynamical system are maintained.

4. The General Problem*

We have seen the persistent recurrence of some basic themes in several dynamical models of the capitalist economy. Analysis of a model with more general assumptions (especially about technology) is called for. As a start, one might study the model put forward by Hahn.** In Hahn's model, there is one consumption good, m capital goods, and $m + 1$ sectors:

$$\begin{aligned} c &= f^0(k_1^0, ..., k_m^0)\, l_0 \\ z_1 &= f^1(k_1^1, ..., k_m^1)\, l_1 \\ &\vdots \\ z_m &= f^m(k_1^m, ..., k_m^m)\, l_m \end{aligned}$$

when all variables are measured in intensive form. Full-employment requires

$$\sum_{j=0}^{m} l_j k_i^j = k_i \quad \text{for } i = 1, ..., m \text{ and } \sum_{0}^{m} l_j = 1 .$$

Capital accumulation is given by

$$\dot{k}_i = z_i - \lambda k_i \qquad i = 1, ..., m \,.$$

If $p_1, ..., p_m$ are the respective consumption prices of the m capital goods, then competition implies that

* The material in this section is related to two contributions of Frank Hahn [6,7].

** This model is general in that there are arbitrarily many capital goods and sectors, but does not allow for joint-production as in the quarter-circle technology.

$$\partial f^0/\partial k_j^0 = p_i(\partial f^i/\partial k_j^i) \quad \text{for} \quad i, j = 1, \ldots, m \ .$$

Rates of return are equalized when

$$\dot{p}_i/p_i + \partial f^i/\partial k_i^i = \dot{p}_j/p_j + \partial f^j/\partial k_j^j$$

for $i, j = 1, \ldots, m$. The consumption function (all wages consumed, all rentals saved) closes the model. If the production functions $f^0(\cdot), f^1(\cdot), \ldots, f^m(\cdot)$ satisfy Cobb-Douglas-like conditions, it is easy to show that there exists a unique balanced-growth equilibrium $(k_1^*, \ldots, k_m^*, p_1^*, \ldots, p_m^*)$. The full dynamic analysis is substantially more difficult.

As Hahn shows, unless restrictive assumptions are made about technology, momentary equilibrium is not necessarily unique. Let k be the m-vector of capital-labor ratios, p be the m-vector of capital goods prices in terms of consumption. In general, the dynamical system of the Hahn model can be written as

$$(\dot{k}, \dot{p}) \in \Omega\,(k, p) \, ,$$

where $\Omega(\cdot)$ is an upper-semicontinuous correspondence in $2m$ arguments. As such, the analysis cannot be cast in terms of the theory of differential equations, but we must instead turn to the general theory of dynamical systems. See Bhatia and Szegö [2] for a general view of this new mathematical field. See also the recent study of Cellina [5], which places special emphasis on the dynamical systems defined by upper-semicontinuous differential correspondences.

In order to avoid the problem of dynamical systems which cannot be written as single-valued differential equation systems, we can, like Kurz [9], begin with the production surface instead of with production functions. Even so, the full dynamic analysis seems to be hard to come by. Nonetheless, in this case, we are definite about the result for which we are searching.

I conjecture that, for the Hahn model with uniqueness of momentary equilibrium, the unique balanced-growth equilibrium (k^*, p^*) is a saddlepoint in the 2m-dimensional (k, p) space. *In particular, the manifold of trajectories tending to (k^*, p^*) is of m dimensions and "covers" the positive half k-hyperplane. I further conjecture that trajectories not tending to (k^*, p^*) will in finite time be revealed to be disequilibrium paths along which asset markets do not clear at every instant.*

5. Remarks on the Inverse-Optimum Problem

Recently Hahn [8] and Kurz [9] have investigated a problem which bears on the analysis of competitive models of heterogeneous capital accumulation. Assume that technology is of the sort described in the previous section, but that the central planner chooses a consumption program to maximize

$$\int_0^{\infty} U(c)e^{-\delta t}\,dt\,,$$

where $U(\cdot)$ is some concave utility function and $\delta \geqslant 0$ is the pure subjective rate of time discount. If momentary equilibrium is always unique (i.e., the production possibility frontier is never flat), then for a consumption program to be optimal there must exist m socially imputed prices $(p_1, ..., p_m) = p$ that satisfy Euler's system of $2m$ first-order differential equations $(\dot{k}, \dot{p}) = \phi\,(k, p)$.

The so-called inverse-optimum problem is: For the competitive system described in section 4, does there exist $U(\cdot)$ and δ such that the Euler differential equations from the above planning problem,

$$(\dot{k}, \dot{p}) = \phi(k, p)$$

"mimic" the differential equations from the short-run-perfect-foresight competitive enterprise economy? The attractiveness of this technique stems from the fact that there are added conditions "closing" the planning model. These transversality conditions,

$$\lim_{t\to\infty} p_i e^{-\delta t} = 0\,, \qquad \text{for} \quad i = 1, ..., m\,,$$

when combined with the inherited capital-labor ratios, give $2m$ boundary conditions to "close" the $2m$ first-order differential equation system. While these transversality conditions* do not apply when $\delta = 0$, similar boundary conditions can be found by comparing the optimal trajectory to any other feasible trajectory.**

* See Shell [11].

** The sufficiency proof for the one-sector model appears in Cass [3]. This proof is easily extended to higher dimensions if the utility function and the production functions are concave.

I would like to argue that the inverse-optimum argument may be somewhat misleading. Even if an Euler system can be found to "mimic" the competitive system, what one really wants to look for are the forces that "close" the model of the enterprise economy. We want to know how the capitalist system avoids "errant" trajectories. We already know, by definition, that an optimally controlled economy avoids inefficiency. [It should also be noted that the inverse-optimum problem is not an extension of the dynamic theory of revealed preference. The revealed-preference exercise would be to infer $U(\cdot)$ and δ by observing how optimal consumption programs vary with changes in technological possibilities.]

One important by-product of the inverse-optimum problem is an intensive study of the $2m$ Euler equations, $(\dot{k}, \dot{p}) = \phi(k, p)$. For example, Kurz shows that a stationary (k^*, p^*) to the Euler equations is either unstable or is a saddlepoint. If the production functions are strictly concave and if $U(\cdot)$ is strictly concave with $\lim_{c \to 0} U'(c) = \infty$, then not only is (k^*, p^*) unique, but also an optimal trajectory, if it exists, tends to (k^*, p^*) and is itself unique.*

Thus assuming the existence of an optimal program, we establish that the unique equilibrium (k^*, p^*) is a saddlepoint. That is, in (k, p) phase space, there is an m-dimensional manifold of trajectories tending to (k^*, p^*). If this manifold were of higher dimensionality then after assignment of the initial endowment vector, k_0, there would be extra degrees of freedom, refuting the fact that the optimal trajectory is unique. If the dimensionality of this manifold were less than m, then after assignment of k_0 the system would be overdetermined, denying the existence of an optimal trajectory. Similarly, existence of an optimal trajectory implies that this m-dimensional manifold covers the positive k-hyperplane.

6. Paper Assets: Government Debt and another Saving Hypothesis**

Again, we take consumption as the *numéraire* so that

$$\dot{W} = p\dot{K} + p_B \dot{B} + CG ,$$

where B (for bonds) is the nominal stock of government debt and p_B is the

* All these assertions are proved by a simple extension of the 2-dimensional proofs of Cass to $2m$ dimensions.

** The analysis of this section is from Shell-Sidrauski-Stiglitz [13].

consumption price of a bond. W is consumption value of wealth, K is capital stock and p is the price of capital, which in the one-sector model is equal to unity when production is not completely specialized. If CG denotes appreciation in the consumption value of assets, then

$$CG = \dot{p}k + \dot{p}_B B \, .$$

Assume that the nominal rate of interest on government bonds is zero. (Indeed, the reader may think of this non-interest-bearing debt as "money" in an economy in which there are no transactions nor liquidity preference demands for money.) Then, for the asset market to be in momentary equilibrium it is required that both assets yield the same rate of return, i.e.,

$$\frac{\dot{p}}{p} + \frac{\max(1,p)f'}{p} = \frac{\dot{p}_B}{p_B} \, .$$

f' is the marginal product of capital. The above equation states that both assets must have the same rental plus price appreciation per consumption unit. If both goods are produced, then the market price of capital must equal the market price of output which in turn must equal the market price of consumption, i.e., $p = 1$. Let $\theta = \dot{B}/B$ be the increase in the nominal supply of bonds and $b = p_B B/L$ be the consumption value of the *per capita* stock of bonds. Then if we assume that the consumption value of the community's savings (change in the value of wealth) is a constant fraction s of the consumption value of Individual Purchasing Power* (output plus the change in the value of wealth),

$$\dot{k}/k = sf(k)/k - (1-s)\,[\theta + (\dot{p}_B/p_B)]\, b/k - n \, .$$

Since we are only considering the case where both goods are produced $p \equiv 1$, we have that

$$f' = \dot{p}_B/p_B$$

and therefore

* Since government expenditures are equal to zero, the rate of increase in the outstanding stock of bonds is equal to the government budget deficit which in turn equals transfers minus taxes. Thus, perceived income or Individual Purchasing Power is $\hat{Y} = \dot{p}K + \dot{p}_B B + p_B \dot{B} + \max(1,p)Q$. If both goods are produced, then $p \equiv 1$ and $\dot{p} = 0$.

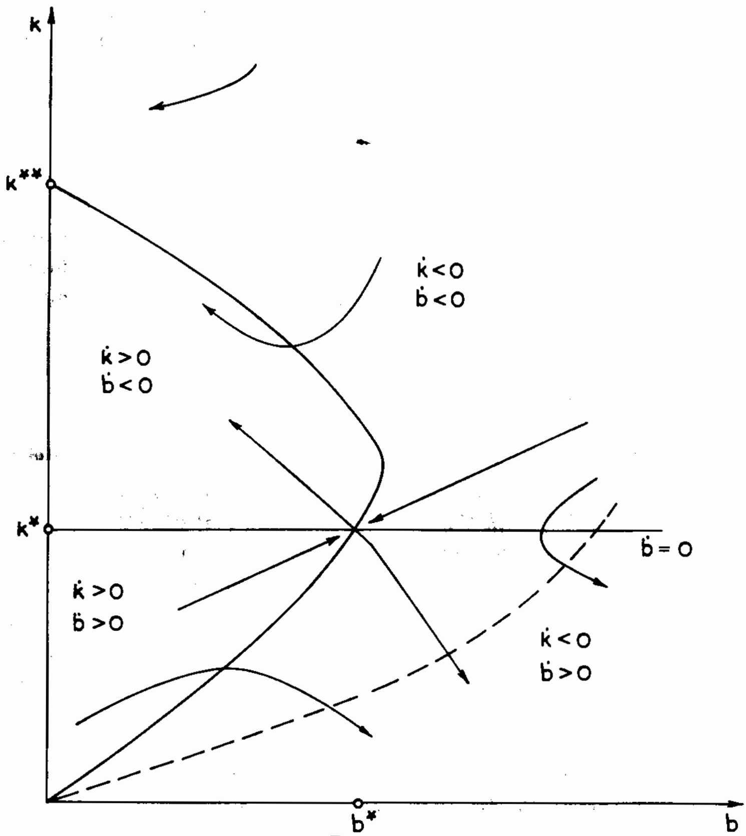

Figure 6.1.

$$\dot{k}/k = sf(k)/k - (1-s)\,[\theta + f'(k)](b/k) - n\ .$$

Logarithmic time differentiation yields

$$\dot{b} = [f'(k) + \theta - n]\,b\ ,$$

from the definition of b.

We analyze the dynamic behavior of the above system assuming that the government pursues a policy of a constant rate of expansion of the nominal supply of bonds, i.e., the case when θ is a given constant. $\dot{b} = 0$ if and only if

$$f'(k) = n - \theta\ .$$

If the production function is neoclassical and satisfies the Inada conditions, then for $n > \theta$, there exists a unique value k^* of the capital-labor ratio that yields a stationary for $b \neq 0$, i.e.,

$$f'(k^*) = n - \theta > 0 .$$

This is indicated in the phase diagram of Figure 6.1. For $k > k^*$, $\dot{b}$ is decreasing, for $k < k^*$, b is increasing. Setting the left-hand side of the $\dot{k}$ equation equal to zero and substituting k^* for k shows that the stationary solution (b^*, k^*) to the differential equations is unique and that b^* is given by

$$b^* = \frac{sf(k^*) - nk^*}{(1-s)n} .$$

If $b = 0$, $\dot{k} = 0$ if $sf(k) = nk$. Therefore, there exists a second nontrivial balanced growth equilibrium which is denoted in Figure 6.1 by the point $(0, k^{**})$, where $sf(k^{**}) = nk^{**}$. It should be noted that $b^* \gtreqless 0$ as $k^* \lesseqgtr k^{**}$.

In what follows, it is assumed that the production function $f(\cdot)$ and the parameters n and θ are such that b^* is positive. Setting $\dot{k} = 0$, implicit differentiation yields

$$\left(\frac{dk}{db}\right)_{\dot{k}=0} = \frac{(1-s)\,[\theta + f'(k)]}{sf'(k) - n - (1-s)bf''(k)} ,$$

which is unsigned, although under our assumptions the $\dot{k} = 0$ locus intersects the $\dot{b}$ = locus exactly once. From Figure 6.1, we see that to the right of the $\dot{k} = 0$ curve, k is decreasing; to the left of the $\dot{k} = 0$ curve, k is increasing. Thus, the equilibrium (b^*, k^*) is a saddlepoint. That is, given initial endowments $K(0), L(0), B(0)$ there exists only one initial assignment of the price of bonds $p_B(0)$ that will lead the economy to the non-trivial balanced growth state with bonds, (b^*, k^*). As in the models of Cagan, Sidrauski, Hahn and Shell and Stiglitz, there is nothing in the model so far presented to ensure that this unique initial price be "chosen" by the economy. The Solow zero bond equilibrium $(0, k^{**})$ is, on the other hand, locally stable.

Paths not converging to the (b^*, k^*) equilibrium either (a) converge to the Solow $(0, k^{**})$ balanced growth equilibrium or (b) in finite time have such large capital gains that real investment goes to zero. In Figure 6.1, we have indicated by a dashed curve the locus of points along which all of output is consumed while $p = 1$. To the right of the dashed curve p must be less than unity. Along the dashed curve

$$b = \frac{sf(k)}{(1-s)[\theta + f'(k)]}$$

and

$$\frac{db}{dk} = \frac{sf'(k)}{(1-s)[\theta + f'(k)]} - \frac{sf(k)f''(k)}{(1-s)[\theta + f'(k)]^2} > 0$$

for $0 < k < k^*$ or $\theta \gtreqless 0$.

Consider a trajectory crossing the dashed line from the left with $p = 1$. The asset market clearing equation is $\dot{p}_B/p_B = (f'/p) + (\dot{p}/p)$ and therefore the savings-investment equation yields

$$\dot{p} = \frac{p\,[(sf/(1-s)b) - \theta] - f'}{(pk/b) + 1} .$$

With $p < 1$, $\dot{k}/k = -n < 0$ and in finite time $k < k^*$ so that in finite time

$$(f'/p) + \theta - n > f' + \theta - n > 0 .$$

Since $(\dot{b}/b) = \theta - n + (f'/p) + (\dot{p}/p)$, $\frac{d(pk/b)}{dt} < 0$. So $p < 1$ falls faster than at a constant *absolute* rate, and thus the price of capital goes to zero in finite time.

But if capital is freely disposable when $p = 0$, $\dot{p} \gtreqless 0$. Since $f' > 0$, the rate of return on capital is then infinite. Remember that asset market clearance requires that

$$(\dot{p}/p) + (f'/p) = \dot{p}_B/p_B .$$

So with $p = 0$ and $p_B > 0$, p and p_B must be discontinuous and expectations about price changes must be frustrated.

The long-run equilibrium (b^*, k^*) is a saddlepoint. This is a property that the government-debt model shares with any competitive model with more than one asset and an asset market clearing equation consistent with the hypothesis that individuals instantaneously adjust their expectations about price changes. There is an important difference between the growth model with government debt and the heterogeneous capital goods models. In the heterogeneous capital goods models, on *all* paths not tending to balanced growth, expectations about price changes are frustrated in finite time. In the model with government debt, expectations are not frustrated on paths tending to the k-axis. This is because we assumed that the own rate of interest on bonds is zero. In the heterogeneous capital goods model given initial endowments, one and only one assignment of initial prices is consistent with non-disappointment of expectations; in the government debt model, many (but not all) initial price assignments are consistent with non-disappointment of expectations.

Even when there is no increase in the nominal supply of bonds, i.e., when $\theta = 0$, in the government-debt economy the inclusion of asset appreciation implies that the long-run equilibrium capital-labor ratio will be less than in the corresponding no-bond economy.

References

[1] A.B.Atkinson, The Time-Scale of Economic Models: How Long is the Long Run? Review of Economic Studies 36 (1969) 137–152.

[2] N.P.Bhatia and G.P.Szegö, Dynamical Systems: Stability and Applications, Lect. Notes Math. 35 (Berlin: Springer-Verlag, 1967).

[3] D.Cass, Optimum Growth in an Aggregative Model of Capital Accumulation, Review of Economic Studies 32 (1965) 233–240.

[4] C.Caton and K.Shell, An Exercise in the Theory of Heterogeneous Capital Accumulation, Discussion Paper No. 145, University of Pennsylvania, January 1970. Also to appear in Review of Economic Studies.

[5] A.Cellina, Multi-valued Differential Equations and Ordinary Differential Equations, Technical Note NV-574, The Institute of Fluid Dynamics and Applied Mathematics, University of Maryland, September 1968.

[6] F.H.Hahn, Equilibrium Dynamics with Heterogeneous Capital Goods, Quarterly Journal of Economics 80 (1966) 633–646.

[7] F.H.Hahn, On Some Equilibrium Paths, The Theory of Economic Growth, Edited by J.A.Mirrless and N.Stern (New York, MacMillan) (for I.E.A.), forthcoming.

[8] F.H.Hahn, On Warranted Growth Paths, Review of Economic Studies 35 (1968) 175–184.

[9] M.Kurz, The General Instability of a Class of Competitive Growth Processes, Review of Economic Studies 35 (1968) 155–174.

[10] P.A.Samuelson, Fundamental Singularity Theorem for Non-Joint Production, International Economic Review 7 (1966) 34–41.

[11] K.Shell, Applications of Pontryagin's Maximum Principle to Economics, Mathematical Systems Theory and Economics I, Lect. Notes Oper. Res. and Math. Econ. 11, edited by H.W.Kuhn and G.P.Szegö (Berlin, Springer-Verlag, 1967) pp. 241–292.

[12] K.Shell and J.E.Stiglitz, The Allocation of Investment in a Dynamic Economy, Quarterly Journal of Economics 81 (1967) 592–609.

[13] K.Shell, M.Sidrauski and J.E.Stiglitz, Capital Gains, Income and Saving, Review of Economic Studies 36 (1969) 15–26.