Economic complexity: chaos, sunspots, bubbles, and nonlinearity

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CHAPTER 1

Sunspot equilibrium in an overlapping-generations economy with an idealized contingent-commodities market

David Cass and Karl Shell

Abstract: We analyze a highly idealized infinite-horizon, overlapping-generations economy in which trading in a full spectrum of contingent commodities takes place before the beginning of economic time. We postulate that participation in the market is unrestricted. Hence, individuals are able to insure against all economic risks, even those that are resolved before their birthdates. We construct an example that possesses a sunspot, nonmonetary equilibrium. It Pareto dominates the nonsunspot, nonmonetary equilibrium, but it is Pareto dominated by a nonsunspot, monetary equilibrium.

In earlier work, we have established that restricted market participation is a source of sunspot equilibria. Because market restrictions are assumed away in the present chapter, our example establishes that the "double infinity" of consumers and dated commodities is another, logically separate source of sunspot equilibria.

Our example suggests that the social contrivance of sunspots can be an imperfect substitute for the social contrivance of money in attenuating the effects of oversaving. Our sunspot equilibrium also provides an example of a weakly Pareto-optimal allocation that can be seen to be not fully Pareto optimal from observing the state-contingent allocation for only a finite number of periods.

1 Introduction and summary

a *Economic complexity*

The subject of this volume is economic complexity, which can be taken to be the analysis of *complicated solutions to relatively simple economic*

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models. Consider, for example, the chapters devoted to economic dynamics. A focus of these contributions is on the nonstationary paths that arise in economies with stationary environments. We know that business cycles (periodic solutions) and even "chaotic" fluctuations (aperiodic solutions) can occur even though all the economic fundamentals are nonfluctuating (stationary). In some of the other chapters, including ours, the focus is on stochastic outcomes in nonstochastic economic environments. We know that economic outcomes can be random even though all the economic fundamentals (such as preferences, endowments, and technology) are nonrandom. In particular, "sunspots" can affect the allocation of resources even though the economic fundamentals are unaffected by sunspots.

b *Stochastic outcomes in nonstochastic economies*

The current chapter is on *sunspot equilibrium;* we analyze the stochastic outcomes to a rational-expectations model of a competitive economy in which the fundamentals are nonstochastic. The fact that economies generate uncertainty – as well as transmitting uncertainty arising from outside the economy – should come as no surprise. An economy is a social system in which the individual participants cannot be certain of the behavior of the others. In seeking to optimize his own actions, an economic actor must attempt to predict the moves of all other economic actors. Because he is uncertain about the moves of others, he is also uncertain about economic outcomes (such as prices), even if he is completely certain about the economic fundamentals. Nor should the existence of sunspot equilibria come as a surprise to the economist conversant with game theory. Pure-strategy equilibrium is not the only type of solution to nonstochastic games. Mixed-strategy equilibrium and, more generally, correlated equilibrium are staples of noncooperative game theory. There is no obvious reason that the stochastic equilibria that arise in game-theoretic models of imperfect competition must vanish in the limit as the economy becomes competitive.

Thinking of an economy as a social system composed of strategic decision makers should lead one to accept as natural the idea that the market economy itself creates uncertainty. The reader of business and financial news is well aware of the important role of market uncertainty. Businessmen face uncertainties that are not based on their uncertainty about

Footnote *(cont.)*
their comments. We take this opportunity to record our debt to Paul Samuelson, whose overlapping-generations model serves as a foundation for intelligent macroeconomics and provides two sources of sunspot equilibria.

economic fundamentals; they are, for example, uncertain about the overall level of consumer confidence, the confidence and financial plans of their customers, the production and marketing plans of their rivals, and the tax and regulatory plans of government agencies. For these reasons, we should expect sunspot equilibria to be the rule rather than the exception.

c *The overlapping-generations model and sunspot equilibrium*

The histories of the OG model and sunspot equilibrium are intertwined.[1] It is our ambition to persuade the reader that, although sunspot equilibrium is an essential aspect in the analysis of overlapping-generations economies, sunspot phenomena are far more general. However, we first return to the original source of sunspots: the overlapping-generations model.

The overlapping-generations model began with Samuelson (1958)[2] who provided an example of perfect-foresight nonmonetary, competitive equilibrium that is not Pareto optimal. For his example, there is also a very simple monetary competitive equilibrium that is Pareto optimal.[3] His example also establishes that the public debt need not be retired in perfect-foresight economies.[4]

What is the source of possible inoptimality in Samuelson's model? The overlapping-generations model differs from the Arrow–Debreu model in two ways:

1. The time horizon is unbounded and the generations overlap. Hence, in the discrete model, there is both a countable infinity of dated commodities and a countable infinity of traders.
2. Trades are restricted by the natural vitality and mortality of the economic actors.

Property 1, the so-called double-infinity property (including the stipulation of the overlapping structure of the generations), is *the* source of the possible failure of Pareto optimality of perfect-foresight (i.e.,

[1] A major reason for this is, of course, that the earliest research on sunspot equilibrium was based on either the overlapping-generations model or closely related models. Perhaps another reason has to do with proximity. At the time of our discovery of sunspots, CARESS was a leading center of research on the overlapping-generations model. Even at the time we discovered sunspots, both of us were listing as our prime research interest "overlapping-generations models."

[2] For a detailed and very enthusiastic account of the many virtues of the OG model for macro and monetary analysis, see our propaganda piece, Cass and Shell (1980).

[3] For an up-to-date welfare analysis of the OG model, see Balasko and Shell (1980, sect. 2, 4, and 5).

[4] For an analysis of fiscal policy, equilibrium, and welfare in the perfect-foresight OG model, see Balasko and Shell (1981). For an analysis of debt retirement and equilibrium in the OG model, see Balasko and Shell (1986).

nonsunspot), competitive equilibrium. See Shell (1971) and Cass (1985). Property 2, the restricted-participation property, has been incorrectly cited in the literature as causing the failure of optimality in perfect-foresight, overlapping-generations economies. Shell (1971) establishes that this property is not binding for perfect-foresight (i.e., nonsunspot) equilibria: The set of perfect-foresight equilibria derived on the assumption that traders are constrained by property 2 is identical to the set of perfect-foresight equilibria derived on the assumption that market participation is unrestricted.

Although property 2 plays no role in the analysis of nonsunspot equilibrium, it is important in the analysis of sunspot equilibrium. In Cass and Shell (1983), we showed how such restrictions on market participation are a source of sunspot equilibrium. Here we show how the double infinity (property 1) of the OG model is yet another source of sunspot equilibrium.

The first work on sunspot equilibrium was reported in Shell (1977), which is based on our joint research efforts on the infinite-horizon, overlapping-generations economy. In this unpublished paper, it was assumed that there is no intrinsic uncertainty. The only randomness is in the level of sunspot activity, which has no effect on the economic fundamentals. We showed that there is an equilibrium in which rational individuals believe that the general price level is affected by the level of sunspot activity and that these beliefs are self-justifying. It was assumed that the economy is "shocked" by sunspots in each period. In this first example of sunspot equilibrium, economic fluctuations are generated within the private sector, and the stabilizing (contrasunspot) fiscal policy must be perpetually active. This is contrary to the notion that erratic behavior in rational-expectations economies is solely the fault of the erratic behavior of the government.

In a very nice paper, Azariadis (1981) translated our pure-exchange overlapping-generations model into the simple production-consumption model popular with macroeconomists. Azariadis and Guesnerie (1982, 1986) and Spear (1984) provided sufficient conditions for the existence of *stationary* sunspot equilibria. Peck (1988) showed how nonstationary sunspot equilibria can be constructed for a very wide class of one-good OG models. His method of construction would seem to extend beyond the one-good case. His result suggests that sunspot equilibria are not flukes in the overlapping-generations model.

d *The sources of sunspot equilibria*

We previously argued (Subsection 1b) that since an economy is a social system in which individuals must attempt to predict the moves of others, it should not be surprising that the economy generates uncertainty. Because extrinsic uncertainty is so likely to affect economic outcomes, it is natural to ask: Which economies, if any, are immunized against the effects of extrinsic uncertainty?

We answer this question in Cass and Shell (1983, sect. VII, pp. 215–18). In an economy with convex preferences and convex technologies, a sunspot-equilibrium allocation is never Pareto optimal. Hence, in a convex environment, if the preconditions of Pareto optimality are met, we know that extrinsic uncertainty cannot matter.

Such is the hold on economists of the basic Arrow–Debreu model that we are frequently asked for "the cause" of or "the reason" for sunspot equilibria. (These questions have the same roots as the idea that failures in Pareto optimality and other nice properties of Arrow–Debreu equilibrium must necessarily be identified with some particular "market failures.") In attempting to respond to these questions, we have pursued the following program: Relax the usual preconditions for Pareto optimality of equilibrium allocations, one at a time, and provide, for each weakening of the assumptions, an example of sunspot equilibrium.[5] In what follows, we list some of these sources of sunspot equilibrium.

(i) *Restrictions on market participation*

In Cass and Shell (1983), we relaxed the perfect market assumption in a way that is motivated by intertemporal economic theory. All markets are assumed to be open and competitive, but participation in these markets is restricted to economic actors who are alive when the market convenes. In particular, the members of the "young" generation cannot participate in a market for securities based on a state of nature that is revealed before their births. It was shown that sunspot equilibria can be generated in an economy that is Arrow–Debreu except for restrictions on market participation. If there is no intrinsic uncertainty, then the Walrasian equilibria are unaffected by the restrictions on market participation. They are thought of as the nonsunspot equilibria. We showed that there can also be new equilibria, the sunspot equilibria, in which the allocation of resources is affected by extrinsic uncertainty. Balasko, Cass, and Shell (1988) provided some comparative statics for the Cass and Shell (1983) economy. In the effect on the existence of sunspot equilibrium, varying the proportion

[5] Our joint program is referred to as the Philadelphia Pholk Theorem by Shell (1987) in his *Palgrave* entry on sunspots. The PPT is the assertion: "In each 'class' of models in which Pareto-optimal allocations are not guaranteed, one can find an example of sunspot equilibrium." The "proof" is based on several examples constructed by us and our co-authors. Of course, time does not stand still. The pholk theorem is now also identified with Ithaca, Geneva, Evanston, New Brunswick, Paris, New York, Pittsburgh, ...

of market-restricted consumers is similar to varying the endowments of the economy.

Natural restrictions on market participation obviously arise in the overlapping-generations model. Hence, in that model, restricted participation is *a* source of sunspot equilibria. There is one other source of sunspot equilibria in OG models - the double infinity of consumers and dated commodities (along with the structure of generational overlap), which is analyzed in Sections 2 and 3.

(ii) *Incomplete markets*

Cass (1989) showed that, in an example with securities markets that are not complete, there is a continuum of sunspot competitive equilibria if the endowments are not Pareto optimal (see also Siconolfi, 1987). Cass's securities are purely financial, but subsequent work has indicated that missing markets in general, whether for financial or real commodities, can cause sunspot equilibria to exist.

One is free to interpret an economy with externalities as nothing more than an economy with incomplete markets. Hence, by showing that incomplete markets can be a source of sunspot equilibria, one also establishes that externalities can be sources of sunspot equilibria. Spear (1988) worked out an interesting dynamic model with an externality in the private investment process. He showed that sunspots can affect the equilibrium outcome, and he generated equilibria that seem to be in agreement with some features of actual economic time series.

(iii) *Imperfect competition*

Peck and Shell (1985) analyzed the role for extrinsic uncertainty in a general-equilibrium model of imperfect competition. The model is like the Arrow-Debreu model except that the price-taking assumption is replaced by the assumption of strategic behavior. For this model, there is a sunspot Nash equilibrium if and only if endowments are not Pareto optimal. Allowing for asymmetric information even further broadens the class of sunspot Nash equilibria. The work of Peck and Shell provided the first formal link between the relatively well-established tradition of extrinsic uncertainty in game theory and the relatively recent tradition of extrinsic uncertainty in formal economic equilibrium models. They showed that, in the market-game environment, every correlated equilibrium is a sunspot equilibrium but that some sunspot equilibria are not correlated equilibria.

(iv) *Nonconvexities*

Guesnerie and Laffont (1987) provided examples of sunspot equilibria in economies with nonconvex preferences. In contrast, Cass and Polemarchakis (1988) proved that sunspots cannot matter in competitive equilibrium if preferences are strictly convex, even if there are nonconvexities in production.

e Sunspot equilibria in the idealized overlapping-generations economy: overview and summary of Sections 2 and 3

We first found sunspot equilibria during the course of our joint research on Samuelson's model of overlapping generations. Two features distinguish the overlapping-generations model from the standard Walrasian model:

1. Participation in markets in the overlapping-generations economy is naturally restricted by the lifetimes of the individual traders.[6] In the standard Walrasian model, market participation is unrestricted.
2. In the overlapping-generations economy, there is the double infinity of traders and dated commodities[7] along with the overlapping structure of generations.[8] The standard Walrasian model is based on a finite number of commodities and a finite number of consumers.

We know that the first feature, restricted market participation, is a source of sunspot equilibrium.[9] To isolate the sources of sunspot equilibria, we construct in Section 3 a highly idealized overlapping-generations economy in which the first feature, restricted market participation, is assumed away. We postulate in the spirit of Shell (1971) that the market for contingent commodities meets before the beginning of economic time and that each consumer can purchase a full spectrum of contingent claims, even those based on states of nature that were revealed before the consumer's birthdate.

A special case of the model developed in Section 2 is further specified in Section 3. For this example, we find three rational-expectations competitive equilibria:

1. a nonsunspot, nonmonetary competitive equilibrium (which is autarky);
2. a nonsunspot, monetary competitive-equilibrium; and
3. a sunspot, nonmonetary competitive equilibrium.

In our example, the nonsunspot, monetary competitive-equilibrium allocation is Pareto optimal. It also Pareto dominates the sunspot, nonmonetary competitive-equilibrium allocation, which in turn Pareto dominates

[6] See Cass and Shell (1983) and Subsection 1d(i).
[7] See Shell (1971). [8] See Cass (1985). [9] See Cass and Shell (1983).

the nonsunspot, nonmonetary competitive-equilibrium allocation. The two nonmonetary equilibria are not Pareto optimal, but they are weakly Pareto optimal. Even though the sunspot equilibrium allocation is weakly Pareto optimal, the planner could not convert it to full Pareto optimality without intervening in the first period.

We conclude that *a* source of sunspot equilibria in the overlapping-generations economy is the second distinguishing feature of the model, the double infinity property.

Our example also suggests that sunspots might conceivably be useful as a coordinating device to attenuate a tendency toward oversaving. Lump-sum transfers of outside money to the private sector attenuate any tendency to oversaving and do so costlessly because these transfers do not distort economic decisions. Wagers on sunspots might, under some circumstances, serve to attenuate oversaving, but they are always costly in strictly convex economies because they introduce individual risks into situations in which there is no aggregate risk.

In Samuelsonian prose: The "social contrivance of sunspots" may serve as an imperfect substitute for the "social contrivance of money."

2 An idealized overlapping-generations model with sunspots

In each generation there is only one consumer, labeled by his birthdate $t=0,1,\ldots$ and referred to as Mr. t. Mr. 0 is active only during period 1, while Mr. $t>0$ is alive and active during both periods t and $t+1$. In each period $t=1,2,\ldots$ there is only one standard commodity. The economy's level of sunspot activity s is a random variable, realized at the beginning of period 1 and taking on the values α or β. All consumers share the same probability beliefs about the occurrence of sunspots, and these common probabilities $\pi(\alpha)$ and $\pi(\beta)$ satisfy $0<\pi(\alpha)<1$ and $\pi(\beta)=1-\pi(\alpha)$.

Let $x_t^\tau(s)\in\mathbb{R}_+$ denote consumption by Mr. t in period τ and state s, and

$$x_0=[x_0^1(\alpha),x_0^1(\beta)] \quad\text{and}\quad x_t=[x_t^t(\alpha),x_t^{t+1}(\alpha),x_t^t(\beta),x_t^{t+1}(\beta)],$$
$$\text{for } t=1,2,\ldots.$$

Assume that Mr. 0 has a utility function $v_0:\mathbb{R}_+^2\to\mathbb{R}$ with the von Neumann–Morgenstern representation

$$\begin{aligned} v_0(x_0)&=v_0[x_0^1(\alpha),x_0^1(\beta)]\\ &=\pi(\alpha)\phi[x_0^1(\alpha)]+\pi(\beta)\phi[x_0^1(\beta)], \end{aligned}$$

where $\phi:\mathbb{R}_+\to\mathbb{R}$ is smooth, strictly increasing, and strictly concave. Similarly, assume that Mr. $t>0$ also has a von Neumann–Morgenstern utility function $v_t:\mathbb{R}_+^4\to\mathbb{R}$, but with the special form

$$v_t(x_t) = v_t[x_t^t(\alpha), x_t^{t+1}(\alpha), x_t^t(\beta), x_t^{t+1}(\beta)]$$
$$= \pi(\alpha)\psi[ax_t^t(\alpha) + bx_t^{t+1}(\alpha)] + \pi(\beta)\psi[ax_t^t(\beta) + bx_t^{t+1}(\beta)],$$

where $0 < a < b$ and $\psi: \mathbb{R}_+ \to \mathbb{R}$ is smooth, strictly increasing, and strictly concave. So, in particular, Mr. t's certainty utility function ψ exhibits a constant marginal rate of substitution that is less than one,

$$\frac{\partial\psi(ax^1 + bx^2)/\partial x^1}{\partial\psi(ax^1 + bx^2)/\partial x^2} = \frac{a}{b} < 1.$$

Let $\omega_t^\tau(s) \in \mathbb{R}_+$ denote the endowment of Mr. t in period τ and state s, and

$$\omega_0 = [\omega_0^1(\alpha), \omega_0^1(\beta)] \quad \text{and} \quad \omega_t = [\omega_t^t(\alpha), \omega_t^{t+1}(\alpha), \omega_t^t(\beta), \omega_t^{t+1}(\beta)]$$
$$\text{for } t = 1, 2, \ldots.$$

Specify that sunspots do not affect endowments and assume that endowments are stationary so that, independent of state and birthdate, Mr. t's endowment is, say, $\omega^1 > 0$ when young and $\omega^2 > 0$ when old. Then we have

$$\omega_0 = (\omega^2, \omega^2) \quad \text{and} \quad \omega_t = (\omega^1, \omega^1, \omega^2, \omega^2) \quad \text{for } t = 1, 2, \ldots.$$

Assume for the moment that there is no outside money in the economy, but that there is a complete set of Arrow securities markets or, what amounts to the same thing, a complete set of contingent-claims markets, and that all consumers can participate on all these markets. Let $p^t(s)$ denote the price (before sunspot activity is observed) of a claim for one unit of commodity in period t if state s is realized. Given these contingent-claims prices and the fundamentals (i.e., preferences and endowments) described earlier, we can proceed to characterize consumer behavior.

Mr. 0's optimal consumption bundle x_0 solves the problem:

$$\begin{aligned}
&\text{maximize} && \pi(\alpha)\phi[x_0^1(\alpha)] + \pi(\beta)\phi[x_0^1(\beta)] \\
&\text{subject to} && p^1(\alpha)x_0^1(\alpha) + p^1(\beta)x_0^1(\beta) \le p^1(\alpha)\omega^2 + p^1(\beta)\omega^2 \qquad (1) \\
&\text{and} && x_0 \ge 0,
\end{aligned}$$

whereas Mr. t's optimal consumption bundle x_t solves the problem:

$$\begin{aligned}
&\text{maximize} && \pi(\alpha)\psi[ax_t^t(\alpha) + bx_t^{t+1}(\alpha)] + \pi(\beta)\psi[ax_t^t(\beta) + bx_t^{t+1}(\beta)] \\
&\text{subject to} && p^t(\alpha)x_t^t(\alpha) + p^{t+1}(\alpha)x_t^{t+1}(\alpha) + p^t(\beta)x_t^t(\beta) + p^{t+1}(\beta)x_t^{t+1}(\beta) \\
& && \le p^t(\alpha)\omega^1 + p^{t+1}(\alpha)\omega^2 + p^t(\beta)\omega^1 + p^{t+1}(\beta)\omega^2 \qquad (2) \\
&\text{and} && x_t \ge 0.
\end{aligned}$$

Interior solutions to these problems are characterized by the Lagrangean conditions, which, after some simplification, can be written as

$$p^1(\alpha)[x_0^1(\alpha)-\omega^2]+p^1(\beta)[x_0^1(\beta)-\omega^2]=0 \tag{3}$$

and

$$\frac{\phi'[x_0^1(\alpha)]}{\phi'[x_0^1(\beta)]}=\frac{p^1(\alpha)/\pi(\alpha)}{p^1(\beta)/\pi(\beta)} \tag{4}$$

for equation (1), and

$$p^t(\alpha)[x_t^t(\alpha)-\omega^1]+p^{t+1}(\alpha)[x_t^{t+1}(\alpha)-\omega^2]$$
$$+p^t(\beta)[x_t^t(\beta)-\omega^1]+p^{t+1}(\beta)[x_t^{t+1}(\beta)-\omega^2]=0, \tag{5}$$

$$\frac{p^t(\alpha)}{p^{t+1}(\alpha)}=\frac{p^t(\beta)}{p^{t+1}(\beta)}=\frac{a}{b}, \tag{6}$$

and

$$\frac{\psi'[ax_t^t(\alpha)+bx_t^{t+1}(\alpha)]}{\psi'[ax_t^t(\beta)+bx_t^{t+1}(\beta)]}=\frac{p^t(\alpha)/\pi(\alpha)}{p^t(\beta)/\pi(\beta)} \tag{7}$$

for equation (2).

Definition. *The contingent-claims price sequence*

$$p^1(\alpha),p^1(\beta),p^2(\alpha),p^2(\beta),\dots,p^t(\alpha),p^t(\beta),\dots \tag{8}$$

is said to be a nonmonetary competitive equilibrium *if we have*

$$x_{t-1}^t(s)-\omega^2=-[x_t^t(s)-\omega^1] \quad \text{for } t=1,2,\dots \tag{9}$$

where x_0 is the optimal solution to (1), *and x_t is the optimal solution to* (2) *for $t=1,2,\dots$. The sequence of these optimal consumption bundles*

$$x_0,x_1,\dots,x_t,\dots$$

is the associated nonmonetary competitive equilibrium allocation.

Equations (9) are the materials balance conditions; and when consumers optimize, they are the (contingent-claims) market-clearing conditions.
Next we provide the corresponding nonsunspot and sunspot concepts.

Definition. *A nonmonetary competitive equilibrium is said to be nonsunspot if, in addition, the associated allocation sequence satisfies*

$$x_0^1(\alpha)=x_0^1(\beta) \quad \text{and} \quad [x_t^t(\alpha),x_t^{t+1}(\alpha)]=[x_t^t(\beta),x_t^{t+1}(\beta)]$$
$$\text{for } t=1,2,\dots. \tag{10}$$

Otherwise, it is said to be a sunspot, nonmonetary competitive equilibrium.

We shall also need the concept of a monetary competitive equilibrium. Suppose that the economy is as described previously with the sole exception that the government transfers to Mr. 0, in period 1, one unit of paper money independent of the state of nature. Let $p^m(s)$ be the money price in state s. Then Mr. 0's problem becomes finding the optimal consumption bundle x_0 that solves

$$\begin{aligned} &\text{maximize} && \pi(\alpha)\phi[x_0^1(\alpha)] + \pi(\beta)\phi[x_0^1(\beta)] \\ &\text{subject to} && p^1(\alpha)x_0^1(\alpha) + p^1(\beta)x_0^1(\beta) \\ & && \quad \le p^1(\alpha)\omega^2 + p^1(\beta)\omega^2 + p^m(\alpha) + p^m(\beta) \\ &\text{and} && x_0 \ge 0. \end{aligned} \tag{1^m}$$

Interior solutions to (1^m) are characterized by the simplified Lagrangean conditions

$$p^1(\alpha)[x_0^1(\alpha) - \omega^2] + p^1(\beta)[x_0^1(\beta) - \omega^2] - [p^m(\alpha) + p^m(\beta)] = 0 \tag{3^m}$$

and (4).

Definition. *A* monetary competitive equilibrium *is a money price vector* $[p^m(\alpha), p^m(\beta)]$ *and a contingent-claims price sequence* (8) *with the property that both the contingent-commodities market-clearing equation* (9) *and the money-market-clearing conditions* $-p^1(s)[x_1^1(s) - \omega_1^1(s)] = p^m(s)$, *for* $s = \alpha, \beta$, *hold when* x_0 *is the optimal solution to* (1^m), *and* x_t *is the optimal solution to* (2) *for* $t = 1, 2, \ldots$. *If, in addition, property* (10) *is satisfied, then we have a* nonsunspot, monetary competitive equilibrium.

In the following section we present three (stationary) competitive equilibria for this example. The nonsunspot nonmonetary and the nonsunspot monetary equilibria are simple and straightforward to display. The sunspot nonmonetary equilibrium, however, requires further specification of the parameters in the example.

3 Three competitive equilibria and their properties

a *A nonsunspot nonmonetary competitive equilibrium*

Using the simplified Lagrangean conditions (3), (4), (5), and (7), it is easily verified that the autarkic (and thus stationary) allocation described b

$$x_0^1(s) = \omega^2 \quad \text{for } s = \alpha, \beta$$

and

$$[x_t^t(s), x_t^{t+1}(s)] = (\omega^1, \omega^2) \quad \text{for } t = 1, 2, \ldots \text{ and } s = \alpha, \beta$$

is a nonsunspot, nonmonetary competitive equilibrium allocation when supporting prices are defined (up to some normalization) by

$$\frac{p^1(\alpha)}{\pi(\alpha)} = \frac{p^1(\beta)}{\pi(\beta)} > 0$$

and (12)

$$p^{t+1}(s) = \left(\frac{b}{a}\right)p^t(s) \quad \text{for } t = 1, 2, \dots \text{ and } s = \alpha, \beta.$$

Because, by hypothesis, we have $b/a > 1$, the second condition in (12) implies that "present-value" contingent-claims prices explode, that is,

$$\lim_{t\to\infty} \frac{p^t(s)}{p^1(s)} = \infty \quad \text{for } s = \alpha, \beta. \tag{13}$$

Now, because it is a competitive equilibrium allocation, we know that the sequence defined by (11) is weakly (or, equivalently, short-run) Pareto optimal. Property (13), however, suggests that this autarkic allocation is not (fully) Pareto optimal. This conjecture is in fact established by the next equilibrium we display.

b *A nonsunspot monetary competitive equilibrium*

Consider the stationary allocation described by

$$x_0^1(s) = \omega^1 + \omega^2 \quad \text{for } s = \alpha, \beta$$

and (14)

$$[x_t^t(s), x_t^{t+1}(s)] = (0, \omega^1 + \omega^2) \quad \text{for } t = 1, 2, \dots \text{ and } s = \alpha, \beta.$$

Because, again by hypothesis, we have

$$\left.\frac{\partial\psi(ax^1 + bx^2)/\partial x^1}{\partial\psi(ax^1 + bx^2)/\partial x^2}\right|_{(x^1, x^2) = (0, \omega^1 + \omega^2)} = \frac{a}{b} < 1,$$

it is easily verified [e.g., by attempting to improve on Mr. 0's welfare by perturbing the sequence (14)] that this particular stationary allocation is Pareto optimal. Moreover, because of the strict monotonicity of ϕ and ψ, we have

$$\phi(\omega^1 + \omega^2) > \phi(\omega^2) \quad \text{and} \quad \psi[b(\omega^1 + \omega^2)] > \psi(a\omega^1 + b\omega^2),$$

so that the allocation (14) clearly Pareto dominates the autarkic allocation (11); this demonstrates directly that the autarkic allocation is not Pareto optimal. Finally, the sequence (14) is also a nonsunspot, monetary competitive equilibrium allocation when supporting prices are defined by

$$\frac{p^m(\alpha)}{\pi(\alpha)} = \frac{p^m(\beta)}{\pi(\beta)} > 0 \quad \text{and} \quad p^m(\alpha) + p^m(\beta) = p^1(\alpha)\omega^1 + p^1(\beta)\omega^1 \tag{15}$$

and

$$\frac{p^1(\alpha)}{\pi(\alpha)} = \frac{p^1(\beta)}{\pi(\beta)} > 0$$

and (16)

$$p^{t+1}(s) = p^t(s) \quad \text{for } t = 1, 2, \dots \text{ and } s = \alpha, \beta.$$

[The last claim is now easily verified by using the simplified Lagrangean conditions ($3'''$) and (4) together with (5),

$$\frac{p^t(s)}{p^{t+1}(s)} > \frac{a}{b} \quad \text{for } s = \alpha, \beta,$$

and (7), because the latter constitutes the (simplified) Kuhn–Tucker conditions characterizing an optimal solution to (2), for which $x_t^t(s) = 0$ and $x_t^{t+1}(s) > 0$ for $s = \alpha, \beta$.]

c A sunspot nonmonetary competitive equilibrium

The aim in this section is to specify in more detail the parameters of our example in order to construct a sunspot nonmonetary competitive equilibrium. In such an equilibrium, at least one consumer chooses (and thus strictly prefers) an optimal consumption bundle different from his endowment. Hence, this construction guarantees that the associated allocation Pareto dominates the (autarkic) nonsunspot nonmonetary equilibrium allocation (11).

Our strategy is to restrict attention to allocations that are stationary after the level of sunspot activity is revealed. (So, in this sense, all three of our displayed equilibria are "stationary.") Consider an allocation of the form

$$x_0^1(s) = x^{2,s} \quad \text{for } s = \alpha, \beta$$

and (17)

$$[x_t^t(s), x_t^{t+1}(s)] = (x^{1,s}, x^{2,s}) \quad \text{for } t = 1, 2, \dots \text{ and } s = \alpha, \beta.$$

Choose the parameters in (17) so that Mr. $t > 0$ will be better off in state α than in autarky, but better off in autarky than in state β. In particular, consider $(x^{1,s}, x^{2,s})$ for $s = \alpha, \beta$ that satisfy the following conditions:

$$0 < x^{1,\alpha} < \omega^1 < x^{1,\beta}, \quad 0 < x^{2,\beta} < \omega^2 < x^{2,\alpha};$$

$$x^{i,\alpha} - \omega^i = -(x^{i,\beta} - \omega^i) \quad \text{for } i = 1, 2; \qquad (18)$$

$$x^{2,s} - \omega^2 = -(x^{1,s} - \omega^1) \quad \text{for } s = \alpha, \beta.$$

The properties described by (18) are illustrated in Figure 1, together with the relationship between the stationary allocation (17) and (18) and the

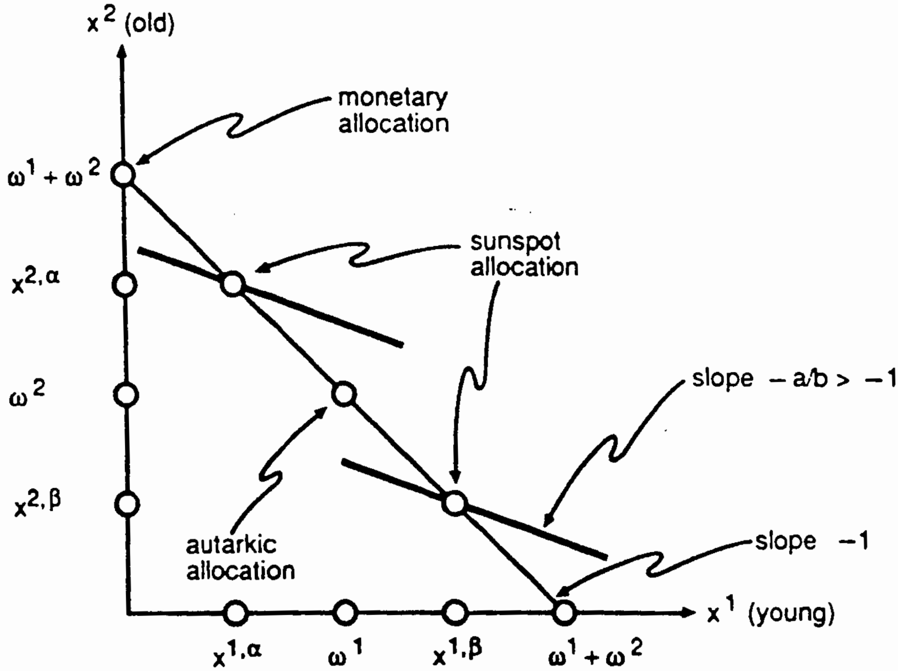

Figure 1

nonsunspot monetary and nonsunspot nonmonetary allocations. Note, especially, that the last equation in (18) implies that the market-clearing conditions (8) are satisfied and [together with the first inequalities in (18)] that we have

$$x^{2,\alpha}-x^{2,\beta}=-(x^{1,\alpha}-x^{1,\beta})>0. \tag{19}$$

If an allocation satisfying conditions (17) and (18) is to be supported as a competitive equilibrium allocation, then from Mr. t's first-order condition (6), the real rate of interest after sunspot activity is revealed must equal his constant marginal rate of substitution a/b, so that

$$p^{t+1}(s)=\left(\frac{b}{a}\right)p^{t}(s) \quad \text{for } t=1,2,\ldots \text{ and } s=\alpha,\beta. \tag{20}$$

Make the further specification that

$$p^{1}(\alpha)=p^{1}(\beta)>0, \tag{21}$$

which in conjunction with (20) yields

$$p^{t}(\alpha)=p^{t}(\beta)>0 \quad \text{for } t=1,2,\ldots. \tag{22}$$

At contingent-claims prices satisfying (22), the middle equation in (18) implies that both budget constraints (3) and (5) are satisfied.

Now substitute from (17) and (22) into Mr. t's other first-order condition, (7), which after simplification yields

$$\frac{\psi'(ax^{1,\alpha}+bx^{2,\alpha})}{\psi'(ax^{1,\beta}+bx^{2,\beta})}=\frac{1-\pi(\alpha)}{\pi(\alpha)}. \tag{23}$$

Condition (19), together with the assumption that $0<a<b$, implies that

$$ax^{1,\alpha}+bx^{2,\alpha}=ax^{1,\beta}+bx^{2,\beta}+(-a+b)(x^{2,\alpha}-x^{2,\beta})>ax^{1,\beta}+bx^{2,\beta},$$

or that, by strict concavity of ψ, the left-hand side of Eq. (23) must be less than one. So we can solve Eq. (23) for $0<\pi(\alpha)<1$, in which case both of Mr. t's first-order conditions (6) and (7), will be satisfied.

Given the foregoing specification of consumption [in (17) and (18)], prices [in (20) and (21)], and probabilities [in (23)], all that remains is to guarantee that Mr. 0's first-order condition (4) is also satisfied. This is accomplished by further restricting ϕ to exhibit the property

$$\frac{\phi'(x^{2,\alpha})}{\phi'(x^{2,\beta})}=\frac{1-\pi(\alpha)}{\pi(\alpha)}<1,$$

which is possible because ϕ is strictly concave whereas, from the first inequalities in (18), we have $x^{2,\alpha}>x^{2,\beta}$.

We have displayed a sunspot nonmonetary competitive equilibrium.

Remark 1: Constructing this sunspot equilibrium required balancing various considerations (e.g., Mr. t's risk aversion against his relative welfare improvement in state α over state β). This balancing does not seem to be so delicate as to lead us to believe that we are dealing with a razor's edge situation. In particular, the simplifying assumption of linear indifference curves is not essential even for our specific example. From Figure 1, one can see that we basically need only the property that the marginal rate of substitution along some line $x^1+x^2=k>0$ be equal and less than unity (for at least two distinct points). Thus, for instance, a more general specification that would justify our specific construction is

$$\psi(x^1,x^2)=f[ax^1+bx^2+cg(x^1+x^2)],$$

where $0<a<b$, $c>0$, and both f and g are smooth, strictly increasing, and strictly concave.

Remark 2: As we mentioned at the beginning of this subsection, the sunspot allocation (17) and (18) Pareto dominates the nonsunspot, nonmonetary competitive equilibrium allocation (11). However, although weakly Pareto optimal, it in turn is itself not Pareto optimal; it also is Pareto dominated by the nonsunspot, monetary competitive equilibrium allocation (14). This fact is easily deduced by employing the relationships displayed in Figure 1, together with the strict monotonicity of ϕ and ψ, which yield

$$\phi(\omega^1+\omega^2) > \max_s \phi(x^{2,s}) > \pi(\alpha)\phi(x^{2,\alpha}) + \pi(\beta)\phi(x^{2,\beta})$$

and

$$\psi[b(\omega^1+\omega^2)] > \max_s \psi(ax^{1,s}+bx^{2,s})$$
$$> \pi(\alpha)\psi(ax^{1,\alpha}+bx^{2,\alpha}) + \pi(\beta)\psi(ax^{1,\beta}+bx^{2,\beta}).$$

d *Sunspots, money, and welfare*

We reemphasize the following welfare considerations:

1 The nonsunspot monetary competitive equilibrium allocation (14) is Pareto optimal; it also (strictly) Pareto dominates both the nonsunspot nonmonetary competitive equilibrium allocation (11) and the sunspot nonmonetary competitive equilibrium allocation (17) and (18).

2 Both nonmonetary allocations are weakly Pareto optimal. Although sunspot allocations cannot be Pareto optimal, the sunspot allocation from our example Pareto dominates the nonsunspot, nonmonetary competitive equilibrium allocation from our example. In other words, our example suggests that, in an overlapping-generations setting, sunspots might be employed as a coordinating device to attenuate a tendency toward oversaving; that is, *sunspots can serve as imperfect substitutes for the social contrivance of money.*

The sunspot allocation has another striking welfare property. Even though it is weakly Pareto optimal, a planner seeking to restore Pareto optimality would have to intervene in the first period.[10] This distinguishes this more complicated OG model from the one-good, one-state OG model. In the latter case the planner could wait any finite number of periods before intervening and still convert a weakly Pareto optimal allocation to Pareto optimality.

To paraphrase Samuelson (1967), one might question the importance of distinguishing between weakly Pareto-optimal allocations and fully Pareto-optimal allocations on the grounds that *based solely on a finite*

[10] The sunspot allocation is not *observationally Pareto optimal* (see Burke, 1987) because we need only observe a *finite* history to determine that it is not (fully) Pareto optimal. Clearly, in the finite history, the complete contingent allocation (*not* merely the realized allocation) must be observed.

history one cannot observe a difference between weak Pareto optima and full Pareto optima. The importance of Samuelson's point aside, it is accurate for the one-good, one-state model. Samuleson's point is, however, not accurate for one sunspot equilibrium allocation. This weakly Pareto optimal allocation can be detected (or *observed*) to be not Pareto optimal immediately after the first-period contingent allocation has been assigned.[11] Once the allocation has been contaminated by sunspots, it cannot be extended to be Pareto optimal.

Perhaps, in future work, one can move from constructing examples to establishing theorems for economies with many commodities per period, many consumers per generation, more general demographic structure, and naturally restricted market participation. Can the existence of sunspot, nonmonetary competitive equilibria in the general case be related as in our special example to the structure of the set of nonsunspot (monetary and nonmonetary) competitive equilibria? If so, the idea that sunspots may partly fill the same role as money would be strengthened.

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[11] Our example need not necessarily be interpreted as a sunspot equilibrium. It is, of course, also an example of equilibrium for a two-good, one-state OG model. The equilibrium is weakly Pareto optimal; and after the first period allocation is made, it is revealed that the intertemporal allocation could not be fully Pareto optimal. If symmetry-breaking occurs in the first period, it can never be repaired.

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