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The Economic Effects of Restrictions on Government Budget Deficits[1]

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In overlapping-generations economies with perfect financial markets and lump-sum taxation, restrictions on the government budget deficit do not limit the set of achievable allocations. For economies in which the tax instruments are distortionary and limited in number, this strong form of irrelevance does not hold even if markets are perfect. We propose a weaker (but natural) definition of irrelevance in which only a finite (but arbitrarily large) number of restrictions near the baseline deficit are considered. We show that if the government can use only anonymous consumption taxes, there is weak irrelevance of the deficit restrictions if the number of tax instruments is large relative to the number of policy goals. *Journal of Economic Literature* Classification Numbers: D51, D91, E32.

1. INTRODUCTION

It has been proposed that the United States constitution be amended to make it unlawful for there to be a deficit in the federal government's budget. For some states in the U.S., there are already constitutional

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prohibitions or restrictions on deficits. The Maastricht Treaty calls for fines to be levied on countries in the European Union that incur deficits beyond the prescribed limits (currently 3% of GDP). Other governments are obliged to limit their budget deficits by the terms of their loan agreements with international agencies.

What are the economic effects of restrictions on government budget deficits? The popular wisdom is that these restrictions do matter, but there is debate about their desirability. The popular view is supported by empirical evidence suggesting that deficit restrictions are effective in reducing government expenditures. See, for example, Alesina and Perotti [1], Bohn and Inman [10], and Poterba [21], studies based on comparisons across the states of the U.S. On the other hand, the thrust of the existing theoretical literature is that deficit restrictions are ineffective in the sense that the set of perfect-foresight equilibrium allocations is independent of the sequence of government budget deficits. See Barro [9], Auerbach and Kotlikoff [2], Auerbach *et al.* [3], Kelly [17], and Kotlikoff [18]. These theoretical results are mainly based on models with a finite number of infinitely-lived consumers or on models with only one (representative) consumer per generation.

In the present paper, we take a fresh look at the theory of budget-deficit restrictions. We adopt a pure-exchange *overlapping-generations* model with *several consumers* per generation and *several commodities* per period. We allow for *distortionary taxes*, focusing on the case of consumption taxes. We also allow for the fact that tax schedules cannot be made perfectly individual-specific. For example, it might be the case that each consumer in a given generation must face the same tax schedule, possibly because of limits on the information of the tax authority or possibly for considerations of fundamental fairness. In this sense, we allow for (at least partially) *anonymous taxation.*

We use the approach of Ramsey [22], Vickrey [26], Diamond and Mirrlees [11], Mirrlees [19], and others in modeling the government. We assume that the government knows the distribution of individual characteristics within a given consumer class (say, the individuals of a given generation), but it either does not know or cannot act upon the characteristics of any particular individual within this class. We also assume that the government cannot tax commodities in the same commodity class at different rates, even though it knows the distribution of commodities in fine detail.

The advantage of lump-sum taxes (if they are feasible) is that they are nondistortionary. If perfectly personalized lump-sum taxes were feasible, then every Pareto optimal allocation could be decentralized. If perfectly personalized taxes are not available, then consumption taxes, while distortionary, have some potential advantages over lump-sum taxes. Everyone in the same consumer tax class must get the same lump-sum transfer or tax. Everyone in the same consumer tax class must also face the same consumption tax rates, but by varying the rates over the commodity tax classes, the government is typically able to “redistribute income” within a given consumer tax class. Of course, if the taxes are distortionary, such redistribution of income is not costless.

We show that if there are lump-sum taxes, then the set of equilibrium allocations is independent of the sequence of budget deficits. Hence, in this case, restrictions on the deficits are *irrelevant*. This is in accord with the existing results of Kelly [17] and others, but our result is stronger in that we have several commodities per period, distortionary tax instruments in addition to the lump-sum instruments, and restrictions that tax rates be identical in the same consumer tax class and in the same commodity tax class. The reasoning behind this so-called irrelevance result is clear. The government's borrowing and lending is restricted in every period. These restrictions are not binding because the government can *in effect* “borrow” from consumers, whose own credit is unrestricted, by increasing their taxes in their youth while “repaying” them through transfers in their old age in such a way as to keep total lifetime taxation unchanged.

The situation is different if lump-sum taxes are not available. It turns out that the definition of deficit-restriction irrelevance used for lump-sum taxation is far too strong for the more general case. In economies with lump-sum taxation, the baseline deficit-restriction sequence is compared to any arbitrary deficit-restriction sequence. In more general economies, the baseline deficit restriction is compared only to deficit restrictions that are close in a finite number of components (no matter how many) to the baseline restriction with the remaining components left unrestricted. The corresponding definition of (weak) irrelevance avoids problems that would result in “exploding” taxes and negative prices.

We analyze an economy in which the only taxes are proportional consumption taxes. In this economy, there is weak irrelevance if the number of independent tax instruments per period is large relative to the number of consumers per generation. When too few tax instruments are available, meeting the restrictions on the budget deficit is likely to require infeasible redistributions of wealth.

In our formal model, we assume that the number of tax instruments and the number of goods are exogenously given. The actual political situation is that the government is likely to be able to expand the number of tax instruments (by introducing, for example, nonproportional taxes, exemptions from tax, and exclusions from tax) while limiting its redistributive aims to broad goals. Hence weak irrelevance is more likely to hold when one allows for the government's role in defining instruments. When the government uses the tools and goals that it has been given and faces deficit-restriction irrelevance, then the government is able to *avoid* the constitutional restrictions. If, on the other hand, the government changes the tools or the goals or the accounting conventions to achieve deficit-restriction irrelevance, then it is *evading* the restrictions. Our formal analysis is on avoidance, not evasion. Both subjects are important.

Typically there is a vast multiplicity of perfect-foresight equilibria to money models and overlapping-generations models. There are even more rational-expectations equilibria (including the sunspot equilibria). The approach taken in the present paper—essentially the approach of Diamond and Mirrlees—implicitly assumes that the government has full power to "select" equilibria. If the government does not have this power, it would seem at first blush that the likelihood of deficit-restriction irrelevance would be substantially reduced, but this requires more careful analysis.

We introduce the basic model in Section 2. Feasible fiscal policies are defined in Section 3. Section 4 is on perfect-foresight, competitive equilibrium. Section 5 contains our analysis of the economy with lump-sum taxation. Section 6, on consumption taxes, contains the heart of our analysis. Our concluding remarks are in Section 7. In the text, our "proofs" are essentially arguments based on careful counting of equations and unknowns. The full matrix rank analysis is left for the Appendix.

2. THE BASIC MODEL

We employ a pure-exchange overlapping-generations model in which there are n different consumers per generation and ℓ perishable commodities per period. We suppose without loss of generality that consumers live for two periods. The government collects taxes and distributes transfers. In the present paper, we focus on two types of instruments: (non-distortionary) lump-sum taxes and (distortionary) consumption taxes. The full spectrum of these taxes is typically not available to the government: individuals in the same consumer tax class must face the same tax schedule; i.e., taxation must be anonymous within a given class of individuals. The government is also constrained to set the same tax rate for each commodity in the same commodity class.

Our set-up is based on the Samuelson [23] overlapping-generations model presented in Balasko and Shell [5–7]; wherever possible we adopt the notation in [6]. Let $m_{th}^{s} \in \mathbb{R}$ be the lump-sum transfer to consumer h of generation t in period s (if m_{th}^{s} is negative, then the consumer is paying a lump-sum tax). We assume that these transfers (and the entire public debt) are in bonds that—like T-bills—bear a zero coupon rate of interest. If all economic actors including the government were unrestricted in their borrowing and lending, then assuming a zero (or any other nominal) coupon return on the government debt is without loss in generality by the superneutrality of money; see, e.g., [6]. In our case, government borrowing is restricted but we will still have superneutrality if the definition of the government budget deficit includes a proper inflation offset for consumer capital losses (i.e., government capital gains) on the public debt.

We add consumption taxes to the government instruments in [6]. Let $\tau^{si}_{th} \in \mathbb{R}$ be the *present* tax rate levied on consumer h of generation t based on his consumption of commodity i in period s; if τ^{si}_{th} is negative then this consumption is being subsidized.

Let $x^s_{th} = (x^{s1}_{th}, ..., x^{si}_{th}, ..., x^{s\ell}_{th}) \in \mathbb{R}^{\ell}_{++}$ be the vector of consumption in period s by consumer h of generation t and $\omega^s_{th} = (\omega^{s1}_{th}, ..., \omega^{si}_{th}, ..., w^{s\ell}_{th}) \in \mathbb{R}^{\ell}_{++}$ be the vector of his endowments in period s for $t = 0, 1, ...,$ $s = 1, 2, ...,$ and $h = 1, ..., n$. Let $m^s_{th} \in \mathbb{R}$ be the money transfer in period s to consumer h of generation t, and $\tau^s_{th} = (\tau^{s1}_{th}, ..., \tau^{si}_{th}, ..., \tau^{s\ell}_{th}) \in \mathbb{R}^{\ell}$ be the vector of his consumption tax rates in period s. Consumers from generation 0 are alive in period 1, while consumers from generation t $(t = 1, 2, ...,)$ are alive in periods t and $t+1$. It is convenient to define the vectors

$$x_{0h} = x^1_{0h} \in \mathbb{R}^{\ell}_{++}, \qquad x_{th} = (x^t_{th}, x^{t+1}_{th}) \in \mathbb{R}^{2\ell}_{++},$$

$$\omega_{0h} = \omega^1_{0h} \in \mathbb{R}^{\ell}_{++}, \qquad \omega_{th} = (\omega^t_{th}, \omega^{t+1}_{th}) \in \mathbb{R}^{2\ell}_{++},$$

$$m_{0h} = m^1_{0h} \in \mathbb{R}, \qquad m_{th} = (m^t_{th}, m^{t+1}_{th}) \in \mathbb{R}^2,$$

and

$$\tau_{0h} = \tau^1_{0h} \in \mathbb{R}^{\ell}, \qquad \tau_{th} = (\tau^t_{th}, \tau^{t+1}_{th}) \in \mathbb{R}^{2\ell}.$$

Let

$$p^s = (p^{s1}, ..., p^{si}, ..., p^{s\ell}) \in \mathbb{R}^{\ell}_{+}$$

be the vector of present before-tax prices for commodities available in period s and let

$$q^s_{th} = (q^{s1}_{th}, ..., q^{si}_{th}, ..., q^{s\ell}_{th}) \in \mathbb{R}^{\ell}_{++}$$

be the present after-tax vector of commodity prices for consumer h of generation t in period s. These prices must satisfy

$$q_{0h} = q^1_{0h} = p^1 + \tau_{0h} \in \mathbb{R}^{\ell}_{++}$$

and (2.1)

$$q_{th} = (q^t_{th}, q^{t+1}_{th}) = (p^t, p^{t+1}) + (\tau^t_{th}, \tau^{t+1}_{th}) \in \mathbb{R}^{2\ell}_{++}$$

for $h = 1, \ldots, n$. Define the sequences

$$x = (x_{0h})_{h=1}^{h=n}, \ldots, (x_{th})_{h=1}^{h=n}, \ldots$$

$$\omega = (\omega_{0h})_{h=1}^{h=n}, \ldots, (\omega_{th})_{h=1}^{h=n}, \ldots$$

$$p = p^1, \ldots, p^t, \ldots$$

$$m = (m_{0h})_{h=1}^{h=n}, \ldots, (m_{th})_{h=1}^{h=n}, \ldots$$

$$\tau = (\tau_{0h})_{h=1}^{h=n}, \ldots, (\tau_{th})_{h=1}^{h=n}, \ldots$$

and

$$q = (q_{0h})_{h=1}^{h=n}, \ldots, (q_{th})_{h=1}^{h=n}, \ldots .$$

We assume that the preferences of consumer h from generation t can be described by the utility function u_{th} defined over the consumption set of all strictly positive x_{th}'s (i.e. $\mathbb{R}^{\ell}_{++}$ or $\mathbb{R}^{2\ell}_{++}$) with the properties:

- u_{th} is twice differentiable with strictly positive first-order derivatives and with corresponding negative definite Hessian
- the closure of every indifference surface of u_{th} is in the consumption set (i.e. $\mathbb{R}^{\ell}_{++}$ or $R^{2\ell}_{++}$).

These rather standard assumptions simplify the comparative statics. See Balasko and Shell [5, 6] for their application in overlapping-generations models.

The behavior of consumer h ($h = 1, 2, \ldots, n$) from generation t ($t = 1, 2, \ldots$) is described by

$$\begin{aligned}
&\text{maximize } u_{th}(x_{th}^{t}, x_{th}^{t+1}) \\
&\text{subject to} \\
&q_{th}^{t} \cdot x_{th}^{t} + q_{th}^{t+1} \cdot x_{th}^{t+1} = p^{t} \cdot \omega_{th}^{t} + p^{t+1} \cdot \omega_{th}^{t+1} + p^{m}\mu_{th} \qquad (2.2) \\
&\text{and} \\
&x_{th} = (x_{th}^{t}, x_{th}^{t+1}) \in \mathbb{R}^{2\ell}_{++},
\end{aligned}$$

where the *present* price of a bond $p^m \in \mathbb{R}_+$ is constant since a no-arbitrage condition yields

$$p^{mt} = p^{m, t+1} \in \mathbb{R}_+, \tag{2.3}$$

where p^{ms} is the present price of the bond in period s, $s = 1, 2, \ldots, t, t+1, \ldots$. Then $p^m\mu_{th} \in \mathbb{R}$, the present value of the lifetime lump-sum subsidy to consumer h from generation t, is defined by

$$p^m\mu_{th} = p^m(m_{th}^{t} + m_{th}^{t+1}).$$

Equilibrium condition (2.3) is quite intuitive once one gets the knack of present prices (used by general-equilibrium types) as opposed to current prices (normally used by macro types). We have assumed that the nominal (coupon) rate of interest on bonds is zero. Hence the only return on holding bonds is their capital gain relative to commodities. Condition (2.3) is thus that bonds appreciate in value relative to any commodity at the commodity rate of interest. (Hence the transfers $m_{th} = (m_{th}^t, m_{th}^{t+1}) \in \mathbb{R}^2$ affect the behavior of the consumer only through his lifetime transfer $\mu_{th} = m_{th}^t + m_{th}^{t+1} \in \mathbb{R}$.) See Balasko and Shell [6].

It remains to describe the behavior of the older generation ($t=0$) in period 1. Consumer $0h$ maximizes his utility subject to his one-period budget constraint:

$$\text{maximize } u_{0h}(x_{0h}^1)$$

$$\text{subject to}$$

$$q_{0h}^1 \cdot x_{0h}^1 = p^1 \cdot \omega_{0h}^1 + p^m \mu_{0h} \tag{2.4}$$

$$\text{and}$$

$$x_{0h}^1 \in \mathbb{R}_{++}^{\ell},$$

where $\mu_{0h} = m_{0h}^1$.

In what follows, we assume that the price of the government bond is positive, i.e. that the price level in terms of bonds is finite. If this were not the case, the government would be unable to finance its deficit. Because of the absence of "money illusion" on the part of consumers, we can normalize taxes and transfers so that $p^m = 1$ without loss of generality.[2]

We assume that government consumption of commodities is exogenously given. Let $g^{ti} \in \mathbb{R}_+$ denote the allocation of commodity i ($i = 1, ..., \ell$) in period t ($t = 1, 2, ...$) to the government, and let

$$g^t = (g^{t1}, ..., g^{ti}, ..., g^{t\ell}) \in \mathbb{R}_+^{\ell}$$

be the vector of government consumption in period t. The government allocation sequence is denoted by g where

$$g = (g^1, ..., g^t, ...) \in (\mathbb{R}_+^{\ell})^{\infty}.$$

[2] This would be restrictive if we allowed for specific individual expectation formation. Expectations might depend on the nominal values of taxes. In this case, it would be illegitimate to fix p^m at unity.

The present value of the government budget deficit in period t, $d^t \in \mathbb{R}$, is defined by

$$d^t = p^t \cdot g^t + \sum_{h=1}^{n} \left[m^t_{t-1,h} + m^t_{th} - \sum_{i=1}^{\ell} \left(\tau^{ti}_{t-1,h} x^{ti}_{t-1,h} + \tau^{ti}_{th} x^{ti}_{th} \right) \right]$$

for $t = 1, 2, ...$. The sequence of present-value government deficits d is defined by

$$d = (d^1, ..., d^t, ...) \in (\mathbb{R})^\infty.$$

Let $D^t \in \mathbb{R}$ for $t = 1, 2, ...$ be the present value of the government debt in period t. Then we have

$$D^t = \sum_{s=1}^{t} d^s + D^0,$$

where D^0 is the initial debt.

3. FISCAL POLICIES

In the simplest case, a fiscal policy could be any sequences of lump-sum transfers m and commodity tax rates τ. This would be the case of full fiscal potency. There are, however, restrictions on the power of the government. It is unlikely that the tax authorities can differentiate individuals sufficiently to use the full range of personalized taxation. Some personalizations of the schedules might require too much detailed information about individuals, be very costly to administer[3], or be deemed unfair.

On the other hand, governments can and do base taxes and other policies on individual demographic characteristics. Age and family size are frequently used in tax policies. To qualify for a government retirement transfer, one must meet an age test. Tax rates for withdrawals from private retirement plans in the U.S. depend on the age of the withdrawer. For U.S. personal income taxation, the "personal exemption" (from gross income) is doubled for those over 65 years; other deductions and exemptions (from income) are based on family size. In Europe, the fare on public transportation is reduced for older people and for people from "large families."

We suppose that the set of consumers $\{1, ..., h, ..., n\}$ is partitioned into N consumer tax classes $C_1, ..., C_H, ..., C_N$, where $N \leqslant n$. If $N = n$, then complete individualization of taxes would be possible. If $N = 1$, every consumer

[3] See Heller and Shell [15] for the effects on tax policy of costs of tax administration.

within the same generation would face the same m's and τ's. Some assumptions are implicit in our formulation. The partition into tax classes is stationary: it is the same for each generation. Furthermore, individuals remain in the same consumer tax class for their lifetimes. Neither of these assumptions affects our results. Note, however, that these assumptions do not prevent taxing the young differently from the old. Indeed, this possibility plays an important role in our proofs and examples.

There are other restrictions on government tax policies. It is in some instances impossible or at least very costly for the government to tax "nearby" commodities at different rates. Imagine taxing white bread differently from whole wheat bread. This might impose unreasonably high compliance costs on bakeries, while also imposing unreasonably high administrative costs on the tax authority.

We suppose that the set of commodities $\{1, ..., i, ..., \ell\}$ is partitioned into L commodity tax classes, $K_1, ..., K_I, ..., K_L$, where $L \leqslant \ell$. If $L=1$, then all commodities must be taxed at the same rate. If $L=\ell$, then the restriction to commodity tax classes is not binding.

We formalize the notion of these restrictions on government policy in the next definition.

Definition 1. A *feasible fiscal policy* $\phi=(m, \tau)$ is a sequence of lump-sum transfers m and a sequence of commodity tax rates τ that satisfies

(1) $m^s_{th}=m^s_{th'}$ and $\tau^s_{th}=\tau^s_{th'}$ for $t=0, 1, ..., s=1, 2, ...,$ and every h and h' in the consumer tax class C_H, for $H=1, ..., N$

(2) $\tau^{si}_{th}=\tau^{si'}_{th}$ for $t=0, 1, ..., s=1, 2, ..., i=1, ..., \ell, i'=1, ..., \ell,$ and every i and i' in the commodity tax class K_I, for $I=1, ..., L$.

The set of feasible fiscal policies is denoted by Φ. If only lump-sum transfers are available, then a feasible fiscal policy is denoted by the sequence m; and the set of feasible fiscal policies is denoted by $\mathscr{M}$. If only consumption taxes are available, then a feasible fiscal policy is denoted by the sequence τ and the set of feasible fiscal policies by $\mathscr{T}$.

If the constitutional restriction on the budget deficit is satisfied, then we have $d^t=\delta^t$, where $\delta^t \in \mathbb{R}$ is the deficit restriction for period t, or

$$p^t \cdot g^t + \sum_{h=1}^{h=n} [m^t_{t-1,h}+m^t_{th} - \sum_{i=1}^{i=\ell} (\tau^{ti}_{t-1,h} x^{ti}_{t-1,h} + \tau^{ti}_{th} x^{ti}_{th})] = \delta^t \quad (3.1)$$

for $t=1, 2, ...$. The budget restriction sequence δ is defined by $\delta=(\delta^1, \delta^2, ..., \delta^t, ...)$.

The restriction (3.1) is complicated when there are consumption taxes. The money transfers enter (3.1) in a relatively simple way, but the consumption tax rates interact with the individual consumptions in determining whether (3.1) is satisfied.

4. EQUILIBRIUM

We maintain throughout this paper some strong assumptions. We suppose perfect foresight on the part of consumers and the government. We do not consider sunspots. We also suppose that the government is able to perfectly commit to its announced fiscal policy.

Next we define equilibrium.

DEFINITION 2. Given the fiscal policy ϕ, the government allocation g, the deficit restriction δ, endowments ω, the behavior of consumers as described by (2.2) and (2.4), the normalization[4] yielding $q_{01}^{11}=1$, and the (further) monetary normalization yielding $p^m=1$, a *constitutional competitive equilibrium* is defined by the nonnegative price sequence p, the positive price sequence q, and the consumer allocation sequence x such that markets clear, i.e., we have

$$g^t + \sum_{h=1}^{h=n} (x_{t-1,h}^t + x_{t,h}^t) = \sum_{h=1}^{h=n} (\omega_{t-1,h}^t + \omega_{t,h}^t),$$

and the deficit restriction (3.1) is satisfied for $t=1, 2, \ldots$.

From Balasko and Shell [5], one might expect that the existence of competitive equilibrium to be guaranteed in nice overlapping-generation models, but this does not extend to our Definition 2. There are three reasons that equilibrium as defined above could fail to exist.

The first reason has to do with commodity taxation. Some component of τ might be too large in absolute magnitude to permit market clearing with both nonnegative p and positive q.

The second reason is that so-called "monetary" equilibria (i.e., equilibria with $p^m>0$) are harder to come by than "nonmonetary" equilibria (i.e., equilibria with $p^m=0$). For a proper monetary equilibrium to exist, the fiscal policy ϕ must be bonafide.[5] In finite economies, bonafidelity requires

[4] Because of our assumptions on preferences, the q's must be positive in equilibrium. Hence choosing the first good as the numeraire is justified. These assumptions also imply that some component of p^t $(t=1, 2, \ldots)$ is positive. Hence $\sum_{i=1}^{\ell} p^{1i}=1$ would also have been a justified normalization choice. Diamond and Mirrlees [11] might have chosen to set the first τ equal to zero. We will stick with $q_{01}^{11}=1$.

[5] If ϕ is not bonafide, then equilibrium p^m must be zero and the government will be unable to finance its deficit. Bonafidelity of the government fiscal policy is basic to our present analysis, but we do not face this thorny problem head on. For the analysis of bonafidelity in the lump-sum tax case, see Balasko and Shell [6, 7, 8].

that the fiscal policy be balanced, i.e., that the sum of all taxes be equal to the sum of all transfers. In overlapping-generations economies, bonafidelity is not so simple. Depending on the economy, unbalanced fiscal policies can be bonafide; the public debt need not always be retired. Consider, for example, the David Gale [12] version of the stationary OG model. In the "classical case," the simple imbalanced transfer sequence $m = (1, 0, ..., 0, ...)$ does not permit a positive p^m; in this case, the fiscal policy is not bonafide. In the "Samuelson case" the same imbalanced transfer sequence is bonafide, and hence in this case a proper monetary equilibrium exists.

Third, equilibrium might fail if the government allocation g is too large in some component. This problem goes away if we assume $g < \omega$.

A discussion of the constitutional restriction (3.1) is in order. The leading example of restrictions on government deficits is the strict balanced-budget requirement: $\delta^t = 0$ for $t = 1, 2, \ldots$. Most actual constitutions are based on aversion to positive deficits with typically no aversion to surpluses, so that a more realistic constraint would be in inequality form, namely $d^t \leqslant \delta^t$. The equality version, $d^t = \delta^t$, is simpler to work with and our basic results are not substantially affected by this choice. A more serious worry is that these constraints might be stated in real terms[6], even perhaps that they might be based on economic performance as in the case of the Maastricht Treaty. This formulation would create some problems in notation, but it would not affect our results. Even so, the leading case is the strict balanced-budget restriction, $\delta = 0$, which is the same in dollar or real terms.

We have required that the government issue only zero-coupon bonds. If the government pays interest on its debt, then two extra terms should appear in the definition of the deficit and on the left hand side of the restriction (3.1): (1) the coupon payments and (2) the resulting capital losses to individual bond holders, which represent capital gains to the government. The coupon payments and the capital losses on the bonds would then be perfectly offsetting for all economic actors including the government. Hence we have "superneutrality" for the government bonds.

Since the government faces the period-by-period "budget constraints" given by (3.1) rather than a single constraint, we are considering a general equilibrium economy in which participation in the financial markets is restricted. In this case, it is the government that cannot borrow freely. In this paper, the other economic actors (the consumers) are unrestricted in their borrowing.

[6] Or, if in dollar terms, at prices different from the "supply," or before-tax, prices p. Valuing government consumption g^t at prices p^t is probably realistic in terms of actual practice, but it does seem to offer obvious possibilities for avoidance. Assume that g^{ti} is large. The government must be tempted to choose the $\tau^{ti}_{t-1,h}$ and the τ^{ti}_{th} to be sufficiently large to reduce private demand so that p^{ti} is either small or even zero, thus reducing the present value d^t.

In general, the restrictions (3.1) cannot be represented by simple restrictions on the set Φ of feasible tax and transfers, because revenues from the consumption taxes depend on the equilibrium allocation x. Indeed, because of the possible multiplicity of equilibria, it might be the case that—fixing preferences, endowments, and the fiscal policy—for some of the equilibria the deficit restrictions (3.1) are satisfied, while for other competitive equilibria based on the same given parameters these deficit restrictions are not satisfied.

The aim of our analysis is to describe the circumstances under which the government is able to avoid the effects of the deficit restrictions.

We say that δ is *irrelevant* if it does not restrict x or g. Our formal definition is next.

DEFINITION 3. *Irrelevance of the deficit restriction*: Let g be the government allocation and x be the consumer allocation that can be implemented as a constitutional competitive equilibrium with some feasible fiscal policy ϕ resulting in deficits $d = \delta$. The deficit restriction δ is said to be *irrelevant* if for any other deficit restriction δ' there exists a feasible fiscal policy ϕ' that implements the allocation x as a competitive equilibrium that is compatible with g, but with the resulting deficits given by the sequence δ'.

Definition 3 is global in nature. First, the comparison restriction $\delta' = (\delta^1)', ..., (\delta^t)', ... \in \mathbb{R}^\infty$ applies to each of an infinite number of periods. Second, the comparison deficit restriction δ' can be far from the baseline restriction δ. The global aspects of Definition 3 make it inapplicable to cases other than lump-sum taxation. With commodity taxation, if δ' is far from δ, then the nonnegativity of p and the positivity of q cannot be assured while, if δ' is infinite in length, the required tax rates might "explode."[7]

We propose a weaker notion of deficit-restriction irrelevance. The weaker notion involves comparisons with only a subset of the set of possible deficit restrictions. The first weakening of the concept involves the number of periods, T, to which the comparison deficit restriction applies. Let $\delta(T) = (\delta^1, \delta^2, ..., \delta^t, ..., \delta^T) \in \mathbb{R}^T$ be a *deficit restriction of (finite) length* T. For a competitive equilibrium to be constitutionally feasible under our weaker notion the deficit d^t in period t must be equal to δ^t if $t = 1, 2, ..., T$, but for $t > T$ the deficit is unrestricted[8]. The second weakening of the concept is related to the "distance" of the comparison deficit restriction from the baseline restriction. Only period-by-period deficits that are not too

[7] See Example 11.

[8] In Balasko and Shell [7], it is shown that every strictly balanced fiscal policy (one for which the public debt is forever retired by some date T) is bonafide. This is an approach to "getting rid" of the tails to infinite budget sequences. The approach taken here is to "ignore" restrictions on the tails of infinite budget sequences.

different from the "pre-reform" deficits are considered. In other words, only a neighborhood of the original sequence is considered (in any topology, since the restriction is finite). According to the weaker notion of irrelevance only restrictions of finite length $\delta(T)$ that belong to a first T-period neighborhood of the deficit restriction $\delta = (\delta^1, \delta^2, ..., \delta^t, ..., \delta^T, ...)$, denoted $\mathfrak{D}^T(\delta)$, are considered.

The formal definition of weak irrelevance follows.

DEFINITION 4. *Weak irrelevance of the deficit restriction*: Let g be the government allocation and x be the consumer allocation that can be implemented as a competitive equilibrium with some feasible fiscal policy ϕ and with the resulting deficits given by the sequence $d = \delta$. The deficit restriction δ is said to be *weakly irrelevant* if *for each* T there exists a neighborhood $\mathfrak{D}^T(\delta)$ of $\delta(T) = (\delta^1, ..., \delta^t, ..., \delta^T)$, the first T components of δ, such that for all $\delta'(T) = ((\delta^1)', ..., (\delta^t)', ..., (\delta^T)') \in \mathfrak{D}^T(\delta)$, there is a fiscal policy ϕ that implements the allocation x compatible with g, but with the resulting first T deficits $d(T) = (d^1, ..., d^T) \in \mathbb{R}^T$ given by $d(T) = \delta'(T)$ (while $d^{T+1}, d^{T+2}, ...$ are not constrained).

Note that the time horizon of the deficit specification is arbitrary and hence includes every positive integer.

The rest of the paper is devoted to obtaining economic conditions under which irrelevance or weak irrelevance holds.

5. IRRELEVANCE OF DEFICIT RESTRICTIONS WITH LUMP-SUM TAXES

For some overlapping-generations economies with a single (representative) consumer per generation, perfect borrowing and lending markets for consumers, and a full range of lump-sum taxes and transfers, restrictions on the government budget deficit have no impact on the set of equilibrium allocations. The reason for this irrelevance is that in these economies only the present value of taxes and transfers, not their timing, matters to consumers. In this case, the government can "borrow" freely from taxpayers by adjusting the timing of individual taxes and transfers. See, for example, Auerbach and Kotlikoff [2] and especially Kelly [17].[9]

The following proposition extends the irrelevance analysis to economies with several (heterogeneous) consumers per generation, several commodities

[9] In Barro [9] the argument is similar but simpler. In infinite-lived-agent models with sufficiently high interest rates, one only has to remark that each individual's wealth depends only on the present value of his taxes, not on the timing of these taxes. Barro clearly recognizes the potential problems that distortionary taxes cause for the irrelevance theorems.

per period, (distortionary) consumption taxes and transfers in addition to (non-distortionary) lump-sum taxes and transfers, and restrictions on taxes to be "measurable" with respect to the consumer tax classes and with respect to the commodity tax classes.

PROPOSITION 5 (Irrelevance of Deficit Restrictions When Lump-Sum Taxes and Transfers Are Available). *Suppose that lump-sum taxes and transfers are available. Then the deficit-restriction sequence $d=\delta$ is irrelevant.*

Proof. We consider the least favorable case, the case in which taxes and transfers must be made completely anonymously within a given generation, i.e., the case of $N=1$. For simplicity, we assume $g^t=0$ for $t=1, 2, \ldots$.

The demand function f_{th} for consumer h of generation t $(t=1, 2, \ldots)$ maps present after-tax consumer prices $(q^t_{th}, q^{t+1}_{th}) \in \mathbb{R}^{2\ell}_{++}$ and present wealth $w_{th}=p^t \cdot \omega^t_{th}+p^{t+1} \cdot \omega^{t+1}_{th}+m^t_{th}+m^{t+1}_{th} \in \mathbb{R}_{++}$ into consumption $(x^t_{th}, x^{t+1}_{th}) \in \mathbb{R}^{2\ell}_{++}$. In the case of generation 0 the demand function f_{0h} maps consumer prices $q^1_{0h} \in \mathbb{R}^{\ell}_{++}$ and income $w_{0h}=p^1 \cdot \omega^1_{0h}+m^1_{0h} \in \mathbb{R}_{++}$ into consumption $x^1_{0h} \in \mathbb{R}^{\ell}_{++}$.

Therefore, f_{th} depends on m_{th} solely through the lifetime sum or present value of transfers $\mu_{th}=m^t_{th}+m^{t+1}_{th} \in \mathbb{R}$ for consumer $t=1, 2, \ldots,$ and $\mu_{0h}=m^1_{0h} \in \mathbb{R}$ for consumer 0. We claim that the deficit restriction δ is irrelevant because the sequence $\mu=(\mu_{0h})^{h=n}_{h=1}, (\mu_{1h})^{h=n}_{h=1}, \ldots, (\mu_{th})^{h=n}_{h=1}, \ldots = (m^1_{0h})^{h=n}_{h=1}, (m^1_{1h}+m^2_{1h})^{h=n}_{h=1}, \ldots, (m^t_{th}+m^{t+1}_{th})^{h=n}_{h=1}, \ldots$ of lifetime transfers is compatible with any deficit restrictions $\delta'=(\delta^1)', \ldots, (\delta^t)', \ldots$.

To establish this claim, consider the constitutional competitive equilibrium with consumer allocation x and government allocation g implemented by the fiscal policy $\phi=(m, \tau)$ with deficit $d=\delta=\delta^1, \ldots, \delta^t, \ldots$.

Consider the alternative deficit restriction $d=\delta'=(\delta^1)', \ldots, (\delta^t)', \ldots$. Construct the fiscal policy $\phi'=(m, \tau)$ defined by

$$\begin{aligned}
(\tau^t_{t-1,h})' &= \tau^t_{t-1,h}, \\
(\tau^t_{th})' &= \tau^t_{th}, \\
(m^1_{0h})' &= m^1_{0h}, \\
(m^t_{th})' &= (\delta^t)'/n-(m^t_{t-1h})',
\end{aligned}$$

and

$$(m^{t+1}_{th})' = \mu_t - (m^t_{th})'$$

for $t=1, 2, \ldots$ and $h=1, \ldots, n$. The tax policy ϕ is anonymous and it implements the allocation x while meeting the sequence of deficit restrictions δ'. ∎

From the proof of Proposition 5, it is easy to see—and this is not surprising—that the irrelevance result still holds when consumption taxes are not included in the set of feasible tax instruments. We also conclude that if lump-sum taxes and transfers are included in the set of feasible instruments, one lump-sum tax instrument per period is sufficient to make the budget restrictions irrelevant.

In most of what follows, we suppose that lump-sum taxes and transfers are not included in the set of tax instruments.

6. CONSUMPTION TAXES

For this section, we assume that only taxes on consumption are available, i.e., the set of feasible tax policies is $\mathscr{T}$. When lump-sum taxes and transfers are available, a balanced budget amendment can be "avoided" by the government at no cost in the sense that at equilibrium all individual and government consumptions remain unchanged. The question is then whether the government is still able to "avoid" the constitutional restriction on deficits even though only distortionary taxation is available. The answer will depend on the number of instruments (taxes) compared to the number of goals (consumers) and on the "duration"—finite or infinite—of the constitutional deficit restrictions.

We start with an example showing that irrelevance may fail because of the shortage of tax instruments. We suppose that before the reform the economy is in a steady state and that government consumption is positive. In the absence of taxes, a non-zero deficit is the only possible outcome of such a policy. Suppose now that a reform is adopted and that it specifies the budget deficit sequence to be zero in all subsequent periods. The question is then whether is it possible for the government to implement the same allocation while running a balanced budget. Note that in this example we focus on stationary equilibria. This is equivalent to assuming that in the first period a suitable transfer is made so that the economy actually "starts" at the steady state and that the deficit specification starts at $t=2$ and goes onward[10].

Example 6 (Relevance of Deficit Restrictions Due to the Scarcity of Tax Instruments). Consider a stationary overlapping-generations economy with one commodity per period ($\ell=1$) and two consumers per generation ($n=2$). Government consumption is assumed to be constant, $g^t=\gamma\in\mathbb{R}_+$

[10] Another equivalent way to view steady states is to consider a model with no beginning as well as no end.

for $t = 1, 2, \ldots$. The two consumers, Mr. $t1$ and Mr. $t2$, have respectively the log-linear utility functions and endowments given by

$$u_{t1}(x_{t1}^{t}, x_{t1}^{t+1}) = (1/3)\log x_{t1}^{t} + (2/3)\log x_{t1}^{t+1},$$
$$\omega_{t1} = (\omega_{t1}^{t}, \omega_{t1}^{t+1}) = (1, 1),$$
$$u_{t2}(x_{t2}^{t}, x_{t2}^{t+1}) = (1/2)\log x_{t2}^{t} + (1/2)\log x_{t2}^{t+1}, \tag{6.1}$$

and

$$\omega_{t2} = (\omega_{t2}^{t}, \omega_{t2}^{t+1}) = (3, 1).$$

For period t, we have from (6.1) the system of demand functions

$$x_{t-1,1}^{t} = \frac{2(p^{t-1} + p^{t})}{3q_{t-1,1}^{t}},$$
$$x_{t1}^{t} = \frac{p^{t} + p^{t+1}}{3q_{t1}^{t}},$$
$$x_{t-1,2}^{t} = \frac{3p^{t-1} + p^{t}}{2q_{t-1,2}^{t}}, \tag{6.2}$$

and

$$x_{t2}^{t} = \frac{3p^{t} + p^{t+1}}{2q_{t2}^{t}},$$

where the p's are before-tax market prices and the q's are after-tax personalized prices. We assume that taxes must be completely anonymous within a given generation, i.e., we have $N = 1$. There is only one commodity per period, i.e., we have $L = \ell = 1$. Because of tax anonymity, we have

$$q_{t-1,1}^{t} = q_{t-1,2}^{t} = q_{t-1}^{t},$$
$$\tau_{t-1,1}^{t} = \tau_{t-1,2}^{t} = \tau_{t-1}^{t},$$
$$q_{t1}^{t} = q_{t2}^{t} = q_{t}^{t}, \tag{6.3}$$

and

$$\tau_{t1}^{t} = \tau_{t2}^{t} = \tau_{t}^{t}.$$

We look at a steady-state competitive equilibrium of the form $p^{t} = \beta^{t}$. The situation prior to reform is one in which taxes are zero (except the initial tax transfer to the old consumer of generation 0). In this case, the steady state is given by

$$\gamma+\frac{2(1+\beta)}{3\beta}+\frac{3+\beta}{2\beta}+\frac{\beta+\beta^2}{3\beta}+\frac{3\beta+\beta^2}{2\beta}=6.$$

The relevant equation is

$$(\beta-1)(5\beta-13)=-6\beta\gamma.$$

This equation admits two solutions as long as γ is not too large, i.e., if we have $\gamma \leqslant 0.31$. To simplify the computations, let $\gamma=\frac{1}{6}$. Then the solutions are $\beta=1.16$ and $\beta=2.24$. The steady-state allocations associated with $\beta=1.16$ are given by

$$\hat{x}_1=(x_{t1}^t, x_{t1}^{t+1})=(0.72, 1.24)$$

and (6.4)

$$\hat{x}_2=(x_{t2,}^t, x_{t2}^{t+1})=(2.08, 1.79).$$

The *current* value of the deficit is $d^t/\beta^t=1/6$.

Is it possible to use anonymous consumption taxes (τ_{t-1}^t, τ_t^t) to meet the deficit requirement $(\delta^t)'=0$ in period t without disturbing the allocations (6.4) and the government consumption? Such a tax scheme must satisfy for each t $(t=2, 3, ...)$ the equations

$$\begin{aligned}\frac{2(p^{t-1}+p^t)}{3(p^t+\tau_{t-1}^t)}&=1.24,\\ \frac{p^t+p^{t+1}}{3(p^t+\tau_t^t)}&=0.72,\\ \frac{3p^{t-1}+p^t}{2(p^t+\tau_{t-1}^t)}&=1.79,\\ \frac{3p^t+p^{t+1}}{2(p^t+\tau_t^t)}&=2.08,\end{aligned} \tag{6.5}$$

and

$$\frac{1}{6}-(3.03)\,\tau_{t-1}^t-(2.80)\,\tau_t^t=0.$$

The only possible solution to the first four equations of (6.5) is of the form $p^t=(\beta)^t\,p^1=(\beta)^t$, $\tau_t^t=(\beta)^t\,\tau^0$, and $\tau_{t-1}^t=(\beta)^t\,\tau^1$, where $\beta\in\mathbb{R}_{++}$ is the interest factor, $\tau^0 \in \mathbb{R}$ is the current value of the tax rate on the young and $\tau^1 \in \mathbb{R}$ is the current value of the tax rate on the old. The second and the fourth equations admit as a unique solution $\tau^0 = 0$ and $\beta = 1.16$. The other two equations yield $\tau^1 = 0$. Then, the last equation in (6.5) cannot be fulfilled. Therefore the government cannot implement the zero-interest-rate steady state allocation as a constitutional equilibrium with $(\delta^t)' = 0$ for $t = 2, 3\ldots$.

Example 6 indicates that budget irrelevance can fail in economies *both* (1) with only distortionary taxes *and* (2) without the power to completely individualize tax rates. In this example, there is only one commodity per period, $\ell = L = 1$. There are two consumers per period, $n = 2$, but there is only one consumer tax class per period, $N = 1$.

The next proposition provides a necessary condition for generic irrelevance of restrictions on the government budget deficits. Roughly speaking, the condition is that the number of instruments exceed the number of goals. This proposition and those that follow it hold only generically —i.e., for an open and dense set of economies. In this way, degenerate cases—principally those in which individual endowments are co-linear—are excluded.

PROPOSITION 7 (A Necessary Condition for Irrelevance of Deficit Restrictions). *Let x be an equilibrium allocation that can be implemented with a fiscal policy $\tau \in \mathcal{T}$ compatible with government deficits δ and government consumption g. Then if we have*

$$(2\ell - 1)\, N + n + 1 > 2LN + \ell,$$

the deficit restriction δ is not weakly irrelevant.

Remark 8. Suppose that the government is unconstrained by commodity tax classes; i.e., we have $L = \ell$. Then Proposition 7 says that generically the deficit restriction matters if the inequality $\ell - 1 < n - N$ holds. In the case of a single consumer tax class, $N = 1$, this reduces to the simple condition that the number of commodities be smaller than the number of consumers, i.e., $\ell < n$ is satisfied.

Proof of Proposition 7. To simplify calculations, we assume that $g^t = 0$ for $t = 1, 2, \ldots$. The demand function f_{th} of consumer h from generation t ($t = 1, 2, \ldots$) is homogenous of degree zero in the after-tax prices $q_{th} = (q^t_{th}, q^{t+1}_{th})$ and his wealth $w_{th} = p^t \cdot \omega^t_t + p^{t+1} \cdot \omega^{t+1}_t$. Renormalize the arguments of the demand function f_{th} by dividing by q^{t1}_{th}. The demand function f_{th} then defines a diffeomorphism between the consumption set and $\mathbb{R}^{\ell-1}_{++} \times \mathbb{R}^{\ell}_{++} \times \mathbb{R}_{++}$. Let $Q_{th} = (Q^t_{th}, Q^{t+1}_{th}) \in \mathbb{R}^{\ell-1}_{++} \times \mathbb{R}^{\ell}_{++}$ and $W_{th} \in \mathbb{R}_{++}$

be the value of the renormalized arguments of f_{th} associated with the constitutional equilibrium x that can be implemented with the budget deficit sequence $\delta = (\delta^1, ..., \delta^t, ...)$. The demands f_{th} remain constant as δ is changed to $\delta' = (\delta^1)', ..., (\delta^t)', ...$ if and only if all the renormalized arguments of f_{th} are unaffected. For each consumer in generation 1 or later, there are $(2\ell - 1)$ conditions coming from the price ratios and one condition coming from his budget constraint. The constitutional restriction on the government budget deficit adds the restriction on government revenue,

$$\sum_{h=1}^{h=n} (\tau^t_{t-1,h} \cdot f^t_{t-1,h} + \tau^t_{th} \cdot f^t_{th}) = -\delta^t.$$

The relevant system of equations for period t is then,

$$\begin{aligned} \frac{1}{q^{t1}_{th}}(\hat{p}^t + \hat{\tau}^t_{th}) &= Q^t_{th} \qquad \text{for} \quad h = 1, ..., n, \\ \frac{1}{q^{t1}_{th}}(p^{t+1} + \tau^{t+1}_{th}) &= Q^{t+1}_{th} \qquad \text{for} \quad h = 1, ..., n, \\ \frac{1}{q^{t1}_{th}}(p^t \cdot \omega^t_{th} + p^{t+1} \cdot \omega^{t+1}_{th}) &= W_{th} \qquad \text{for} \quad h = 1, ..., n, \end{aligned} \tag{6.6}$$

and

$$\sum_{h=1}^{h=n} \sum_{i=1}^{\ell} [\tau^{ti}_{th} f^{ti}_{th}(Q^t_{th}, Q^{t+1}_{th}, W_{th}) + \tau^{ti}_{t-1,h} f^{ti}_{t-1,h}(Q^{t-1}_{t-1,h}, Q^t_{t-1,h}, W_{t-1,h})] = -\delta^t,$$

where the right hand sides (Q^t_{th}, Q^{t-1}_{th}, W_{th}, and δ^t) are fixed and $\hat{p}^t \in \mathbb{R}^{\ell-1}_{++}$ and $\hat{\tau}^t_{th} \in \mathbb{R}^{\ell-1}$ are respectively the vectors p^t and τ^t_{th} without the first coordinate ($p^{t1} \in \mathbb{R}_{++}$ or $\tau^{t1}_{th} \in \mathbb{R}$). The first two lines in system (6.6) provide $(2\ell - 1)\, n$ restrictions on the prices, but because of the limited potency of the government only $(2\ell - 1)\, N$ of these restrictions are independent. Assume for the moment that the p^t and $\tau^t_{t-1,h}$ $(h = 1, ..., n)$ are predetermined. (We will justify this assumption when we consider consumer 0.) Multiplication of (6.6) by $q^{t1}_{th} \in \mathbb{R}_{++}$ creates a linear system of equations in $2LN + \ell$ unknowns, i.e., $2LN$ independent tax rates, $(\tau^t_{th}, \tau^{t+1}_{th})$, and ℓ prices.

Comparing the number of equations and unknowns suggests that a necessary condition for there to be a solution to the system (6.6) is that we have

$$(2\ell - 1)\, N + n + 1 \leqslant 2LN + \ell.$$

Consumers from generation 0 are considered next. The individual demand $f_{0h} \in \mathbb{R}^{\ell}_{++}$ of consumer h of generation 0 has the form $f^1_{0h}(q^1_{0h}, w_{0h})$ with $w_{0h} = p^1 \cdot \omega^1_{0h} + m^1_{0h}$. Note that the pure monetary term m^1_{0h} is added for generality, the result would be the same with $m^1_{0h} = 0$ for all h. From homogeneity and the diffeomorphism property, the relevant system of equations for such a consumer is

$$\hat{p}^1 + \hat{\tau}^1_{0h} = (p^{11} + \tau^{11}_{0h})\, R^1_{0h}$$

and

$$p^1 \cdot \omega^1_{0h} + m^1_{0h} = (p^{11} + \tau^{11}_{0h})\, W_{0h}$$

for $h = 1, ..., n$ where $R^1_{0h} \in \mathbb{R}^{\ell-1}_{++}$ and $W_{0h} \in \mathbb{R}_{++}$. For each consumer in generation 0, there are $\ell - 1$ equations coming from the prices and 1 equation coming from income. Ignoring the redundant equations due to the limited potency of the government, the system consists then of $(\ell - 1)\, N + n$ equations (the constraint on the government budget is not included at this stage). On the other hand, there are LN taxes and ℓ prices. However, for consumer 1 there is a further equation due to the normalization $q^{11}_{01} = p^{11} + \tau^{11}_{01} = 1$ (of course $p^m = 1$ is also fixed). Then only $\ell - 1$ prices are free. The necessary "counting condition" for existence of a solution is then $(\ell - 1)\, N + n \leqslant LN + \ell - 1$. From $(2\ell - 1)\, N + n + 1 \leqslant 2LN + \ell$ and $L \leqslant \ell$, it follows that $(\ell - 1)\, N + n \leqslant LN + \ell - 1$ and hence the necessary condition for the theorem is fulfilled. (Note that if $L = \ell$ both conditions coincide.)

Of course, the full argument must include the ranks of the relevant matrices. In the appendix, we show that provided the endowments of the individual consumers are not colinear, the appropriate rank conditions are fulfilled. ▮

Remark 9. Some authors (see, e.g., Kotlikoff [18]) either explicitly or implicitly adopt a less stringent notion of the constitutional restrictions on budget deficits. For these authors, a restriction is placed on the deficit in each period except the first period (giving the government "one last chance" at unfettered deficit financing). As can readily be seen from the above proof, even in this weaker version, the government budget deficit restrictions can be relevant.

An important fact has to be pointed out at this point. Even under the conditions specified in the above proposition, it is not assured that $q^{si}_{th} - \tau^{si}_{th}$ is nonnegative, i.e., we could have for some s $(s = 1, 2, ...)$ and some i $(i = 1, ..., \ell)$ that $p^{si} < 0$. This would be consistent with the formal mathematical equations presented above, but it is, of course, inconsistent with free disposal of endowments. Hence, if we admit free disposal, the conditions in Proposition 7 are not sufficient for irrelevance of restrictions on the government budget deficit. The next two examples show that having "enough" tax instruments may or may not be sufficient to obtain strong irrelevance in the sense of Definition 3.

The following example is of a case in which the deficit restriction does not matter.

EXAMPLE 10 (Irrelevant Deficit Restriction). Consider an overlapping-generations economy with one commodity per period and one consumer per generation after period 1, i.e., $\ell = L = 1$ and $n = N = 1$. Consumption of the commodity is taxed. The consumer is of the same type as consumer 1 in Example 6. Government consumption is $g^t = \gamma$ for $t = 1, 2, \ldots$. Consider the steady-state sequence of before-tax prices $p = 1, \beta, (\beta)^2, \ldots, (\beta)^{t-1}, \ldots,$ where $p^t = (\beta)^{t-1}$ for $t = 2, 3, \ldots$ for $\beta \in \mathbb{R}$. If markets clear, we must have

$$\gamma + \frac{2(\beta^{t-1} + \beta^t)}{3(\beta^t + \tau^t_{t-1})} + \frac{\beta^t + \beta^{t+1}}{3(\beta^t + \tau^t_t)} = 2.$$

In the absence of taxes and for γ sufficiently small, there are two solutions for β. If $\gamma = 0.04$, then the two values are $\beta = 1.71$ and $\beta = 1.17$. The steady-state consumer allocations associated with $\beta = 1.17$ are $(0.72, 1.24)$. The government budget deficit, in current value terms, is equal to the current expenditures, i.e., $(d^t/\beta^t) = \gamma = 0.04$.

First, we consider the case in which a zero budget deficit is required in every period, i.e., $d^t = (\delta^t)' = 0$ for $t = 2, 3, \ldots$. (This means that—for the time being—the transition from the no-tax situation to the tax situation is ignored.) Let $\tau^t_{t-1} = (\beta)^t \tau^1$ and $\tau^t_t = (\beta)^t \tau^0$. Then the equations for the first "monetary" steady-state are

$$\frac{2(1+\beta)}{3\beta(1+\tau^1)} = 1.24,$$

$$\frac{1+\beta}{3(1+\tau^0)} = 0.72,$$

and

$$0.04 - 1.24\tau^1 - 0.72\tau^0 = 0.$$

The two solutions to this set of equations are $\tau^0 = (0.38, -0.08)$. These solutions give $\beta = 2.19\tau^0 + 1.19 = 2.00$ and 1.00, respectively. The budget equation $0.04 - 1.24\tau^1 - 0.72\tau^0 = 0$ gives the corresponding taxes $\tau^1 = -0.19$ and 0.08. Note that all prices are positive.

The above example shows that it is indeed possible to implement the original allocations with zero budget deficits. However, one of the features of the above example is that it only deals with a stationary budget policy and does not treat the transition to the steady state. The next example allows for a *non-stationary* sequence of government deficit restrictions. This example also shows that the necessary condition in Proposition 7 is not sufficient. It also prepares us for the difficulties caused by constitutional deficit restrictions that last forever.

EXAMPLE 11 (Relevance of Deficit Restrictions Due to Exploding Taxes). Consider the economy of Example 10. For simplicity we suppose $g^t = 0$ for $t = 1, 2, \ldots$ so that the steady-state is (2/3, 4/3). The situation prior to reform is one in which taxes and transfers are zero except the initial transfer to the old consumer of generation 0 (either as a lump-sum transfer or as a transfer proportional to consumption). The associated budget deficit is then $\delta_0^1 = 1/3$ while $\delta^t = 0$ for $t = 2, 3 \ldots$. Suppose also that at time 1 a further budget restriction is imposed on the government $\delta^1 = \delta_0^1 + \underline{\delta}$, $\underline{\delta} \neq 0$, for period 1 while $\delta^t = 0$ for $t = 2, 3 \ldots$. The government wishes to change its fiscal policy without disturbing the steady-state allocation. It will begin by taxing the consumer of generation 1 at the rate τ_1^1, while leaving the old consumer untaxed. In order to fulfill the budget restriction, the tax should be such that $\frac{2}{3}\tau_1^1 = -\underline{\delta}$. Then $\tau_1^1 = -\frac{3}{2}\underline{\delta}$. The present price of second-period endowment, p^2, is obtained using the demand of the young of generation 1; hence we have $p^2 = 1 - 3\underline{\delta}$. The old age demand yields $\tau_1^2 = \frac{3}{2}\underline{\delta}$. In period 2, the government has to fulfill $d^2 = \delta^2 = 0$. Therefore we have $(2/3)\,\tau_2^2 + 4/3[(^3/_2)\,\underline{\delta}] = 0$ from which we get $\tau_2^2 = -3\underline{\delta}$ and $p^3 = p^2 + 2\tau_2^2 = 1 - 9\underline{\delta}$. The same procedure can be repeated in general for $t = 2, 3, \ldots$ to yield

$$p^{t+1} = p^t + 2\tau_t^t,$$

$$2\tau_t^{t+1} + \tau_{t+1}^{t+1} = 0,$$

and

$$\tau_t^{t+1} = -\tau_t^t.$$

The first terms in the sequence of prices are $p^4 = 1 - 21\underline{\delta}$, $p^5 = 1 - 45\underline{\delta}$ and $p^6 = 1 - 93\underline{\delta}$. The sequence is in fact defined by $p^{t+1} = p^t - 3 \cdot 2^{t-1}\underline{\delta}$. It is easily seen that for any $\underline{\delta} > 0$ the condition on the positivity of prices is violated after a finite number of periods. Therefore the tax sequence described is infeasible. On the other hand, restrictions to a positive government surplus are always feasible to implement in this example.

With lump-sum taxation, exploding taxes can also occur, but in this case they are not a cause of any technical problems. For example, taxes on the young could become very large while transfers to the old are becoming large but in such a way that lifetime taxation is stable. Hence incomes would be stable and therefore there would be no price infeasibilities in this (lump-sum) case. On the other hand, if consumption taxes explode, then the sign restrictions on the p's and q's will be violated.

The above two examples illustrate the fact that the set of policies that give the same allocation as some given policy is usually not empty, but that unfortunately this set is not easy to characterize—even as a neighborhood of the original policy. There may not exist an open neighborhood in the sup-norm topology of the original sequence of deficits such that all sequences in that neighborhood are feasible and support the original allocation.

One way out is to weaken the notion of deficit restriction to one that applies only to a finite number of periods. For instance, in Example 11, if the set of budget restrictions is of finite length, there is a neighborhood of the original deficits for which we have generic deficit irrelevance.

We adopt the notion of weak irrelevance given in Definition 4.

PROPOSITION 12 (A Sufficient Condition for Weak Irrelevance). *Let x be an equilibrium consumer allocation with a government allocation g that can be implemented with a feasible fiscal policy τ compatible with government deficits $d = \delta$. If we have*

$$(2\ell - 1)\,N + n + 1 \leqslant 2LN + \ell,$$

then the government deficit restriction δ is weakly irrelevant.

Proof. The rank conditions for the relevant matrices used in the proof of Proposition 7 and established in the appendix show that the set of solutions is nonempty because of the assumption $(2l - 1)\,N + n + 1 \leqslant 2LN + l$. ▮

Example 11 suggests that at least for some cases (e.g., "stationary environments"), our definition of weak irrelevance might be strengthened. Perhaps for some environments, the deficit restrictions could be binding for all but a finite number of periods—or even, all but the first period. Furthermore, if the only restrictions were to balanced budgets (i.e., $\delta' = 0$), then some strengthening of the definition might also be possible. We have not fully investigated these issues.

To conclude this section: We have shown that, when the only instruments are consumption taxes, an exogenously given sequence of government budget restrictions can be fulfilled without changing the equilibrium allocation, but that this requires that there be a sufficiently rich spectrum of different tax instruments. Furthermore, the necessary condition for irrelevance cannot be turned into a global sufficiency result. For sufficiency, the post-reform restrictions must be close to the baseline deficits and the post-reform restrictions can only bind for a finite (although arbitrarily long) duration.

The effective impact of deficit restrictions on social welfare depends on the precise specification of the welfare function. However, one expects that the optimum social welfare based on individual utilities will generally be reduced by budget deficit restrictions unless there is irrelevance. That is, deficit restrictions are likely to have welfare costs in the cases where they are relevant.

7. CONCLUDING REMARKS

Politicians and bureaucrats seem to be convinced that constitutional and other restrictions on the government's budget deficits will matter a lot —perhaps for good, perhaps for ill. On the other hand, following Barro [9], the existing theoretical literature suggests that these restrictions are irrelevant in the sense that the governments can perfectly avoid them. If there are nondistortionary taxes, then deficit restrictions can be completely avoided even if these taxes are completely or partially anonymous. This is because non-distortionary taxes affect prices and allocations solely through their effects on lifetime incomes. The government avoids the constitutional restrictions by in effect "borrowing" from the young.

We go beyond simple models with only non-distortionary taxation or only one consumer per generation or only one commodity per period. Consumption and other distortionary taxes affect allocations and incomes through their effects on prices. The requirement that before-tax prices be non-negative and after-tax prices be positive makes avoiding deficit restrictions more difficult for the government. Global changes in budget restrictions are for this reason likely to be binding. Hence, we introduce a weaker (but economically meaningful) notion of irrelevance. In this weaker notion, only restrictions close to the base-line budget deficit sequence are considered. "Closeness" involves only finite-horizon comparisons, but arbitrarily long horizons are allowed. We show that weak irrelevance holds—i.e., the government can avoid all "nearby" finite horizon deficit restrictions—if the number of tax instruments is large relative to the number of tax goals. At first glance, it might seem to be unrealistic to assume that the number of instruments exceeds the number of goals, but it should be noted that the government has incentives to expand the available set of instruments. The government might also be able to focus only on a few aggregate goals.

There is a sense in which our definitions of irrelevance are too strong. For irrelevance, we require the government to be able to reproduce some given allocation of goods. In principle, it would be less restrictive to require the government to be able to reproduce the given allocation of utilities or even the given social welfare. We are unable to say how such considerations would affect our formulas except to say that they cannot make irrelevance less likely.

We have assumed throughout this paper that capital markets (borrowing and lending markets) are perfect for the individual consumers, even though restrictions are placed on government finance. In a parallel study, we are analysing a model in which some consumers face borrowing restrictions. In this realistic and important case, irrelevance of deficit restrictions seems more likely with consumption taxes than with lump-sum taxes, because commodity taxes can help the government in providing liquidity to some particular member of a consumer class.

It is important to expand the model to include production. If the p's were fixed by efficient production through a perfectly "smooth" technology, then deficit irrelevance would be impossible. The actual fact is that there is not likely to be complete "smoothness" of production. As Diamond and Mirrlees [11] point out, some endowments are consumed directly while other goods are consumed and produced but are not included in consumer endowments. Furthermore, efficiency of production is unlikely in this second-best world. Taxation of intermediate goods and intentionally inefficient government production are possibilities. Hence there are several unexplored policy tools on the production side. The issues associated with production merit further study.

The assumption of perfect commitment by the government is very strong. It assumes away some crucial aspects of the problem. Suppose, for example, that the government taxes some individual very heavily in his youth "in exchange for" large subsidies in his old age. If the government's "promise" is not perfectly credible, the individual may be unable to borrow in the private capital market against the future subsidy. The individual might then seek an IOU from the government. If the government complies, it has moved from deficit-restriction *avoidance* to deficit-restriction *evasion* since the private IOU should be counted as part of the government's current deficit. The present paper is about deficit-restriction avoidance, but deficit-restriction *evasion* is at least half of the problem. As Kotlikoff and others point out, deficit-restriction evasion is very easy for the government. Often, governments merely "redefine" taxes and expenditures to evade the constitutional restrictions.

There is, however, another quite subtle, but very important sense in which our definition of irrelevance is inappropriate. In this paper and virtually all others on the subject, we look at all competitive equilibria with no restrictions on individual expectations except that they satisfy perfect foresight. There are typically at least a continuum of "non-monetary" (i.e., $p^m = 0$) perfect-foresight equilibria in this overlapping-generations economy. If ϕ is bonafide, then there are also many more "monetary" (i.e., $p^m > 0$) perfect-foresight equilibria. Typically there are many, many proper sunspot equilibria. In our irrelevance analysis, we implicitly assume that the government is able to "select" from the plethora of possible equilibria the one or ones most suitable to its goals. If instead, we took the expectations of the individual consumers as *given parameters* of the economy, our results would be quite different. In this case, irrelevance would seem to be more difficult to achieve. For example: What if the initial value of money is believed to be positive only if the expected sequence of government budgets is in perfect balance? Or, what if individuals believe that the equilibrium is nonsunspot if strict balanced-budget restrictions are expected, while the equilibrium is otherwise affected by sunspots? In neither of these (very special) cases could balanced-budget restrictions ever be irrelevant. One approach to the "selection" problem would be to focus on more fully specified (although rational) beliefs in order to close the model. Some examples of such beliefs specification in OG macro modelling appear in Shell [24]. Another approach would be to consider *all* rational-expectations equilibria while placing some emphasis on the "worst" selections. This would be in the spirit of the "fragility" literature. See, for example, Grandmont [14], Woodford [27], Smith [25], Goenka [13], and Keister [16]. See also the approaches to "fragility" taken in the bank-runs literature, especially Peck and Shell [20].

APPENDIX: RANK COMPUTATIONS

Our analysis is for the economy with only consumption taxes. In order to obtain the relevant rank conditions, we will consider the two polar cases consisting of completely anonymous taxes, $N = 1$ and $L = l$, and of completely individualized taxes, $N = n$ and $L = l$. The extension to the general case is then straightforward.

1. *Anonymous Consumption Taxes.* Here we suppose that the government cannot discriminate among the consumers in a given generation, but it is able to use a different tax rate for each commodity, i.e., $N = 1$ and $L = l$.

The linear system based on the behavior of the n consumers from generation 0 can be written as $A_1 z_1 = b_1$ or

$$\begin{bmatrix} I_{l-1} & 0 & I_{l-1} \\ & -\omega_{01}^{11} & 0 \\ \hat{\omega}_0^1 & \vdots & \vdots \\ & -\omega_{0n}^{11} & 0 \end{bmatrix}_{l+n-1 \times 2l-1} \begin{bmatrix} \hat{p}^1 \\ \tau_0^{11} \\ \hat{\tau}_0^1 \end{bmatrix}_{2l-1 \times 1} = \begin{bmatrix} R_0^1 \\ w_1 - \omega_{01}^{11} - m_{01}^1 \\ \vdots \\ w_n - \omega_{0n}^{11} - m_{0n}^1 \end{bmatrix}_{l+n-1 \times 1}$$

with

$$\hat{\omega}_0^1 = \begin{bmatrix} \omega_{01}^{1,2} & \omega_{01}^{1,3} & \cdots & \omega_{01}^{1,l} \\ \omega_{02}^{1,2} & \omega_{02}^{1,3} & \cdots & \omega_{02}^{1,l} \\ \vdots & \vdots & \vdots & \vdots \\ \omega_{0n}^{1,2} & \omega_{0n}^{1,3} & \cdots & \omega_{0n}^{1,l} \end{bmatrix}_{n \times l-1}.$$

Note that the normalization $q_{01}^{11} = p^{11} + \tau_{01}^{11} = 1$ allows us to express p^{11} as a function of τ_{01}^{11} and therefore exclude p^{11} from the set of unknowns. The rank of the matrix A_1 is equal to the rank of the matrix

$$\begin{bmatrix} & -\omega_{01}^{1,l} \\ \hat{\omega}_0^1 & \vdots \\ & -\omega_{0n}^{1,l} \end{bmatrix}_{n \times l}$$

plus $l-1$. For generical economies the above matrix has maximal rank. Therefore, the matrix A has rank $l-1+\min(n, l)$ while there are $2l-1$ unknowns. For $n \geqslant l$ there are $2l-1$ equations, therefore a solution always exists although the dimension of the solution set is zero. The unknowns $\hat{p}^1$ and τ_0^1 are determined, independently of any budget specification.

Consider now the consumers of generation $t = 1, 2, ...,$ with the constraint that the price in period t is already fixed. In matrix form, the relevant system of equations that include the n budget restrictions can be written as $A_t z_t = b_t$ or

$$\begin{bmatrix} 0 & -Q_t^t & I_{l-1} & 0 \\ I_l & -Q_t^{t+1} & 0 & I_l \\ \omega_t^{t+1} & -W_t \cdot J_n & 0 & 0 \\ 0 & \sum_{h=1}^n f_{th}^{t1} & \sum_{h=1}^n \hat{f}_{th}^t & 0 \end{bmatrix} \begin{bmatrix} p^{t+1} \\ \tau_t^{t1} \\ \hat{\tau}_t^t \\ \tau_t^{t+1} \end{bmatrix}$$

$$= \begin{bmatrix} -\hat{p}^t + p^{t1} Q_1^t \\ p^{t1} Q_t^{t+1} \\ -\omega_t^t \cdot p^t + p^{t1} W_t \cdot J_n \\ -\delta^t - \sum_{h=1}^n \sum_{i=1}^l \tau_{t-1}^{ti} f_{t-1,h}^{ti} \end{bmatrix}$$

where $Q_t^t \in \mathbb{R}^{l-1}$, $Q_t^{t+1} \in \mathbb{R}^l$, $\hat{\tau}_t^t \in \mathbb{R}^{l-1}$and $\tau_t^{t+1} \in \mathbb{R}^l$ and the matrices ω_t^t, ω_t^{t+1} and W_t, and the vectors J_n are defined by

$$J_n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}, \qquad \omega_t^s = \begin{bmatrix} \omega_{t1}^{s,1}, & \omega_{t1}^{s,2} & , ..., & \omega_{t1}^{s,l} \\ \omega_{t2}^{s,1}, & \omega_{t2}^{s,2} & , ..., & \omega_{t2}^{s,l} \\ \vdots & \vdots & \vdots & \vdots \\ \omega_{tn}^{s,1}, & \omega_{tn}^{s,2} & , ..., & \omega_{tn}^{s,l} \end{bmatrix}_{n \times l},$$

and

$$W_t = \begin{bmatrix} w_{t1} & 0 & \cdots & 0 \\ 0 & w_{t2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & w_{tn} \end{bmatrix}_{n \times n}.$$

The notation for the individual demands is given by

$$f_t^{t1} = (f_{th}^{t1}(Q_t^t, Q_t^{t+1}, w_{th}))_{h=1}^n,$$
$$\hat{f}_t^t = (f_{th}^{ti}(Q_t^t, Q_t^{t+1}, w_{th}))_{h=1,\ldots,n}^{i=2,\ldots,l},$$

and

$$f_t^{t+1,i} = (f_{th}^{t+1}(Q_t^t, Q_t^{t+1}, w_{th}))_{h=1,\ldots,n}^{i=1,\ldots,l}.$$

Similar notation is used for taxes τ_t^{t1}, $\hat{\tau}_t^t$, and τ_t^{t+1}.

The rank of the matrix A_t is equal to the rank of the matrix

$$\begin{bmatrix} 0 & -Q_t^t & I_{l-1} \\ \omega_t^{t+1} & -W_t \cdot J_n & 0 \\ 0 & \sum_{h=1}^n f_{th}^{t1} & \sum_{h=1}^n \hat{f}_{th}^t \end{bmatrix}$$

plus l. By a sequence of manipulations involving the first $l-1$ rows of this last matrix, the rank of the matrix A_t is seen to be equal to the rank of the $(n+1) \times (l+1)$ matrix

$$\begin{bmatrix} 0 & \Sigma_t \\ \omega_t & -W_t \cdot J_n \end{bmatrix}_{n+1 \times l+1}$$

plus $2l-1$, where $\Sigma_t = \sum_{h=1}^n \sum_{i=2}^l f_{th}^{ti} Q_t^{ti} + \sum_{h=1}^n f_{th}^{t1}$. The terms in the sum over consumers in Σ_t represent the first period wealths of the consumers; therefore, generically Σ_t will not be zero. Then the matrix A_t has maximal rank if the matrix ω_t^{t+1} has maximal rank, which follows if the initial endowments are not colinear. The rank of the matrix A is then $\min(2l+n, 3l)$.

The system has no solution if $\text{Rank}(A_t) < \text{Rank}(A_t, b_t)$. The rank of the augmented matrix is equal to the rank of the matrix

$$\begin{bmatrix} 0 & -Q_t^t & I_{l-1} & -p^t + p^{t1} Q_t^t \\ \omega_t^{t+1} & -W_t \cdot J_n & 0 & -\omega_t^t \cdot p_t + p^{t1} W_t \cdot J_n \\ 0 & \sum_{h=1}^n f_{th}^{t1} & \sum_{h=1}^n \hat{f}_{th}^t & -\delta^t - \sum_{h=1}^n \sum_{i=1}^l \tau_{t-1}^{ti} f_{t-1,h}^{ti} \end{bmatrix}_{n+l \times 2l+1}$$

plus l. If $n = l+1$ this is a square matrix. It has full rank $2l+1$ for an open and dense set of values of δ^t provided that the coefficient of δ^t in the polynomial expression representing the determinant of (A_t, b_t) is nonzero. Since this coefficient is the determinant of

$$\begin{bmatrix} 0 & -Q_t^t & I_{l-1} \\ \omega_t^{t+1} & -W_t \cdot J_n & 0 \end{bmatrix}_{2l \times 2l},$$

the relevant condition is that the determinant of

$$[\omega_t^{t+1} - W_t \cdot J_n] = \begin{bmatrix} \omega_{t-1,1}^{t1}, & \omega_{t-1,1}^{t2} & , ..., & \omega_{t-1,1}^{tl} & -w_{t1} \\ \omega_{t-1,2}^{t1}, & \omega_{t-1,2}^{t2} & , ..., & \omega_{t-1,2}^{tl} & -w_{t2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \omega_{t-1,n}^{t1}, & \omega_{t-1,n}^{t2} & , ..., & \omega_{t-1,n}^{tl} & -w_{tn} \end{bmatrix}_{l+1 \times l+1}$$

is nonzero. This determinant is generically nonzero because W_h involves both ω_{th}^t and ω_{th}^{t+1}. Therefore, for a dense and open set of budget specifications when $n = l+1$, $\text{Rank}(A_t) = 3l < 3l+1 = \text{Rank}(A_t, b_t)$, i.e. the system has no solution. Of course this is the knife edge case. For $n \leqslant l$, $\text{Rank}(A_t) = \text{Rank}(A_t, b_t)$ and the system has a solution, while for $n \geqslant l+1$ there is no solution.

2. *Individualized Taxes.* Here we suppose that the government can perfectly distinguish the consumers and is able to apply a different tax rate to every commodity, i.e. $N = n$ and $L = l$. For the n consumers of generation $t = 1, 2, ...,$ the relevant linear system (6.6) can be written in matrix form

$$A_t z_t = \begin{bmatrix} 0 & -Q_t^t & I_{n(l-1)} & 0 \\ \bar{J}_{ln} & -Q_t^{t+1} & 0 & I_{nl} \\ \omega_t^{t+1} & -W_t & 0 & 0 \\ 0 & f_t^{t1} & \hat{f}_t^t & 0 \end{bmatrix} \begin{bmatrix} p^{t+1} \\ \tau_t^{t1} \\ \hat{\tau}_t^t \\ \tau_t^{t+1} \end{bmatrix}$$

$$= \begin{bmatrix} -\hat{P}_t + p^{t1} Q_t^t \cdot J_n \\ p^{t1} Q_t^{t+1} \cdot J_n \\ -\omega_t^t \cdot p^t + p^{t1} W_t \cdot J_n \\ -\delta^t - \sum_{h=1}^n \sum_{i=1}^l \tau_{t-1h}^{ti} f_{t-1,h}^{ti} \end{bmatrix},$$

where the quantities ω_t^t, ω_t^{t+1}, W_t and J_n are as previously defined while now τ_t^s, $\bar{J}_{ln}$, Q_t^s and $\hat{P}_t$ are defined by

$$\bar{J}_{ln} = \begin{bmatrix} I_l \\ I_l \\ \vdots \\ I_l \end{bmatrix}_{ln \times 1}, \qquad \hat{P}_t = \begin{bmatrix} \hat{p}^t \\ \hat{p}^t \\ \vdots \\ \hat{p}^t \end{bmatrix}_{(l-1)\,n \times 1}, \qquad Q^s_t = \begin{bmatrix} Q^s_{t1} & 0 & \cdots & 0 \\ 0 & Q^s_{t2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & Q^s_{tn} \end{bmatrix},$$

$$\tau^s_t = \begin{bmatrix} \tau^s_{t1} \\ \tau^s_{t2} \\ \vdots \\ \tau^s_{tn} \end{bmatrix}_{ln \times 1}.$$

Similar notation is needed for taxes $\hat{\tau}^t_t$.

The rank of the $(2nl+1) \times (l(2n+1))$ matrix A_t is equal to the rank of the matrix

$$\begin{bmatrix} 0 & -Q^t_t & I_{n(l-1)} \\ \omega^{t+1}_t & -W_t & 0 \\ 0 & f^{t1}_t & \hat{f}^t_t \end{bmatrix}_{nl+1 \times l(n+1)}$$

plus ln. By a sequence of manipulations involving the first $n(l-1)$ rows of this last matrix, the rank of the A_t matrix is seen to be equal to the rank of the $(n+1) \times (n+l)$ matrix

$$M_t = \begin{bmatrix} 0 & \Sigma_t \\ \omega^{t+1}_t & -W_t \end{bmatrix}_{n+1 \times n+l}$$

plus $2nl-n$, where Σ_t is given by

$$\Sigma_t = \left(\sum_{i=2}^{l} f^{ti}_{t1} Q^{ti}_{t1} + f^{t1}_{t1}, \ldots, \sum_{i=2}^{l} f^{ti}_{tn} Q^{ti}_{tn} + f^{t1}_{tn} \right).$$

Note that, since the coordinates of Σ_t represent the first-period wealths of the consumers, we generally have $\Sigma_t \in (\mathbb{R} \backslash \{0\})^n$.

Suppose that we have $n \leqslant l$, then the matrix M_t has rank $n+1$ when ω^{t+1}_t has maximal rank n, a generic property for non-colinear endowments. In this case, the matrix A_t has rank $2nl+1$ and the set of solutions of $A_t x_t = b_t$ is nonempty.

If we have $n > l$, the rank of the matrix M is equal to $n - l + 1$ plus the rank of the square matrix

$$\begin{bmatrix} \omega^{t+1,1}_{t1} & \omega^{t+1,2}_{t1} & \cdots & \omega^{t+1,l}_{t1} \\ \omega^{t+1,1}_{t2} & \omega^{t+1,2}_{t2} & \cdots & \omega^{t+1,l}_{t2} \\ \vdots & \vdots & \vdots & \vdots \\ \omega^{t+1,1}_{t,l-1} & \omega^{t+1,2}_{t,l-1} & \cdots & \omega^{t+1,l}_{t,l-1} \\ \Omega^1_t & \Omega^2_t & \cdots & \Omega^l_t \end{bmatrix}_{l \times l}$$

with $\Omega_t^i = (\Sigma_t^n)^{-1} w_n \sum_{s=0}^{m-l-1} \omega_{t,l+s}^{t+1,i} w_{t,l+s}^{-1} \Sigma_t^{l+s}$. The rank of this last matrix can be seen to be maximal for an open and dense set of endowments. Hence the matrix M_t has rank $n-l+1+l=n+1$ and the set of solutions of $A_t x_t = b_t$ is also nonempty. To complete the proof, notice that the results obtained for the consumers from generation 0 in the case with anonymous taxation can be replicated here.

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