Notes on the Economics of Infinity

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This is an attempt to expose the essence of Samuelson's consumption-loan paradox. It is maintained that the double infinity of traders and dated commodities allows for competitive equilibria that are not Pareto-optimal. While such models are most interesting in the dynamic setting, the fact that generations do not meet is not essential. The chain-letter aspect of the model reminds us that the appropriate form of the budget constraint is not obvious for the potentially infinitely long-lived economic entity (such as the corporation or the family). The analysis is related to recent contributions in the theories of general equilibrium, economic planning, and decentralization.

1. Paul Samuelson's (1958) paper on consumption loans is to my mind one of the most original and stimulating contributions to modern economic theory. In each period, there are assumed to be a finite number of individuals and one homogeneous commodity (say chocolate). Individuals are assumed to live for three periods: first, as dependent youths; second, as breadwinning[1] adults; and third, as retirees. There are no externalities. Samuelson shows that—in a world without end—the competitive equilibrium allocation is not necessarily Pareto-optimal. Some imposed reallocation can be found, making no individual worse off while making at least one individual better off.

2. A variety of attempts at explaining this "Samuelson paradox" have been made. Most such attempts concentrate on one of two general points:

A. In an economy with births and deaths, all souls cannot meet in a single market. Since Spiro Agnew cannot haggle over chocolate with George Washington and Buck Rogers, the usual assumptions of general equilibrium theory are violated.

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[1] To be precise, I should say "chocolate-winning adults."

B. There are an infinite number of (dated) commodities and an infinite number of souls engaged in Samuelson's endless economic process. This violates the usual axioms of general equilibrium analysis as stated in, say, Debreu (1959).

3. I maintain that the "double infinity" (of traders and commodities) is the essential ingredient of the Samuelson result. While such "infinite" models are probably most interesting in the dynamic setting, the fact that Washington, Agnew, and Rogers will never meet is not essential. I will design a *Gedankenexperiment* in which they do meet; even then, the competitive allocation is not necessarily Pareto-optimal. Looked at in this light, the consumption-loan model is much like the (timeless) problem of allocation of hotel beds in Gamow's (1961) *One Two Three . . . Infinity.* I then attempt to relate these problems to some of the most recent contributions in general equilibrium theory and planning theory. The chain-letter aspect of the "double infinity" model reminds us that the appropriate form of the budget constraint is not obvious for any potentially infinitely long-lived economic entity (such as a corporation or a family).

4. To make matters concrete, we can consider a special form of the consumption-loan model.[2] Individuals live for two periods. In each generation there is only one person (or equivalently n persons). In each period there are then two persons (or equivalently $2n$ persons) alive. Thus, the population growth rate is zero. Let the individual born in year t be called the tth individual (he is alive in periods t and $t+1$). Nature endows him with one chocolate in period t and one chocolate in period $t+1$. To simplify the analysis, assume that individuals have simple linear utility functions of the form:

$$u^t\,(c_t^t\,, c_{t+1}^t) = c_t^t + c_{t+1}^t,$$

where $u^t(\cdot)$ $(t = 0, 1, \ldots)$ is the utility of individual t, c_s^t is consumption of chocolate by individual t in period s $(s = 1, 2, \ldots)$. We can make whatever assumption we like about the durability of chocolates. To be somewhat specific, we can assume that they are perfectly nondurable. The endowment matrix of figure 1 recapitulates the story.

5. In this pure-exchange model, it is easily demonstrated that the zero-interest-rate price system is a competitive-equilibrium price system. Choosing chocolate in period 1 as the numeraire, if for each period the interest rate is zero, then

$$p_t = p_1 \equiv 1 \qquad (t = 1, 2, \ldots),$$

where p_t is the price of chocolate in period t (in terms of chocolate in

[2] I have been using this model in the classroom for the last several years. I do not know its ultimate author, but it came to me from Marcel Richter by way of Daniel McFadden. My version of the consumption-loan model resembles that used by Starrett (1969).

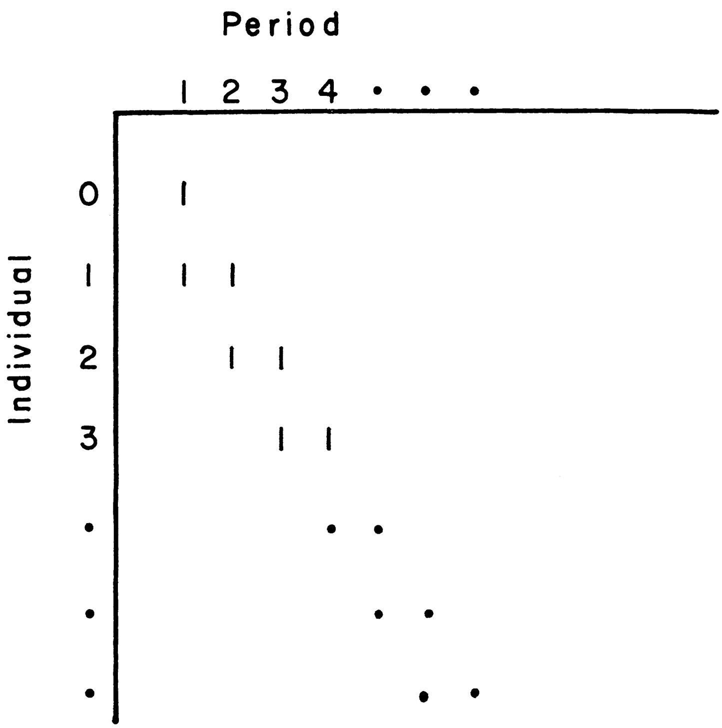

FIG. 1.—The endowment matrix

period 1). The budget constraint for individual t ($t = 1, 2, \ldots$) is $p_t c_t^t + p_{t+1} c_{t+1}^t \leqslant p_t + p_{t+1}$.

Since $p_{t+1} = p_t = 1$ and there is no satiation of consumption,

$$c_t^t + c_{t+1}^t = 2 \qquad \text{for} \qquad t = 1, 2, \ldots. \tag{1}$$

If chocolates cannot be stored without spoiling, then in each period total chocolate consumption must be two units,

$$c_{t+1}^t + c_{t+1}^{t+1} = 2 \qquad t = 0, 1, 2, \ldots. \tag{2}$$

Combining equations (1) and (2) yields that in equilibrium

$$c_t^t = c_{t+1}^{t+1} \qquad \text{for} \qquad t = 1, 2, \ldots. \tag{3}$$

But since man zero does not benefit from trade, $c_1^o = 1$ and therefore from equation (2) in equilibrium $c_1^1 = 1$. Thus from equations (1), (2), and (3) we know that in equilibrium

$$c_1^0 = c_t^t = c_{t+1}^t = 1 \qquad \text{for} \qquad t = 1, 2, \ldots$$

$$c_s^t = 0 \quad t \neq s \neq t+1 \quad t = 0, 1, \ldots; \quad s = 1, 2, \ldots.$$

It is clear that the above solution is a competitive equilibrium solution. Budget constraint (1) and resource limitation (2) are satisfied. Subject to budget constraints, utilities are maximized as long as equation (1) holds, since c_t and c_{t+1} substitute one for one in supply ($p_t = p_{t+1}$), while c_t and c_{t+1} substitute one for one in demand ($u^t = c_t^t + c_{t+1}^t$).

It should be noted that under the competitive allocation, the present value of society's wealth V_C is given by

$$V_C = 1 + \sum_{t=1}^{\infty} 2\alpha^t = (1 + \alpha)/(1 - \alpha),$$

where α is the discount factor. If the interest rate is zero, then $\alpha = 1$, and V_C is infinite.

6. Although the autarchic solution is in this case also a competitive-equilibrium allocation, it is not a Pareto-optimal allocation. If in period 1, a chocolate is transferred from man 1 to man 0, and in period 2 a chocolate is transferred from man 2 to man 1, and in period 3 a chocolate is transferred from man 3 to man 2, and so forth, then man 0 is made better off while no man is made worse off (since $u^t = 2$ $[t = 1, 2, \ldots]$ under the competitive allocation *and* under the imposed transfer scheme). Formally, the above scheme is

$$c_{t+1}^t = 2 \qquad \text{for} \qquad t = 0, \ldots,$$

$$c_s^t = 0 \qquad \text{for} \qquad s \neq t+1 \quad s = 1, 2, \ldots \quad t = 1, 2, \ldots.$$

While this imposed transfer scheme is a Pareto-optimal allocation, it is not the only Pareto-optimal allocation. In this economy, any allocation which is not wasteful and "passes" one chocolate backward from the infinite future is Pareto-optimal.[3] (This point will be elaborated when money is introduced into the model.) Under the imposed allocation above, the present value of society's wealth V_I is given by

$$V_I = 2 + \sum_{t=1}^{\infty} 2\alpha^t = 2/(1 - \alpha),$$

which is infinite when the discount factor $\alpha = 1$. If, however, $0 < \alpha < 1$, then $\infty > V_I > V_C > 0$. This suggests that the "Samuelson paradox"

[3] If society could control the infinite past as well as the infinite future, then an imposed allocation could net the present generation two additional chocolates—one "passed backward" from the infinite future and one "passed forward" from the infinite past.

arises for competitive equilibria where the interest rate is zero (more generally, when the interest rate is equal to the growth rate, or "biological interest rate"), since in this case the present value of society's wealth is infinite.

7. Although the model's most natural interpretation is in a dynamic setting, it can be interpreted in a static setting in which each of a denumerably[4] infinite number of individuals trades at the same market. Under this interpretation, there are also denumerably infinite varieties of chocolates. Individual t, under this interpretation, is endowed with one chocolate of type t and one chocolate of type $t+1$. While he is able to consume any type of chocolate, he craves only types t and $t+1$. In particular, $u^t(c_1^t, c_2^t, \ldots) = c_t^t + c_{t+1}^t$. Individual t's budget constraint is $p_1c_1^t + p_2c_2^t + \ldots = p_t + p_{t+1}$, where $p_1, p_2, \ldots p_t, p_{t+1}, \ldots$, are the unit prices of the various chocolate varieties.

Therefore, even when all souls are able to transact business in the same Walrasian market, the absence of Pareto-optimality persists in the competitive-equilibrium model. This absence of Pareto-optimality is implicit in Gamow's hotel problem.[5] An innkeeper has committed each of the denumerably infinite number of beds on a certain rainy night. A guest asks for a bed when all are occupied, but a bed can be found if the innkeeper requires each guest to move down one bed. In our little chocolate game, the imposed allocation can produce one extra chocolate. In the hotel problem, on the other hand, the innkeeper by imposing an allocation will be able to produce a denumerable infinity of extra beds.

8. Back to our problem of chocolate allocation. Assume man 0 declares that the paper wrapper from his chocolate is money. If he is able to convince the future generation to accept his wrapper as legal tender, he will be able to make a chocolate profit called seigniorage. Let π_t be the price of money in period t in terms of chocolate in period t. Assume that there are no liquidity or transactions demands for money, so that the only reason money has a place in portfolios is as a store of value. As before, competitive equilibrium chocolate prices must be such that: $p_t = 1$ for $t = 1, 2, \ldots$, while the chocolate price of money must satisfy $\pi_t \leqslant 1$ for $t = 1, 2, \ldots$. The competitive equilibrium in this money-chocolate world is Pareto-optimal if and only if "one chocolate is brought backward," that is (formally) if and only if

$$\lim_{t \to \infty} \pi_t = 1.$$

[4] A set is said to contain a *denumerable* (or *countable*) infinity of elements if there exists a one-to-one correspondence between the elements of that set and the integers. Thus the set of all odd numbers is denumerable, the set of all rational numbers is denumerable, while the set of all irrational numbers is *nondenumarable.* (See Gamow 1961, pages 14–23.)

[5] Gamow (1961, p. 17–18). Thanks to Avinash Dixit for this reference.

The competitive monetary economy can, of course, mimic the imposed allocation in which each generation passes exactly one chocolate backward. In this case, $p_t = \pi_t = 1$ for $t = 1, 2, \ldots$.

The fragility of intergenerational social contracts shows up in our example. If the kth generation repudiates the money of its elders, it can avoid passing chocolate backward. Furthermore, the kth generation can gain seigniorage by printing new money. Under this monetary reform the reforming generation may have chocolate passed to it, but itself passes none backward. If all souls had met in an atemporal Walrasian market, the monetary reform would have been revealed to be a nonequilibrium solution. In the actual dynamic world, "monetary reform" merely leads to frustration of the $(k-1)$ th generation's expectations. The most notable instance of such frustration is when the peasants' sons leave home for the city and fail to provide adequately for their parents' old-age.

9. Diamond (1965) extends the Samuelson model to include production. He shows that competitive-equilibrium sequences in which each individual attempts to maximize his own utility may be Phelps-Koopmans inefficient[6] and thus not Pareto-optimal. Cass and Yaari (1967) duplicate the Diamond results in a model with continuous (rather than discrete) time.[7]

10. There is a different way to look at the problem of intertemporal allocation in the competitive economy. The approach taken by Arrow and Kurz (1969) and others is to focus on family decision making. This approach can run into the difficulties noted in the Strotz (1956) myopia paper. In special cases, the Strotz problem[8] is absent and the problem is equivalent to that of a finite number of infinitely long-lived, utility-maximizing individuals.

Even so, there are still difficulties with this approach. Consider first the planning problem of maximizing the sum of the utilities of consumption,

$$\sum_{t=1}^{\infty} u(c_t),$$

where c_t is aggregate consumption in period t and $u(\cdot)$ is an increasing

[6] An infinite program is inefficient in the sense of Phelps and Koopmans if another program can be found which starts with the same factor endowment and provides equal or greater consumption at every instant, while providing strictly greater consumption in some period. See Phelps (1965) where it is shown for the one-sector model that programs with the capital-labor ratio forever bounded above the golden-rule ratio are inefficient.

[7] In the model with production, certain assumptions ensure that the infinite-horizon competitive-equilibrium model is efficient. See, for example, Gale (1967).

[8] Strotz (1956) studies the problem of intertemporal utility-maximization. He shows that in a world of certainty, the myopic maximizer, planning on the basis of today's tastes and constraints, will not alter his plan tomorrow, if and only if the pure rate of subjective time discount is constant over time.

concave utility function. To make the analysis simple, assume no population growth (although there are offsetting births and deaths). If the technology is neoclassical, then the maximand

$$\sum_{1}^{\infty} u(c_t),$$

may be unbounded, and to proceed we must either adopt the Weizsäcker overtaking criterion[9] or following Ramsey (1928) consider a new (additive, zero-discount) welfare function,

$$\sum_{t=1}^{\infty} [u(c_t) - u(c^*)],$$

where c^* is Golden Rule consumption. As Koopmans (1965) shows in the continuous-time problem, there is an optimal consumption program to the amended problem.

The planning problem possesses a Golden Rule turnpike property. Indeed, if the initial capital-labor ratio is such that the net (of depreciation) interest rate is zero (= the population growth rate), then the optimal policy is to maintain the initial capital endowment while setting $c_t = c^*$ for $t = 1, 2, \ldots$.

Assume now that instead of the planned economy, we have a competitive economy with n families. Each family is originally endowed with $(1/n)$th the capital and inelastically supplies the same labor in each period. The net rate of interest is assumed to be zero. Each family has the same type of maximand,

$$\sum_{t=1}^{\infty} [u(c_t^i) - u(c^*/n)], \qquad i = 1, 2, \ldots, n,$$

where c_t^i is consumption of family i in period t.

Will this competitive process yield the same Golden Rule growth as in the planning model? The answer seems to be no. Taking prices as given, each family would consume all of its wages to achieve a Golden Rule

[9] We may say that the program c_t^1 is better than c_t^2, if there exists a date T_0 such that for $T \geqslant T_0$:

$$\sum_{1}^{T} [u(c_t^1) - u(c_t^2)] > 0.$$

Accordingly (by the overtaking criterion), a feasible program c_t^1 is called optimal if for any feasible program c_t there exists T_0 such that for $T \geqslant T_0$:

$$\sum_{1}^{T} [u(c_t^1) - u(c_t)] \geqslant 0.$$

path. In the neoclassical one-sector model, the price of consumption must equal the price of capital, since both investment goods and consumption goods are produced. The net rate of return on holding capital is zero, so each family would be better off if it could exchange all its capital for consumption goods. Supply of capital exceeds demand and there seems to be no price system that will support the (efficient) Golden Rule consumption program.[10]

11. The above analysis is incomplete. In order to argue that no zero-interest-rate competitive equilibrium exists, I implicitly assumed a special form of the budget constraint for each of the n families, to wit: As t approaches infinity, each family's wealth at time t must approach some nonnegative number. More precisely, I might have asumed that:

$$\lim_{t\to\infty} \inf_{t\leq\tau<\infty} W^i_\tau \geqslant 0 \qquad i=1,\ldots,n,$$

where W^i_τ is the market value of tangible wealth for family i in period τ ($\tau = 1, 2, \ldots$).[11] This form of the budget constraint, while limiting the freedom of families to write chain letters, is rather arbitrary. The family must "ultimately" retire its debt. Why not postulate instead that

$$\lim_{t\to\infty} \inf_{t\leq\tau<\infty} W^i_\tau \geqslant \beta,$$

where β is some negative number? This would mean that we allow the family unit to be an "ultimate" net debtor but only to a limited extent. On the other hand, we could make β a positive number, insisting that it is each family's duty "ultimately" to hold at least this prespecified amount of wealth.

Indeed, if in our previous example, we set $\beta = K^*/n$, where K^* is the Golden Rule capital stock, then for this form of the budget constraint, competitive equilibrium will exist. With $\beta = K^*/n$, the equilibrium solution yields Golden Rule growth.

12. A mapping L from the vector space V to the real line R, ($L{:}V \to$ R), is said to be a linear function[12] if $L(x+y) = L(x) + L(y)$ for each x and y in the vector space V, and $L(\lambda x) = \lambda L(x)$ for each x in the vector space V and each (scalar) λ in the real line R.

[10] A zero rate of interest implies that for equilibrium the marginal rate of transformation between consumption in any two periods is -1. But there seems to be no motivation for the family to hold capital in the zero interest state, since (taking prices as given) it could increase consumption over its Golden Rule value (= wages) by selling all assets.

[11] The assumption is that the limit *inferior* of family wealth must be nonnegative. Family wealth could oscillate with nonexistence of a simple limit. We might insist that the limit of the lower "envelope" of family wealth (or limit inferior) then be nonnegative.

[12] See Dunford and Schwartz (1957, pp. 35–38).

If V is of finite dimensionality, say m, then L must be representable as an inner product, that is,

$$L(x) = \sum_{i=1}^{m} \omega_i x_i,$$

for each vector $x \equiv (x_1, \ldots, x_i, \ldots, x_m)$ in V, where $\omega_i (i = 1, \ldots, m)$ is a scalar. If, on the other hand, V is not a finite dimensional vector space, then the linear functional L may not be representable as such an inner product.[13] In a model with a denumerable infinity of different commodities but only a finite number of agents, Debreu (1954) shows that in a special sense, every Pareto-optimal allocation is a valuation equilibrium. Now this is the usual separating-hyperplane theorem we look for in "finite" economies, with one important difference. In this Debreu model, the budget constraints and "separating hyperplanes" are written as general linear functionals—not necessarily representable as inner products of price vectors and quantity vectors.

In the model with infinitely long-lived families, we have already asked what meaning can be attached to this more general budget constraint. Similarly, what is the economic interpretation to be given to a separating hyperplane that does not provide a price vector for the decentralized economy?

13. Even in infinite-horizon planning problems, the separating-hyperplane[14] theorems are in terms of a linear functional which is not necessarily representable by the inner product of some infinite dimensional vector of prices and an infinite dimensional vector of quantities. See, for example, Radner (1967). As I show in the Varenna Lectures (Shell 1969), this is closely related to the zero-interest-rate counterexample to the alleged transversality conditions used by some authors writing on the theory of optimal growth.

14. There is a basic difficulty with the notion of Pareto-optimality when one passes from the discrete-time model of (say) Samuelson to a continuous-time analog. Whether there is an infinite horizon or not, the number of individuals in the discrete model is countable, while the number of individuals in the continuous model is not countable. When the number of individuals is not countable, then the definition of Pareto-optimality is altered: The allocation A^* is said to be Pareto-superior to the allocation A if under A^* no individual is worse off than under A and the set of individuals better off under A^* is of measure greater than zero; A is a

[13] See, e.g., Dunford and Schwartz (1957).

[14] A hyperplane is a flat of deficiency 1. A flat is a translate of a linear subspace of V. A subspace W in V has deficiency h if there exists an h-dimensional subspace X such that $V = W + X$ and such that $W \cap X$ is empty.

Pareto-optimal allocation if there is no allocation which is Pareto-superior to A.

If the set of individuals can be represented by the real unit interval, then an allocation may be Pareto-optimal even if another allocation in which (the countably infinite number of) individuals represented by rational numbers can be made better off and no one made worse off.

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