# An Exercise in the Theory of Heterogeneous Capital Accumulation [1,2]

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The basic result of Hahn's seminal contribution to the descriptive theory of heterogeneous capital accumulation [2] is that the short-run-perfect-foresight balanced-growth equilibrium is a saddlepoint in the space of capital-labour ratios and capital goods prices. In the context of a special model, Shell and Stiglitz [5] show further that on paths not tending to balanced growth, price ratios become zero (or infinite) in finite time. If machines are freely disposable, asset markets cannot be in equilibrium on such " errant " trajectories.

A recent paper by Atkinson [1] has pointed out some very special features of the Shell-Stiglitz technology, casting some doubt on the generality of the conclusions. In the present paper, we investigate the " Hahn problem " in the context of a more regular model of production, the " quarter-circle " technology employed by Samuelson in his study of joint production [4]. As in the previous studies, we find that the unique balanced-growth equilibrium is a saddlepoint, and on paths not tending to balanced growth the price ratio becomes zero or infinite in finite time.

## 1. THE MODEL

As in [5], we study a one-sector, two-capital model in which a homogeneous output, $Y$, is produced by the cooperation of labour, $L$, machines of the first kind, $K_1$, and machines of the second kind, $K_2$. Assuming constant returns-to-scale and denoting quantities per unit labour by lower case letters, the production relation can be written as $y = f(k_1, k_2)$. For ease of analysis we assume that the production function is linear in logarithms, so

$$y = f(k_1, k_2) = k_1^{\alpha_1} k_2^{\alpha_2}, \qquad \text{...(1)}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$, and $1 - \alpha_1 - \alpha_2 > 0$. If we assume that all rentals are saved, while all wages are consumed, demand for consumption per head is given by

$$f - k_1 f_1 - k_2 f_2 = (1 - \alpha_1 - \alpha_2) k_1^{\alpha_1} k_2^{\alpha_2},$$

when factor markets are competitive. If the product markets are in momentary equilibrium so that demand and supply of consumption goods are equal,

$$c = f - k_1 f_1 - k_2 f_2 = (1 - \alpha_1 - \alpha_2) y > 0, \qquad \text{...(2)}$$

where $c$ is consumption per man. At every instant saving and gross investment per man is equal to $(\alpha_1 + \alpha_2) y \equiv z > 0$.

Following the example of Samuelson [4], we assume a " quarter-circle " technology

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is available for costlessly transforming undifferentiated (gross) investment, $z$, into gross investment in machinery of the first kind, $z_1$, and gross investment in machinery of the second kind, $z_2$. That is,

$$z^2 = z_1^2 + z_2^2, \qquad \ldots(3)$$

with $z_1 \geqq 0$ and $z_2 \geqq 0$. We choose the consumption good as *numéraire*. In momentary equilibrium, the price of a unit of consumption, $p_c$, must equal the price of a unit of investment, $p_z$, since both goods are produced by the same technology, and if $p_y$ is the price of a unit of undifferentiated output,

$$p_y = p_z = p_c \equiv 1, \qquad \ldots(4)$$

since both $c$ and $z$ are positive.

Under competition, profits in the conversion of undifferentiated investment into differentiated investment must be zero. Thus,

$$p_1 z_1 + p_2 z_2 = p_z z = z, \qquad \ldots(5)$$

where $p_1$ and $p_2$ are the unit prices of the respective investment goods. Assume that machinery of either kind depreciates at the constant proportionate rate $\mu$ and that the labour force is growing at the constant proportionate rate $n$. Then capital accumulation can be described by the equations:

$$\dot{k}_1 = z_1 - \lambda k_1, \qquad \ldots(6)$$

and

$$\dot{k}_2 = z_2 - \lambda k_2, \qquad \ldots(7)$$

where

$$\lambda \equiv \mu + n > 0.$$

## 2. MOMENTARY EQUILIBRIUM

Unlike [5], in which the production possibility frontier is assumed to be flat, momentary equilibrium in the present model is always unique. That is, given endowments and prices, the outputs of the consumption good and the two capital goods are uniquely determined. This is easily seen because from (1) and (2), $k_1$ and $k_2$ uniquely determine $y$, $c$, and $z$. Given $z$, $p_1$ and $p_2$, the profit-maximizing values of $z_1$ and $z_2$ are uniquely determined by (3) and

$$z_1/z_2 = p_1/p_2. \qquad \ldots(8)$$

Equation (8) states that production on the quarter circle with radius $z$ will take place at the value of $(z_1, z_2)$ at which the marginal rate of technical transformation is equal to the price ratio. By assumption (3), equation (8) holds for all positive but finite values of $p_1$ and $p_2$. This is another important distinction between the present model and that of Shell-Stiglitz [5]. In the latter paper, the allocation of investment is completely specialized whenever the prices of the two investment goods differ. In the present model, with $0 < p_1 < \infty$ and $0 < p_2 < \infty$, momentary equilibrium is uniquely determined and the allocation of investment is nonspecialized.

Squaring both sides of (5) and (3) and combining yields

$$p_1^2(1-p_1^2) + p_2^2(1-p_2^2) - 2p_1^2 p_2^2 = 0,$$

which can be rewritten as

$$(p_1^2 + p_2^2)(p_1^2 + p_2^2 - 1) = 0.$$

Ignoring the extraneous root, we have that $p_1$ and $p_2$ must lie on the unit circle, i.e.,

$$p_1^2 + p_2^2 = 1. \qquad \ldots(9)$$

Next we turn to the equilibrium conditions in the used-machinery market. *Rentiers* hold machines of the first and second kinds. Expected rates of return on the two types of machinery must be equal, for otherwise all wealth holders will attempt to specialize to the capital good with the higher rate of return. If $\dot{p}_1^e$ and $\dot{p}_2^e$ are the expected rates of price change for the respective capital goods, then the expected consumption rates of return must be equalized, or

$$\dot{p}_1^e/p_1+r_1/p_1 = \dot{p}_2^e/p_2+r_2/p_2, \qquad \text{...(10)}$$

where $r_1$ and $r_2$ are the respective rental rates on machinery. If expectations are always realized, $\dot{p}_1^e = \dot{p}_1$, and $\dot{p}_2^e = \dot{p}_2$, then from (10),

$$\dot{p}_1/p_1+f_1/p_1 = \dot{p}_2/p_2+f_2/p_2, \qquad \text{...(11)}$$

since under competition rental rates are equal to marginal products, i.e., $r_1 = f_1$ and $r_2 = f_2$.

## 3. BALANCED GROWTH

Combining (3), (5), and (8) yields

$$z_1 = p_1 z \text{ and } z_2 = p_2 z. \qquad \text{...(12)}$$

In light of (9), it is legitimate to cast the analysis in terms of the price ratio $p \equiv p_2/p_1$. From (9)

$$p_1 = (1+p^2)^{-\frac{1}{2}} \text{ and } p_2 = p(1+p^2)^{-\frac{1}{2}}. \qquad \text{...(13)}$$

Substituting (12) and (13) in (6) and (7) and using (2) gives the two capital accumulation equations

$$\dot{k}_1 = (\alpha_1+\alpha_2)y(1+p^2)^{-\frac{1}{2}}-\lambda k_1 \qquad \text{...(14)}$$

and

$$\dot{k}_2 = p(\alpha_1+\alpha_2)y(1+p^2)^{-\frac{1}{2}}-\lambda k_2. \qquad \text{...(15)}$$

Since in our model $f_1 = \alpha_1 y/k_1$ and $f_2 = \alpha_2 y/k_2$, asset-market-clearing equation (11) can be rewritten as

$$\dot{p} = y(1+p^2)^{\frac{1}{2}}(p\alpha_1/k_1-\alpha_2/k_2), \qquad \text{...(16)}$$

using (13).

A balanced-growth equilibrium is then defined by $\dot{k}_1 = \dot{k}_2 = \dot{p} = 0$. If $(k_1^*, k_2^*, p^*)$ is the value of $(k_1, k_2, p)$ in balanced growth, then from (14)-(16) it must solve the system

$$k_2 = p(\alpha_1+\alpha_2)y(1+p)^{-\frac{1}{2}}/\lambda, \qquad \text{...(17)}$$

$$p = \alpha_2 k_1/\alpha_1 k_2 = (\alpha_2/\alpha_1)^{\frac{1}{2}}. \qquad \text{...(18)}$$

Equation (18) shows that $p^*$ and the ratio $(k_1^*/k_2^*)$ are uniquely determined. Substituting (18) in (17) yields

$$k_2 = p(\alpha_1+\alpha_2)[(\alpha_1/\alpha_2)^{\frac{1}{2}}k_2]^{\alpha_1}k_2^{\alpha_2}/\lambda. \qquad \text{...(19)}$$

Since $0<\alpha_1+\alpha_2<1$, the right-hand side of (19) is an increasing concave function of $k_2$ with a first derivative that is infinite when $k_2$ is zero and is zero when $k_2$ is infinite. Therefore, the balanced-growth equilibrium $(k_1^*, k_2^*, p^*)$ is uniquely determined.

Notice from (17) and (18) that if $\alpha_1 = \alpha_2$, then $p^* = 1$ and $k_1^* = k_2^*$. In this case the marginal product $f_1(k_1^*, k_2^*)$ equals the marginal product $f_2(k_1^*, k_2^*)$ in balanced growth. If $\alpha_1>\alpha_2$, then $p^*<1$ and $k_1^*>k_2^*$. But from (1) and (17)

$$f_1(k_1^*, k_2^*)/f_2(k_1^*, k_2^*) = (\alpha_1/\alpha_2)^{\frac{1}{2}}>1,$$

by hypothesis. Thus, in balanced growth where $\alpha_1>\alpha_2$, more of the first kind of machinery is produced than the second, but the stock of machines of type one is not deepened sufficiently to equate marginal products.

It will be useful to note that, across steady-states ($\dot{k}_1 = 0 = \dot{k}_2$), consumption per man, $c$, is maximized when $(k_1, k_2) = (k_1^*, k_2^*)$. From (6) and (7), $z_1 = \lambda k_1$ and $z_2 = \lambda k_2$ in steady states. If $\theta$ is an undetermined Lagrange multiplier, the appropriate Lagrangean form is

$$k_1^{\alpha_1}k_2^{\alpha_2} - z + \theta[z^2 - \lambda^2(k_1^2 + k_2^2)]$$

in light of (3). The first-order conditions are:

$$\alpha_1 y/k_1 = 2\lambda^2\theta k_1,$$
$$\alpha_2 y/k_2 = 2\lambda^2\theta k_2,$$
$$2z\theta = 1,$$

which together yield the solution

$$k_1/k_2 = (\alpha_1/\alpha_2)^{\frac{1}{2}}, \qquad \text{...(20)}$$

which reduces to the real equation in (18). Combining (20) with the condition that $\dot{k}_1 = 0 = \dot{k}_2$, yields the same result as is achieved when $p$ is eliminated from (17) and (19) by using (18). This maximal-consumption result should come as no surprise. In balanced growth, the competitive allocation of investment is efficient. Furthermore, reminiscent of the usual Golden Rule exercises, the Marxian saving hypothesis ensures that the rate of interest equals the rate of growth.

## 4. DYNAMIC ANALYSIS

We have found that, in $(k_1, k_2, p)$ phase space, there is a unique rest point $(k_1^*, k_2^*, p^*)$, which we call balanced-growth equilibrium. We turn now to a full dynamic analysis of the system of differential equations (14)-(16).

We can approximate (14)-(16) by its linear Taylor expansion at the rest point, so that

$$\begin{bmatrix} \dot{k}_1 \\ \dot{k}_2 \\ \dot{p} \end{bmatrix} \doteq \begin{bmatrix} \lambda(\alpha_1 - 1) & \lambda p\alpha_1 & \dfrac{-p\alpha_1 y}{(1+p^2)^{\frac{1}{2}}} \\ \lambda p\alpha_1 & \lambda(p\alpha_1 - 1) & \dfrac{\alpha_1 y}{(1+p^2)^{\frac{1}{2}}} \\ \dfrac{-p\lambda^2}{\alpha_1 y(1+p^2)^{\frac{1}{2}}} & \dfrac{\lambda^2}{\alpha_1 y(1+p^2)^{\frac{1}{2}}} & \lambda \end{bmatrix} \begin{bmatrix} (k_1 - k_1^*) \\ (k_2 - k_2^*) \\ (p - p^*) \end{bmatrix} \qquad \text{...(21)}$$

where the $3 \times 3$ matrix above is evaluated at $(k_1^*, k_2^*, p^*)$. The characteristic equation for the linear system (21) is then

$$\begin{vmatrix} \lambda(\alpha_1 - 1) - x & \lambda p\alpha_1 & \dfrac{-p\alpha_1 y}{(1+p^2)^{\frac{1}{2}}} \\ \lambda p\alpha_1 & \lambda(p\alpha_1 - 1) - x & \dfrac{\alpha_1 y}{(1+p^2)^{\frac{1}{2}}} \\ \dfrac{-p\lambda^2}{\alpha_1 y(1+p^2)^{\frac{1}{2}}} & \dfrac{\lambda^2}{\alpha_1 y(1+p^2)^{\frac{1}{2}}} & \lambda - x \end{vmatrix} = 0, \qquad \text{...(22)}$$

where $x$ is the characteristic root and terms in (22) are evaluated at $(k_1^*, k_2^*, p^*)$. Solving cubic equation (22) for $x$ yields the three characteristic roots

$$x = \lambda(\alpha_1 + \alpha_2 - 1),\ \lambda\sqrt{2},\ -\lambda\sqrt{2}. \qquad \text{...(23)}$$

All roots are real, two are negative and one is positive. For the system (14)-(16) in

$(k_1, k_2, p)$ phase-space, there exists a two-dimensional surface along which all trajectories tend to $(k_1^*, k_2^*, p^*)$ as $t \to \infty$. Trajectories tending to $(k_1^*, k_2^*, p^*)$ as $t \to -\infty$ form a manifold of dimension 1 (a curve). Furthermore, since the real parts of the characteristic roots are nonzero no limit cycles can exist.[1] Therefore, in the neighbourhood of $(k_1^*, k_2^*, p^*)$ for every endowment vector, $(k_1, k_2)$, there exists one and only one price ratio, $p$, such that the system tends to $(k_1^*, k_2^*, p^*)$ as $t \to \infty$.

This "saddlepoint" result is reminiscent of the dynamical system studied in [5]. There are, however, two important differences. First, differential equations (14)-(16) are single-valued, whereas in [5] the dynamical system is of the form $(\dot{k}_1, \dot{k}_2, \dot{p}) \in \Omega(k_1, k_2, p)$, with $\Omega(\cdot)$ an upper-semicontinuous correspondence. Second, since (14)-(16) is a properly Lipschitzian system, on trajectories tending to the rest point, equilibrium can only be achieved asymptotically. In [5], on *some* trajectories the economy proceeds to the balanced-growth equilibrium in *finite* time.

We now turn to the *global* analysis of the differential equation system (14)-(16). The straightforward phase portrait for (14)-(16) must be drawn in three-dimensional space. To avoid this complexity, we adopt a trick used by Atkinson [1] which reduces the qualitative analysis to two dimensions.

Setting $k \equiv k_2/k_1$, we can study the development of our model economy in the $(k, p)$ "phase" plane of Fig. 1. From (14) and (15), when $k > 0$, $\dot{k} = 0$ if and only if $k = p$, so $k$ is stationary on a ray with unitary slope. From (16), when $y > 0$, $\dot{p} = 0$ if and only if $p = \alpha_2 k_1/\alpha_1 k_2 = \alpha_2/\alpha_1 k$. Thus, $p$ is stationary on a unique hyperbola in $(k, p)$ space. The unique intersection of the hyperbola with the ray is denoted by $(k^*, p^*)$.

Notice that $\dot{k}$ and $\dot{p}$ are not in general uniquely determined by $k$ and $p$ alone. From (14)-(16), we see that to determine $\dot{k}$ and $\dot{p}$ given $k$ and $p$ we must also know either $k_1$ or $k_2$. Nonetheless, on the ray in Fig. 1, $\dot{k} = 0$ for all values of $k_1$ and $k_2$ such that $k_2/k_1 \equiv k = p$. Similarly on the hyperbola, $\dot{p} = 0$, independent of the particular values of $k_1$ and $k_2$. To the northwest of the $\dot{k} = 0$ ray, $k$ is rising, to the southeast $k$ is falling. To the northeast of the $\dot{p} = 0$ hyperbola, $p$ is rising, to the southwest $p$ is falling. It should be noted that while knowledge of $k$ and $p$ is not sufficient to determine the speeds of motion, $\dot{k}$ and $\dot{p}$, Fig. 1 will serve well enough for *qualitative* analysis. Substituting $k$ in (14)-(16) and rearranging yields

$$\dot{k} = \left(\frac{y}{k_1}\right)\left[\frac{(\alpha_1+\alpha_2)k}{(1+p^2)^{\frac{1}{2}}}\left(\frac{p}{k}-1\right)\right], \qquad \ldots(24)$$

and

$$\dot{p} = \left(\frac{y}{k_1}\right)\left[(1+p^2)^{\frac{1}{2}}\left(p\alpha_1 - \frac{\alpha_2}{k}\right)\right]. \qquad \ldots(25)$$

In both (24) and (25) the terms in brackets are solely dependent upon the variables $k$ and $p$. The term $(y/k_1)$ depends upon $k_1$ and $k_2$ and cannot be written merely as a function of $k$. The right-hand side of (24) gives the horizontal speed of our model economy, while the right-hand side of (25) gives the vertical speed. Since in a "resolution-of-forces" rectangle the term $(y/k_1)$ will "cancel," we know that from (24) and (25), $k$ and $p$ will uniquely determine the direction of development, although we must also know $k_1$ or $k_2$ to ascertain the speed of development.[2]

Some important propositions are immediate from Fig. 1. Since for $k < k^*$, the separatrix (heavy curve) lies below the $\dot{p} = 0$ hyperbola, which in turn is asymptotic to the vertical axis, we can see that the separatrix covers the $k$ half-line. That is, for each $k > 0$, there exists a unique $0 < p < \infty$, which will ensure that the economy tends to the rest

[1] See Nemytskii and Stepanov [3], especially Theorem 1.41, page 233, Theorem 1.48, page 242, and Lemma 1.51, page 244.

[2] Along any trajectory in the "phase plane" of Figure 1, $(dk/dp) = (\dot{k}/\dot{p}) = \psi(k, p)$.

point $(k^*, p^*)$. For all other assignments of $p$, the system (14)-(16) diverges from the steady-state equilibrium.

As in [5], it is essential to study further the development of the economy on trajectories not tending to the $(k^*, p^*)$ equilibrium. From Fig. 1, we see that trajectories not tending to $(k^*, p^*)$ ultimately enter either Region A, where both $k$ and $p$ are falling, or Region B

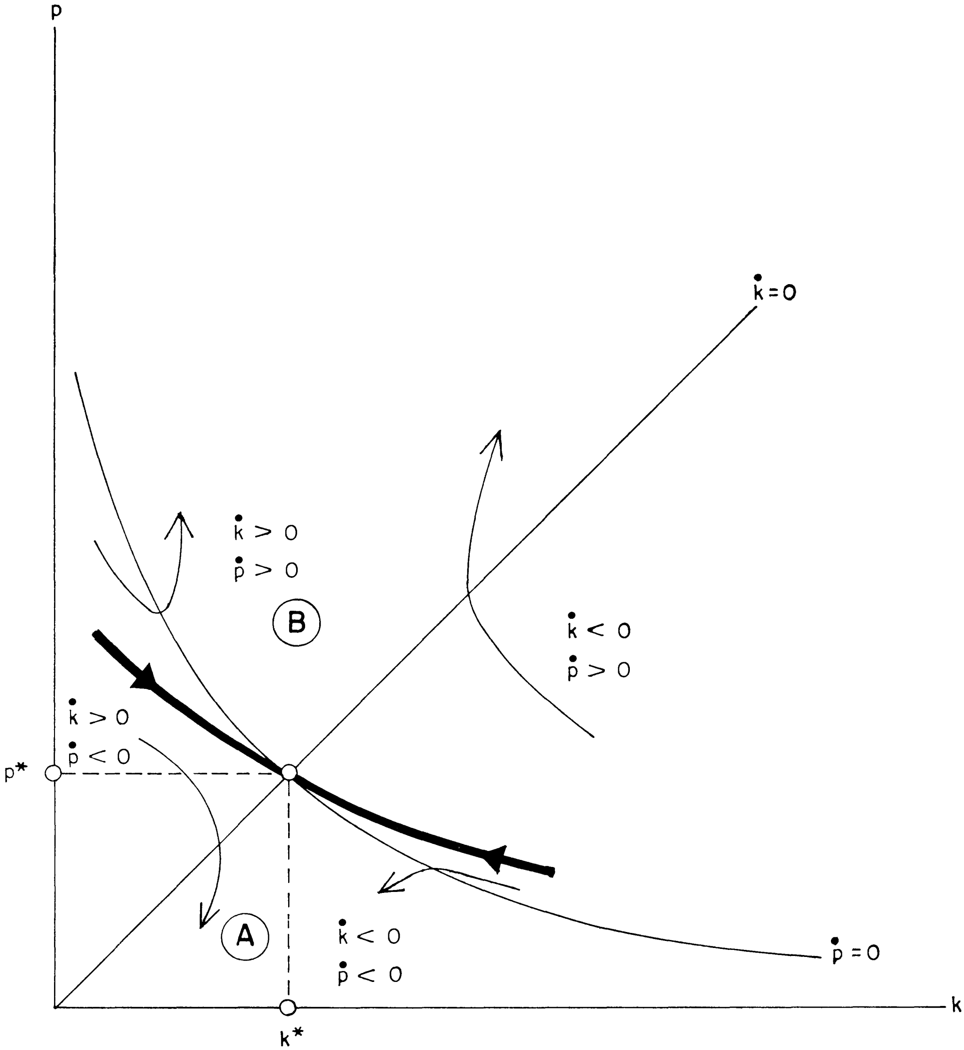


FIGURE 1

where both $k$ and $p$ are rising. Consider, for example, a trajectory in Region A. From (16)

$$\dot{p} = (1+p^2)^{\frac{1}{2}}(f_1/\alpha_1)(\alpha_1 p - \alpha_2/k), \qquad \text{...(26)}$$

where $f_1$ is the marginal product of the first capital good. In Region A, $\alpha_1 p < \alpha_2/k$. Therefore, in this region,

$$\dot{p} < (pf_1 - f_2) < 0, \qquad \text{...(27)}$$

since $(1+p^2)^{\frac{1}{2}} > 1$ for $p \neq 0$. Defining $\beta \equiv f_1/f_2$, (27) yields that

$$\dot{p} < (p\beta - 1)f_2 < 0. \qquad \text{...(28)}$$

In Region A, $p$ is falling and since $k$ is falling, $\beta = \alpha_1 k/\alpha_2$ is falling.

Next we show that, on a trajectory in Region A, the marginal product of the second capital good, $f_2$, is bounded from below. Since $f_2 = \alpha_2 y/k_2$, we can write

$$f_2 = \frac{\alpha_2}{k^{1-\alpha_2}k_1^{1-\alpha_1-\alpha_2}}. \qquad \text{...(29)}$$

Since in Region A, $\dot{k}<0$, we need only demonstrate that there is an upper bound to $k_1$. From equation (14)

$$\dot{k}_1 = \frac{(\alpha_1+\alpha_2)k_1^{\alpha_1}k_2^{\alpha_2}}{(1+p^2)^{\frac{1}{2}}} - \lambda k_1,$$

or

$$\dot{k}_1 = \frac{(\alpha_1+\alpha_2)k_1^{\alpha_1+\alpha_2}k^{\alpha_2}}{(1+p^2)^{\frac{1}{2}}} - \lambda k_1,$$

which yields

$$\dot{k}_1 < (\alpha_1+\alpha_2)k_1^{\alpha_1+\alpha_2}k^{\alpha_2} - \lambda k_1 \qquad \text{...(30)}$$

for $p \neq 0$. In Region $A$, $k$ is falling and since $\alpha_1+\alpha_2<1$, the right-hand side of (30) becomes negative for $k_1$ sufficiently large. In Region A, $k_1$ is bounded from above, and thus $f_2$ is bounded from below.

Therefore, from (28) we know that in Region A, $p$ falls to zero in finite time. From zero-profit condition (5), when $p_2 \leqq 0$ we have that $p_1 = 1$ since $z_2 = 0$ and $z_1 = z$, i.e., investment is specialized to machinery of the first kind. If we assume *for the moment* that differential equation (11) also holds for non-positive prices,[1] then from (11)

$$\dot{p} = pf_1 - f_2 < -f_2$$

when $p \leqq 0$. We have shown that in Region A marginal product $f_2$ is bounded from below, so that starting from Region A, $p \to -\infty$ and $k \to \infty$ as $t \to \infty$.

By symmetry, for trajectories in Region B, $p$ becomes infinite in finite time. Again, assuming for the moment that differential equation (11) holds everywhere, on trajectories starting in Region B, $p(t)$ has a pole at finite $t$; to the right of the pole, $p$ is negative, and $p \to 0$ with $k \to \infty$ as $t \to \infty$.

Since all " errant " trajectories (i.e., trajectories not tending to balanced growth) enter either Region A or Region B, we know that on such " errant " trajectories in finite time $p = 0$ or $p = \infty$. The development of the price ratio, $p$, is summarized in Fig. 2. For given initial relative capital intensity, $k_0$, there exists exactly one initial price ratio, $p_0$, such that the system (14)-(16) develops to ($k^*$, $p^*$). This unique price trajectory is shown by the heavy curve in Fig. 2. For smaller initial price ratios (see, for example, the light curve in Fig. 2), $p$ becomes zero in finite time and tends to minus infinity as $t \to \infty$. For initial price ratios greater than the unique assignment tending to balanced growth (see, for example, the dashed curve in Fig. 2), $p(t)$ rises to a pole at some finite date, then ultimately tends to zero as $t \to \infty$.

If, on the other hand, *capital goods are freely disposable*, the differential equation system (14)-(16) holds only for finite and positive $p$. Consider, for example, an " errant " trajectory in Region A. In finite time, $p = 0$. Therefore, by (13), in finite time $p_1 = 1$ and $p_2 = 0$. At this point asset-market-clearing equation (11) can be written as

$$\dot{p}_1 + f_1 = \dot{p}_2/p_2 + f_2/p_2. \qquad \text{...(31)}$$

Because of free disposability, $(\dot{p}_2/p_2) \geqq 0$ when $p_2 = 0$. Since $f_2 > 0$, the right-hand side of (31) must be $+\infty$. But, unless $p_1$ is discontinuous and thus frustrates the most acute adaptive expectations (myopically correct expectations), asset-market-clearing equation (31) will not hold. Then, the second capital good will have an infinite rate of return while

[1] Essentially assuming non-disposability of capital goods.

the first capital good bears a finite rate of return. All asset-holders will desire to specialize to capital of the second kind: the capital goods market does not clear.

Like Shell and Stiglitz [5], we now ask: What forces are there in capitalism that prevent

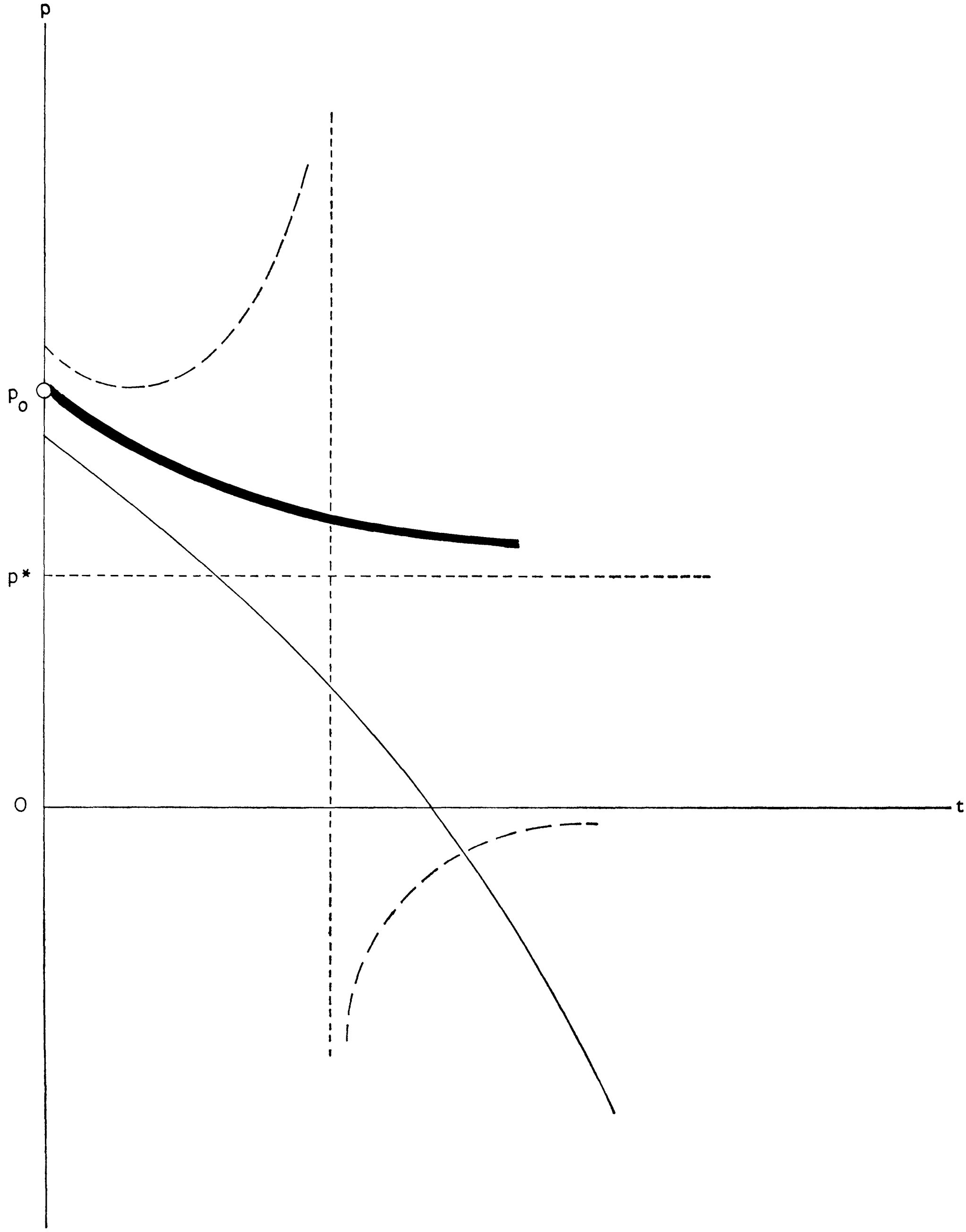


FIGURE 2

the economy from following " errant " trajectories? Our conclusions are very similar to those of [5]. Walrasian (i.e., essentially atemporal) futures markets extending indefinitely into the future ensure that the economy will be stable—that is, ensure that it will develop to $(k^*, p^*)$ independent of the initial capital intensity, $k_0$, since the initial competitive price ratio, $p_0$, will be a function of $k_0$ – lying on the heavy curve in Fig. 1. The Walrasian auctioneer rejects all price ratios but the unique ratio, $p_0(k_0)$, since all other price ratios imply that the spot capital goods market will not clear at some (finite) future date. Even in the absence of futures markets, capitalists possessing *long-run* perfect foresight will tend to bid up the current price of the asset which would eventually bear the infinite rate of return. Errant trajectories may be avoided on this score as well.

On the other hand, if there are no markets for selling and renting machinery, then the economy is stable since there are no capital gains. As in [5], it is furthermore true (although we do not show it here) that if we assume that capitalists possess static expectations, i.e., $\dot{p}_1^e = 0 = \dot{p}_2^e$, then development always tends to the balanced-growth equilibrium $(k_1^*, k_2^*, p^*)$.

In [5], for given initial capital-labour ratios, an economy approaching balanced growth equilibrium will in finite time equate marginal products of machinery, $f_1 = f_2$. This in turn implies that for given initial endowments there is a finite upper limit to the time required for the price on an errant path to reach zero (or infinity).[1] In the present model this is not true. For any date, $T$, that is chosen, there is always an initial $p$ sufficiently close to $p_0$ (in Fig. 2) to ensure that the time required for the price to reach zero is greater than $T$. This difference stems from the difference in curvature of the production possibility frontier. This is important because in [5] we only need futures markets for some finite period to rule out errant paths, whereas in the present model we need futures markets for an infinite period.

## CONCLUDING REMARKS

We have seen the persistent recurrence of some basic themes in several dynamical models of the capitalist economy. Analysis of a model with more general assumptions (especially about technology) is called for. As a start, one might study the model put forward by Hahn [2]. In Hahn's model, there is one consumption good, $m$ capital goods, and $m+1$ sectors. If all wages are consumed, all rentals are saved, expectations are myopically correct, and the production functions are linear in logarithms, it is easy to show that there exists a unique balanced-growth equilibrium $(k_1^*, \ldots, k_m^*, p_1^*, \ldots, p_m^*)$. The full dynamic analysis is substantially more difficult. Unless restrictive assumptions are made about technology, momentary equilibrium is not necessarily unique. Let $k$ be the $m$-vector of capital-labour ratios, $p$ be the $m$-vector of capital goods prices in terms of consumption. In general, the dynamical system of the Hahn model can be written as

$$(\dot{k}, \dot{p}) \in \Omega(k, p),$$

where $\Omega(\cdot)$ is an upper semicontinuous correspondence in $2m$ arguments. As such, the analysis cannot be cast in terms of the theory of differential equations, but we must instead turn to the general theory of dynamical systems.

Based on our knowledge of particular cases, we conjecture that, for the Hahn model, the unique balanced-growth equilibrium $(k^*, p^*)$ is a generalized saddlepoint in the $2m$-dimensional $(k, p)$ space. In particular, the manifold of trajectories tending to $(k^*, p^*)$ is of $m$ dimensions and " covers " the positive orthant of the $k$-hyperplane. We further conjecture that trajectories not tending to $(k^*, p^*)$ will in finite time be revealed to be disequilibrium paths along which asset markets do not clear at every instant.

## REFERENCES

[1] Atkinson, A. B. " The Timescale of Economic Models: How Long is the Long Run? " *Review of Economic Studies*, **36** (1969), 137-152.

[1] This point was noted by Atkinson [1] and was the basis for one possible worry that some of the basic points made in [5] do not carry over to more general models.

[2] Hahn, F. H. " Equilibrium Dynamics with Heterogeneous Capital Goods ", *Quarterly Journal of Economics*, **80** (1966), 633-646.

[3] Nemytskii, V. V. and Stepanov, V. V. *Qualitative Theory of Differential Equations* (translated from the Russian) (Princeton, New Jersey: Princeton University Press, 1960).

[4] Samuelson, P. A. " Fundamental Singularity Theorem for Non-Joint Production ", *International Economic Review*, **7** (1966), 34-41.

[5] Shell, K. and Stiglitz, J. E. " The Allocation of Investment in a Dynamic Economy ", *Quarterly Journal of Economics*, **81** (1967), 592-609.