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FINANCIAL INSTRUMENTS IN THE DYNAMIC THEORY OF AGGREGATE INVESTMENT ALLOCATION

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The theory of financial instruments and intermediation has been largely neglected by general equilibrium economists. Despite interesting attempts by Hahn, Foley, Heller, Starr and others, the very concept of money seems to have alluded us. Unable to reach first base (or is it home plate?) in static theory, it is not surprising that there is so little dynamic theory. It may be true, however, that many important features of the transaction-costs model can be understood in the intertemporal setting.

It is an understatement to say that there are loose ends in the descriptive theory of heterogeneous capital accumulation. From a mathematical viewpoint, the theory is especially unsatisfactory; for the general case – as we shall see – one must make do with conjecture based on the rigorous treatment of several special cases. Nonetheless, it seems to me that some of the underlying *economics* of multi-asset accumulation are revealed by these examples. Analysis of the heterogeneous-capital model suggests – without the explicit introduction of paper assets – some of the most basic problems in social policy toward financial intermediation. Capital gains drive the differential equations of the many-capital-goods model; thus, allowing us to talk about problems such as tulip manias and other speculative booms, orderliness in securities exchange, and the Keynesian divergence between private and social gain.

Growth models have been extended in the simplest ways to accommodate money, bonds, and other paper assets. In the present lecture, I shall describe economies which allow for financial intermediation in various forms. I shall attempt to isolate basic economic differences between these economies and their multi-capital cousins. The lecture concludes with an illustration of some of the magical properties of paper assets that arise when the time-horizon is infinite.

First, a sketch of the descriptive theory of heterogenous-capital accumula-

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tion.[1] In the absence of taxation, transactions costs, liquidity preference, and so forth, anticipated rates of return are equalized across assets in momentary equilibrium. Thus for an n-asset economy

$$r_i + (CG)_i = \rho \text{ for } i = 1, \ldots, n, \tag{1}$$

where r_i is the rental rate (per dollar) of the ith asset, $(CG)_i$ denotes anticipated capital gains per dollar of that asset, and ρ is the common rate-of-return. Individuals must forecast price change to arrive at $(CG)_i$. For simplicity, abstract from the stochastic nature of these forecasts and assume that individuals' beliefs are essentially alike. One forecasting hypothesis – the one to be adopted here – is the Cagan adaptive expectations process,

$$\dot{\xi} = b(\theta - \xi),$$

where ξ is the expected rate of price change, θ is the actual rate of price change, and b is the coefficient of adaptation. If $b = 0$, we are studying the case of static expectations. As b becomes large ($b \to \infty$), the case of instantaneously adjusted expectations is approached, wherein the (expected) right-hand time derivative of price is set equal to the (actual) left-hand time derivative, so that on continuously differentiable paths, $\xi = \theta$. As we shall see, the price system that is generated from the system (1) tends to be unstable when expectations are instantaneously adjusted.

Let us review a simple pedagogic example that I introduced last summer in Varenna.[2] The technology is described by

$$\begin{aligned} & y \equiv c + z = k_1^{\alpha_1} k_2^{\alpha_2} \equiv f(k_1, k_2) \\ & z = \max(z_1, z_2) \\ & \dot{k}_1 = z_1 - \lambda k_1 \\ & \dot{k}_2 = z_2 - \lambda k_2 \\ & c \geqslant 0, z_1 \geqslant 0, z_2 \geqslant 0, \alpha_2 > 0, \alpha_2 > 0, 1 - \alpha_1 - \alpha_2 > 0, \end{aligned} \tag{2}$$

where y denotes output per man, c consumption per man, and z is undifferentiated gross investment per man. Continuing in intensive units, k_1 denotes machines of type 1, k_2 machines of type 2, while z_i ($i = 1, 2$) is gross investment in machinery of type i, $\lambda > 0$ is the sum of the constant proportionate

[1] The basic references: Hahn [2], Shell and Stiglitz [6], Samuelson [3], Caton and Shell [1].

[2] See Shell [4], esp. pp. 466–468.

population growth and depreciation rates. Assume that laborers consume all wages while capitalists invest all profits. Then in momentary equilibrium

$$c = f - k_1 f_1 - k_2 f_2 = (1-\alpha_1-\alpha_2)y. \tag{3}$$

Under the assumptions of profit-maximization and instantaneous adjustment, momentary equilibrium condition (1) can be rewritten as

$$\frac{\dot{p}_1}{p_1} + \frac{f_1}{p_1} = \frac{\dot{p}_2}{p_2} + \frac{f_2}{p_2}, \tag{4}$$

where p_i is the market price of a machine of type i and f_i is the marginal product of machinery of type i ($i = 1, 2$). If prices are positive and finite then an efficiency condition is that

$$z_1 = z_2 = z = (\alpha_1 + \alpha_2)y. \tag{5}$$

Under competition, profits in 'cracking' investment are zero, i.e.,

$$p_1 z_1 + p_2 z_2 = z, \tag{6}$$

and therefore the price system must obey the dual constraint

$$p_1 + p_2 = 1. \tag{7}$$

Letting $p = p_2/p_1$, (7) yields

$$p_1 = \frac{1}{1+p} \text{ and } p_2 = \frac{p}{1+p}. \tag{8}$$

Therefore, market-clearing eq. (4) can be reduced to

$$\dot{p} = (1+p)(pf_1 - f_2). \tag{9}$$

Assume that the two capital-labor ratios are initially equal to their long-run equilibrium values ($k_1 = k_1^*$, so that $\dot{k}_1 = 0$ and $k_2 = k_2^*$, so that $\dot{k}_2 = 0$), but $p < p^*$, i.e., $p < f_2/f_1$. In this case,

$$\dot{p} < pf_1 - f_2 < 0,$$

where f_1 and f_2 remain at their equilibrium values. Therefore, p is falling at a rate which is faster than a constant absolute rate and must become zero in finite time.

By analogy to this simple example and others, we can conjecture as to the underlying dynamical structure in general models of this type. Let k be the m-vector of capital-labor ratios, p the m-vector of capital-goods prices in terms of consumption,

$$(\dot{k},\dot{p}) \in \Omega(k,p),$$

where $\Omega(\cdot)$ is an upper-semicontinuous correspondence. If the rest point (k^*,p^*) is unique, then (k^*,p^*) is a saddlepoint. In particular, the manifold of trajectories tending to the rest point is of dimension m and 'covers' the positive half k-hyperplane. Paths not tending to (k^*,p^*) will in finite time be revealed to be disequilibrium paths.

Figure 1 is a schematic representation of the fuller $2m$-dimensional phase diagram. For each quantity vector, one and only one price vector assigment is consistent with a tendency toward balanced growth. It is our general experience that along paths not tending to (k^*, p^*) the allocation of investment

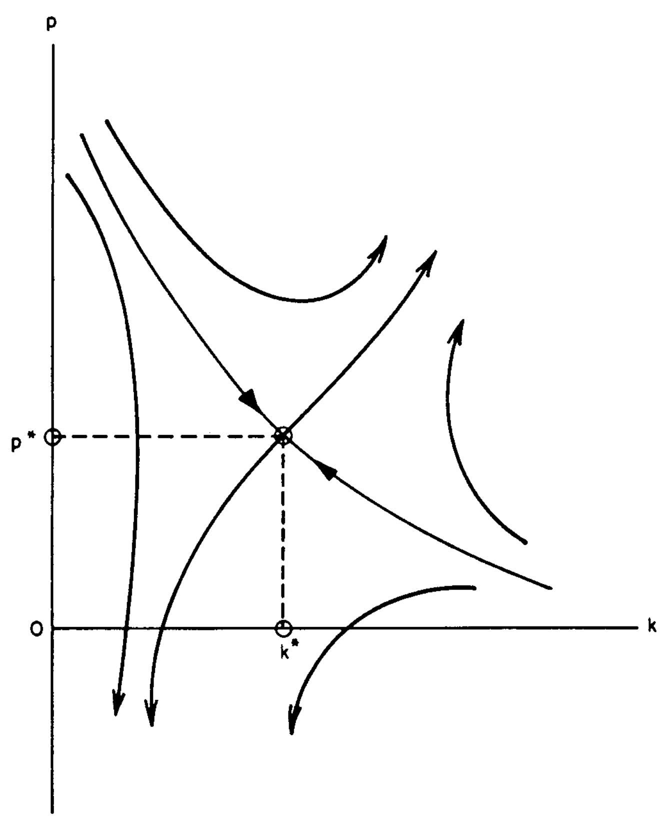

Fig. 1. Schematic representation of phase diagram.

is inefficient – for example, such economies ultimately specialize to production of machines with *lowest* marginal products.

What saves the model from errant development? The answer must be related to the fact that in finite time on errant trajectories some capital-goods price becomes zero or infinite. With free disposability, this is a disequilibrium path since asset markets cannot clear in such a situation.

Are there forces in the real capitalist world that militate against this tendency to errant development? The existence and perfection of all possible futures markets ensures nonerrant development. Even without futures markets, when capitalists possess long-run foresight there is a tendency toward stable development. On the other hand, imperfect short-term foresight and imperfect used machinery markets *also* tend to stabilize development and tend to avoid the grossest inefficiencies.

In reality – while used-capital-goods markets may be imperfect – securities (shares of equity) markets are probably the best organized markets in the capitalist economy. To the extent that perfect securities markets behave qualitatively as perfect used machinery markets, then the heterogeneous-capital model captures the important features of capitalist development.

It should be noted that there are certain aspects of financial markets that play a stabilizing role. There are all the 'frictions' and 'orderliness' imposed upon securities markets by the exchanges and the securities commissions. For example, when the price of a particular share changes 'very rapidly' then trading in that issue is likely to be suspended. Of course, institutions like the Central Bank and the Treasury are designed to play stabilizing roles in financial markets.

Our next exercise is that of a simple one-sector growth model in which there is non-interest-bearing government debt. We may call this debt "money', but remember that it competes perfectly with private capital in portfolios. For simplicity, we abstract from transactions costs, liquidity preference, and so forth. Take consumption as the numeraire; p to be the price of a machine, and p_m to be the consumption price of money. For the asset market to be in momentary equilibrium, rates of return must be equalized, i.e.,

$$\frac{\dot{p}}{p} + \frac{\max(1,p)f'(k)}{p} = \frac{\dot{p}_m}{p_m}, \tag{10}$$

when expectations are instantaneously adjusted. [3] $f'(\cdot)$ is the physical marginal product of capital which depends on capital intensity, k. Since output can be used for consumption or investment, $\max(1, p) f'(k)$ is the value marginal product of capital.

If government expenditures are zero, then the rate of increase in the money supply is equal to the goverment's budget deficit which is in turn equal to net transfers to individuals. Thus, perceived income to individuals is

$$\hat{Y} = \dot{p}K + \dot{p}_m M + p_m \dot{M} + \max(1, p) Y,$$

where M is money supply, K capital stock, and Y output. If both consumption and investment are produced, then $p \equiv 1$ and $\dot{p} = 0$ and (10) reduces to

$$f'(k) = \dot{p}_m / p_m. \tag{11}$$

Let $\theta = \dot{M}/M$ be the constant proportionate rate of increase of the money supply. Assume that a constant fraction, $s \in [0, 1]$, of income, $\hat{Y}$, is saved so

$$\dot{k} = sf(k) - (1-s)[\theta + \dot{p}_m/p_m]\, m - nk, \tag{12}$$

where $m \equiv p_m M/L$ and $\dot{L}/L = n > 0$, the constant proportionate rate of labor-force growth. Substituting (11) in (12) yields

$$\dot{k} = sf(k) - (1-s)[\theta + f'(k)]\, m - nk. \tag{13}$$

From the definition of m

$$\dot{m}/m = f'(k) + \theta - n. \tag{14}$$

by logarithmic differentiation. If $n - \theta > 0$ and $f(\cdot)$ is regular and 'Inada', then there exist stationary solutions to (14) defining the unique capital intensity, k^*, by

$$f'(k^*) = n - \theta > 0.$$

[3] This is the model reported in Shell et al. [7].

This is indicated in the phase diagram of figure 2. For $k>k^*$, $\dot{m}<0$; for $k>k^*$, $\dot{m}<0$. Set the LHS of (13) to zero and substitute for k^*, showing the long-run equilibrium (m^*,k^*) is unique since

$$m^* = \frac{sf(k^*)-nk^*}{(1-s)m} .$$

If $m=0$, $\dot{k}=0$ iff $sf(k)=nk$. Denote $(0,k^{**})$ as this Solow point. The full dynamics are shown in figure 2.

Once again – a saddlepoint! But this time there are some important differences. Paths that diverge to the southeast are perfectly good competitive equilibrium paths ultimately settling down to the no-money equilibrium, k^{**}.

Paths diverging to the northwest are more complicated. With m large, demand for consumption finally exhausts output (along, say, the dashed curve) and the consumption price of capital must fall, i.e., $\dot{p}<0$. Eventually $p<1$ falls faster than a constant absolute rate. In finite time, $p=0$, so with free disposal, $\dot{p}\geqslant 0$ and eq. (10) cannot hold.

In the monetary economy, disequilibrating disturbances from (m^*, k^*) *will have one of two effects*: (1) *readjustment to a non-monetary equilibrium, or* (2) *a runaway high-monetization speculative bubble that must burst in finite time.*

The reason that there are two equilibria – (m^*, k^*) and $(0,k^{**})$ – is that 'money' was assumed to be unnecessary for production, portfolio balance, or

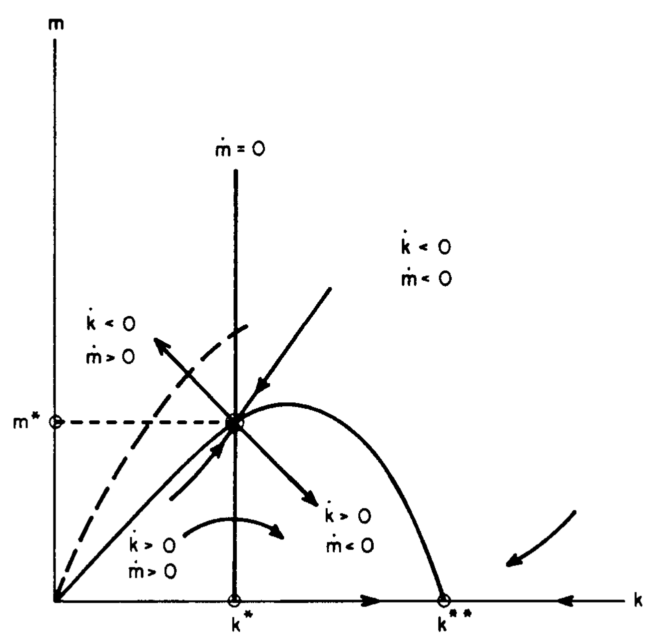

Fig. 2.

transactions. If, on the other hand, the demand for money is given by

$$f'(k)-\dot{p}_m/p_m = L(m/k),$$

where $L(\cdot)$ is the 'liquidity preference' function and $p\equiv 1$, we have under reasonable assumptions on the shape of $L(\cdot)$, that the unique equilibrium is a saddlepoint and paths not converging to equilibrium have either a zero price of money or a zero price of capital in finite time. The student should satisfy himself that this assertion is correct. In figure 2, the vertical $\dot{m}$=0 locus is replaced by a curve with positive slope.

Next consider the magical properties of money. To make matters concrete, consider a special form of the Samuelson consumption-loan model. [4] Individuals live for two periods. In each generation there is only one person equivalently n persons). In each period there are two persons (equivalently $2n$ (persons) alive. The population growth rate is zero. Let the individual born in year t be called the tth individual – he is alive in periods t and t+1. Nature endows him with one chocolate in period t and one chocolate in period t+1. Assume the simple utility function

$$u^t(c_t^t, c_{t+1}^t) = c_t^t + c_{t+1}^t,$$

or any transform in which indifference curves have slopes of minus unity. $u^t(\cdot)$ (t=0, 1, . . .) is the utility of man t, c_s^t is consumption of chocolate by man t in period s (s = 1, 2, ...). Assume for convenience that chocolates are perfectly non-durable.

For this exchange economy, the zero-interest-rate price system is clearly a competitive-equilibrium system. If the interest rate is zero,

$$p_t = p_1 \equiv 1 \ (t{=}1, 2, \ldots.),$$

where p_t is the price of chocolate in period t, and first-period chocolate is the numeraire. The budget constraint for individual t is

$$p_t c_t^t + p_{t+1} c_{t+1}^t \leqslant p_t + p_{t+1}.$$

Since $p_{t+1} = p_t = 1$ and there is no satiation of consumption,

$$c_t^t + c_{t+1}^t = 2.$$

[4] Based on Shell [5].

Because chocolates cannot be stored,

$$c_{t+1}^{t} + c_{t+1}^{t+1} = 2.$$

Combining gives

$$c_t^t = c_{t+1}^{t+1} .$$

But the *ur*-father (man zero) does not benefit from trade; $c_1^0 = 1$. In equilibrium there is no trade,

$$c_1^0 = c_t^t = c_{t+1}^t = 1 \quad t=1, 2, \ldots,$$

$$c_s^t = 0 \text{ otherwise.}$$

Although the autarchic solution is competitive, it is not Pareto-optimal. If in period 1 a chocolate is transferred from man 1 to man 0, and in period 2 a chocolate is transferred from man 2 to man 1, and in period 3 a chocolate is transferred from man 3 to man 2, the *ur*-father has been made better off while no man has been harmed – utility is held at 2 for each succeeding man.

Assume that the *ur*-father declares that the paper wrapper on *his* chocolate is money. Let π_t be the current chocolate price of money. The competitive equilibrium in the money-chocolate economy is Pareto-optimal if $\lim_{t\to\infty} \pi_t = 1$. The money economy can mimic (in a *competitive* fashion) our imposed allocation where each man (except man zero) passes exactly one chocolate backward, by setting

$$p_t = \pi_t = 1 \ (t=1, 2, \ldots) .$$

Figure 3 may be of some help in visualizing the chocolate game. Units in the matrix represent initial chocolate endowment and its distribution. These units also represent the competitive allocation without money. The arrows show how a Pareto-superior allocation can be imposed – forcing sons to pass their chocolates to their fathers. This is also a competitive and Pareto-optimal solution for the 'monetary' economy in which $p_t = \pi_t = 1$ for $t=1, 2, \ldots$.

Individual	Period 1	2	3	4	5	•
0	1	0	0	0	0	•
1	1	1	0	0	0	•
2	0	1	1	0	0	•
3	0	0	1	1	0	0
4	0	0	0	1	1	0
5	0	0	0	0	1	1
•	•	•	•	•	•	1
•	•	•	•	•	•	0

Fig. 3.

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