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Further Evidence of the Necessity of Sunspots

by

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Abstract

We describe some static economies for which a (certainty) Walrasian equilibrium does not exist but a sunspot equilibrium does exist. Our work depends on the indivisibility of a freely produced good in economies with similar consumers.

1 Introduction

Sunspot equilibria are equilibria in which allocations depend on the outcome of an extrinsic random variable, or sunspot. When the sunspot equilibrium concept was first introduced, see Shell [4] and Cass and Shell [1] as the seminal references, the excitement centered on the discovery of new equilibria that where previously missed by ignoring extrinsic uncertainty. Competitive equilibria, that existed before the introduction of extrinsic uncertainty, reemerged as degenerate sunspot equilibria. More recently, Shell and Wright [5] demonstrate that in economies with non-convex consumption sets, nondegenerate sunspot equilibria can exist that Pareto dominate certainty competitive allocations, and that certainty competitive equilibria do not necessarily reappear as degenerate sunspot equilibria once extrinsic uncertainty is introduced. Also, Guesnerie and Laffont [2] give an example of an economy, where one consumer's utility function is not concave, for which there exists a unique sunspot equilibrium that Pareto dominates the certainty competitive equilibrium. Finally, Pietra [3] shows that for intertemporal financial economies with incomplete markets, sunspot equilibria may be the *only* possible equilibria.

Here we describe some static economies for which a (certainty) Walrasian equilibrium does not exist but a sunspot equilibrium does exist. The economies studied are the same as those considered in Shell and Wright [5]. The economies are such that, before the introduction of extrinsic uncertainty, consumers have identical demands. However, due to the presence of an indivisible commodity a symmetric equilibrium solution is not possible. With the introduction of extrinsic uncertainty, however, consumers are able to receive allocations that are ex-ante identical before sunspots are revealed.

2 Model and Results

The economies studied have two consumers and two commodities. The first commodity is divisible and there is one unit of the commodity which is held as endowments by the two consumers. The second commodity is indivisible and is produced freely by a firm owned by the two consumers. The firm's production of commodity 2, is denoted by $y \in Y = \{0, 1\}$.

Denote e_1^i to be i's endowment of the divisible commodity (commodity 1) and let T_2^i be i's ownership (i.e., i's share of the profits, to be denoted by Π) of the firm. For simplicity, we assume

$$e_1^i = \frac{1}{2},\ T_2^i = \frac{1}{2},\ i = 1, 2.$$

However, this specification of endowments and ownership of the firm may be generalized to any points $(e_1^i, T_2^i) >> 0$, $i = 1, 2$, satisfying $e_1^i + T_2^i = 1$ and $e_1^1 + e_1^2 = T_2^1 + T_2^2 = 1$.

The consumption set of consumers is given by

$$X^i = \left\{x^i \in \Re^2 : x_1^i \in [0, 1],\ x_2^i \in \{0, 1\}\right\}.$$

Preferences for the two consumers are defined over elements in their consumption sets by separable utility functions

$$U^i(x_1^i, x_2^i) = u^i(x_1^i) + v^i(x_2^i),$$

where u^i is a strictly increasing, strictly concave function from $[0,1] \to \Re$, v^i is a function from $\{0,1\} \to \Re$ and u^i, v^i satisfy

$$(i)\ u^i(0) = 0,\ v^i(0) = 0,\ i = 1,2,$$

$$(ii)\ 0 < u^i(1) < v^i(1),\ i = 1,2,$$

and

$$(iii)\ u^i(\frac{1}{2}) \geq \frac{1}{2}v^i(1),\ i = 1,2.$$

These conditions define economies which differ according to the specifications of u^i and v^i. Condition (iii) is not required for the non-existence of a Walrasian equilibrium (shown below). Its importance will become clear in the next section where extrinsic uncertainty is introduced.

Let prices be represented by vectors $p = (p_1, p_2) \in \Re^2_+$. Consumers solve

$$\max_{x^i} U^i(x^i_1, x^i_2) \tag{1}$$

$$\text{subject to } p_1 x^i_1 + p_2 x^i_2 \leq p_1 e^i_1 + T^i_2 \Pi = \frac{p_1}{2} + \frac{\Pi}{2} \tag{2}$$

$$x^i \in X^i. \tag{3}$$

Since production is free the profit of the firm is equal to $p_2\, y_2$, i.e., $\Pi = p_2\, y_2$. A *Walrasian equilibrium (WE)* is an allocation (x^{i*}), a production y^*_2 and a price vector p^* satisfying (a) x^{i*} solves equations (1)-(3) for $i = 1,2$, (b) y^*_2 solves max $p^*_2 y_2$ subject to $y_2 \in Y$, and (c) $\sum_{i=1}^{2} x^{i*} \leq (1, y^*_2)$.

Claim 1 *There does not exist a (certainty) Walrasian equilibrium for these economies.*

Proof. Set $p_2 = 1$. It is trivial that $y^*_2 = 1$. Suppose $p_1 < 1$, then neither consumer can afford one unit of the indivisible commodity so each consumer will demand $x^i_1 = 1$ which is not feasible. Suppose $p_1 \geq 1$. Then, by condition (ii), each consumer will demand $x^i_2 = 1$ (and $x^i_1 \geq 0$) which is not feasible. □

Now introduce a set of extrinsic states of nature with two states; $S = \{\alpha, \beta\}$. Suppose probabilities of the two states are given by $\pi(\alpha) = \pi(\beta) = \frac{1}{2}$.

With trade in state contingent commodities the consumption set of consumers becomes

$$\hat{X}^i = \left\{\hat{x}^i \in \Re^4 : \hat{x}_1^i(s) \in [0,1],\ \hat{x}_2^i(s) \in \{0,1\},\ s = \alpha, \beta\right\}.$$

Meanwhile, the production set of the firm is redefined as

$$\hat{Y} = \left\{\hat{y}_2 \in \Re^2 : \hat{y}_2(s) \in \{0,1\},\ s = \alpha, \beta\right\}.$$

State contingent prices are denoted by $\hat{p} = (\hat{p}_1(\alpha), \hat{p}_2(\alpha), \hat{p}_1(\beta), \hat{p}_2(\beta))$.

The expected utility of consumer i is written:

$$\hat{U}^i(\hat{x}^i) = \pi(\alpha)\left(u^i(\hat{x}_1^i(\alpha)) + v^i(\hat{x}_2^i(\alpha))\right) + \pi(\beta)\left(u^i(\hat{x}_1^i(\beta)) + v^i(\hat{x}_2^i(\beta))\right),$$

$i = 1, 2$. The assumptions on u^i and v^i are maintained. Endowments and ownership of the firm do not depend on the state of nature. Thus

$$\hat{e}_1^i(\alpha) = \hat{e}_1^i(\beta) = \frac{1}{2} \text{ and } \hat{T}_2^i(\alpha) = \hat{T}_2^i(\beta) = \frac{1}{2},$$

$i = 1, 2$. The consumer solves

$$\max_{\hat{x}^i} \hat{U}^i(\hat{x}_1^i, \hat{x}_2^i) \tag{4}$$

$$\text{subject to } \sum_{s=\alpha,\beta} \hat{p}_1(s)\hat{x}_1^i(s) + \hat{p}_2(s)\hat{x}_2^i(s) \leq \sum_{s=\alpha,\beta} \frac{\hat{p}_1(s)}{2} + \frac{\hat{\Pi}(s)}{2} \tag{5}$$

$$\hat{x}^i \in \hat{X}^i \tag{6}$$

where $\hat{\Pi}(s) = \hat{p}_2(s)\hat{y}_2(s)$ denotes profits in state $s = \alpha, \beta$. A *sunspot equilibrium (SE)* is an allocation $(\hat{x}^{i*})$, a production $\hat{y}_2^*$, and a price vector $\hat{p}^*$ satisfying (a) $\hat{x}^{i*}$ solves equations (4)-(6) for $i = 1, 2$, (b) $\hat{y}_2^*$ maximizes $\sum_{s=\alpha,\beta} \hat{p}_2^*(s)\hat{y}_2(s)$ subject to $\hat{y}_2(s) \in Y$, and (c) $\sum_{i=1}^{2} \hat{x}^{i*}(s) \leq (1, \hat{y}_2^*(s))$, $s = \alpha, \beta$.

Claim 2 *There exists a sunspot equilibrium for each of these economies.*

Proof. Let $(\hat{p}_1^*(\alpha), \hat{p}_2^*(\alpha), \hat{p}_1^*(\beta), \hat{p}_2^*(\beta)) = (1,1,1,1)$. It is trivial that $\hat{y}_2^*(\alpha) = \hat{y}_2^*(\beta) = 1$. Consider the allocation

$$\hat{x}^{i*} = (\hat{x}_1^{i*}(\alpha), \hat{x}_2^{i*}(\alpha), \hat{x}_1^{i*}(\beta), \hat{x}_2^{i*}(\beta))$$
$$= \begin{cases} (\frac{1}{2}, 0, \frac{1}{2}, 1) \text{ if } i = 1 \\ (\frac{1}{2}, 1, \frac{1}{2}, 0) \text{ if } i = 2 \end{cases}.$$

Clearly this allocation is affordable by each consumer i. It remains to show that this allocation is (weakly) preferred by each consumer i to all other affordable consumption bundles in $\hat{X}^i$, at these prices.

First, consider consumption bundles of the types

$$\hat{x}' = (a, 0, b, 1) \text{ and } \hat{x}'' = (a, 1, b, 0),$$

where $a + b = 1$. Consumption bundles of these types are preferred to similar consumption bundles with $a + b < 1$, since u^i is strictly increasing. By the strict concavity of u^i, $\hat{U}^i(\hat{x}^{i*}) \geq \hat{U}^i(\hat{x}') = \hat{U}^i(\hat{x}'')$ for all a and b such that $a + b = 1$ and for $i = 1, 2$.

Second, the consumption bundle

$$\hat{x}''' = (0, 1, 0, 1)$$

is strictly preferred by each consumer i to

$$\hat{x}'''' = (1, 0, 1, 0)$$

since $v^i(1) > u^i(1)$ (Condition (ii)).

Finally, $\hat{U}^i(\hat{x}^{i*}) \geq \hat{U}^i(\hat{x}''')$, for each consumer i, by condition (iii). □

3 Parametric Representation of Consumer Preferences

For the economies considered in the previous section set $v^i(1) = 1 + c^i$, $c^i > 0$ and $u^i(x_1^i) = (x_1^i)^{d^i}$, $0 < d^i < 1$, $i = 1, 2$. By condition (iii) we require

$$\frac{1}{2^{d^i}} > \frac{1 + c^i}{2}.$$

It follows that

$$d^i \ \ln(\frac{1}{2}) > \ln(\frac{1}{2}) + \ln(1 + c^i)$$

or

$$d^i < 1 + \frac{\ln(1 + c^i)}{\ln(\frac{1}{2})}.$$

Thus, Claims 1 and 2 hold for any preferences in the sets

$$D^i = \left\{ (c^i, d^i) \in \Re^2 : c^i > 0, 0 < d^i < 1, d^i < 1 + \frac{\ln(1 + c^i)}{\ln(\frac{1}{2})} \right\},$$

$i = 1, 2$. The sets D^1 and D^2 are open and non-empty (See Figure 1).

References

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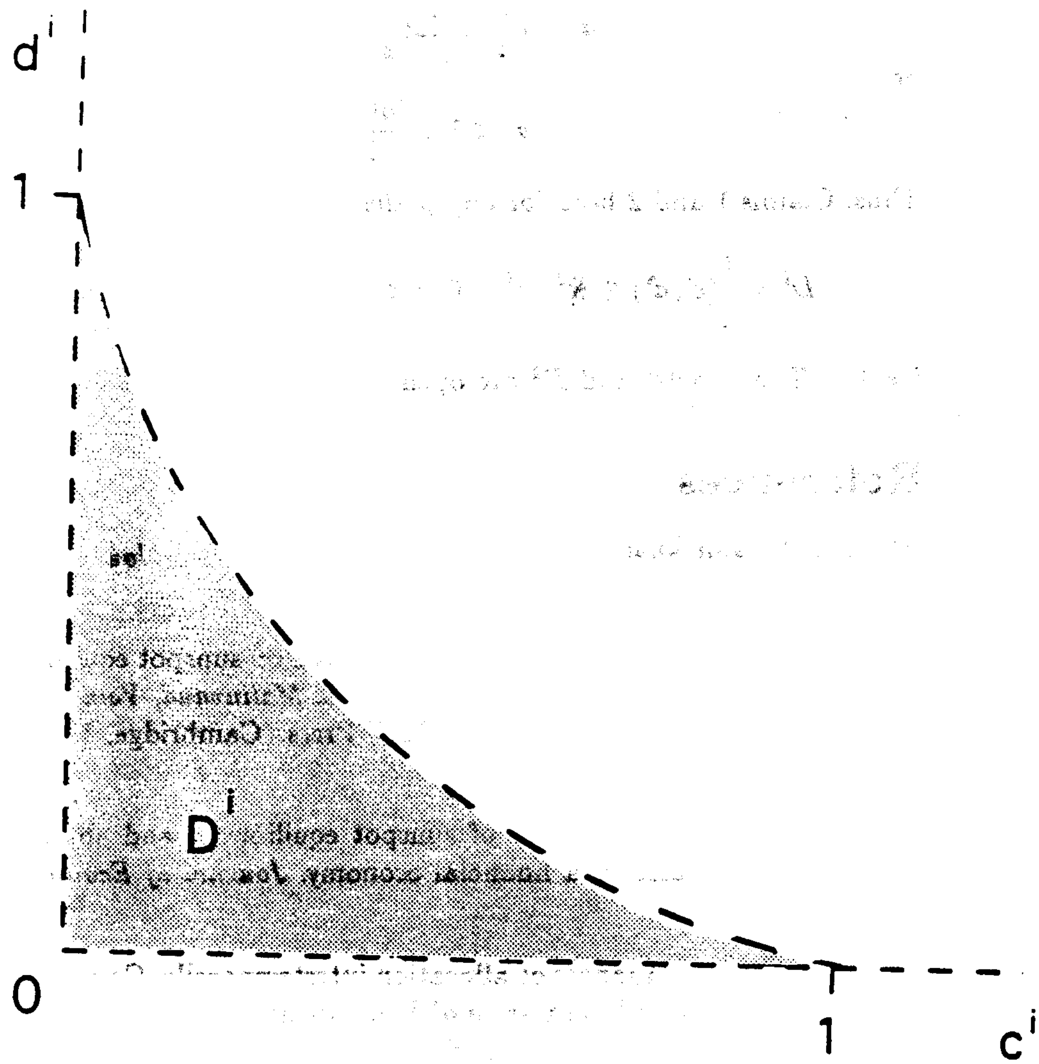

Figure 1