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Introduction to Hamiltonian Dynamics in Economics

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Economics during the fifties and sixties was marked by a substantial resurgence of interest in the theory of capital. While the advances during this period were very impressive, there was also an uneveness in the development of the subject. One-good models were studied in detail, as were many-good models of production–maximal growth and many-good models of consumption–optimal growth for the special case in which there is no social impatience. When treating heterogeneous capital, the literatures on decentralized or descriptive growth and consumption–optimal growth with positive time discounting were dominated by special cases and examples.

Reliance on examples and special cases proved to have some unfortunate consequences. The Battle of the Two Cambridges, ostensibly an argument over approaches to modeling distribution and accumulation, often seemed to focus on the robustness (or lack of robustness) of certain "fundamental" properties of the one-sector model and other worked-out examples when extended to more general heterogeneous-capital models. Furthermore, in large part because growth theory appeared to be an enterprise based only on proliferating special cases, the attention of the young able minds in the profession turned elsewhere, for example, to the—at least seemingly—more evenly-developed general equilibrium tradition.

This is a shame. Intertemporal allocation and its relationship with the wealth of societies is one of the most important problems in our discipline. Growth models are natural vehicles for the study of what is called "temporary equilibrium." Dynamic models of multi-asset accumulation provide the theoretically most satisfactory environment for modeling the macroeconomics of income determination, employment, and inflation.

The papers in this volume can be thought of as attempts at providing some unification of the theory of heterogeneous capital. The major topic, although by no means the only one, which we investigate here is the stability of long-run steady-state equilibrium in models of heterogeneous capital accumulation. However, the basic techniques used in our investigations have wide application to economic dynamics in all its manifestations. This follows from the observation that the economic growth models which we treat each belong to a class of general intertemporal economic models which is essentially representable by what we call a Hamiltonian dynamical system.

The static technology of our growth models can in general be described by an instantaneous technology set, T, with feasible production satisfying

$$(c, z, -k, -l) \in T \subset \{(c, -k, -l) : (c, k, l) \geqslant 0\},$$

where c denotes the vector of consumption-goods outputs, z the vector of net investment-goods outputs, k the vector of capital-goods inputs, and l the vector of primary-goods inputs. There is an alternative representation of static technological opportunities that is more congenial to dynamic analysis, i.e., the representation of the technology by its Hamiltonian function.

Let p be the vector of consumption-goods prices and q the vector of investment-goods prices. Define the Hamiltonian function,

$$H(p, q, k, l) = \sup_{(c', z')} \{pc' + qz' : (c', z', -k, -l) \in T\}.$$

H is typically defined on the nonnegative orthant $\{(p, q, k, l) : (p, q, k, l) \geqslant 0\}$, and is precisely interpreted as the maximized value of net national product at output prices (p, q) given input endowments (k, l).

If the set T is closed, convex, and permits free disposal of outputs, then technology is also completely characterized by a (unique) continuous Hamiltonian function which is convex and linear homogeneous in output prices (p, q), and concave in input stocks (k, l). For each such T-representation, there is a unique H-representation of technology and, conversely, for each such H-representation there is a unique T-representation of technology. While we focus on macroeconomic dynamics, the Hamiltonian function is of more general usefulness. Thus, in his paper in this volume, relating properties of the set T to properties of the function $H(p, q, k, l)$, Lau quite properly refers to the Hamiltonian function as the "restricted profit function."

Lau studies the case where technology can be described by production functions. For the case in which the production functions are twice differentiable, Lau develops identities linking the Hessians of the production function with those of the Hamiltonian (or restricted-profit) function. Lau also analyzes some special cases in which increasing returns to scale are allowed. For these cases, the Hamiltonian function may not be concave but is quasiconcave in the input stocks (k, l).

Representation of static technology by the Hamiltonian function allows us to describe all the growth models in this volume as *Hamiltonian dynamical systems*. In continuous time, $t \in [0, \infty)$, the system follows the laws of motion

$$\begin{cases} \dot{k}(t) \in \partial_q H(p(t), q(t), k(t), l(t)), \\ \dot{q}(t) \in -\partial_k H(p(t), q(t), k(t), l(t)) \end{cases} \tag{HDS}$$

where $\dot{k}(t)$ and $\dot{q}(t)$ are time derivatives of $k(t)$ and $q(t)$, respectively; $\partial_q H$ and $\partial_k H$ are respectively the subdifferentials (or generalized gradients) with respect to q and k. The first correspondence in (HDS) follows from the definition of net investment since it is equivalent to the equation $\dot{k}(t) = z(t)$. The second correspondence in (HDS) is somewhat more subtle and can be rewritten (employing static duality theory) as $\dot{q}(t) + r(t) = 0$, where $r(t)$ is the vector of competitive capital–goods rental rates. Thus, the second line in (HDS) states that the return to asset holders (including rentals and capital gains) are equal across capital goods. If there is (short-run) perfect foresight about price changes, this is the asset-market-clearing equation that follows from a simple arbitrage argument for competitive economies. The existence of prices $(p(t), q(t))$ satisfying (HDS) also follows from the necessary conditions in the optimal and maximal growth problems.

For discrete time, $t = 0, 1, \ldots$, the Hamiltonian dynamical system can be represented by

$$\begin{cases} k_{t+1} \in k_t + \partial_q H(p_t, q_t, k_t, l_t), \\ q_{t+1} \in q_t - \partial_k H(p_{t+1}, q_{t+1}, k_{t+1}, l_{t+1}). \end{cases} \tag{HDS}'$$

Line one of (HDS)′ follows from the definition of net investment, since it is equivalent to $k_{t+1} = k_t + z_t$. Line two is the (short-run) perfect-foresight, competitive, asset-market-clearing equation, since it is equivalent to $q_{t+1} - q_t + r_{t+1} = 0$, where r_{t+1} is the vector of capital–goods rentals in period $t + 1$. (HDS)′ can also be derived from the discrete-time necessary conditions in optimal and maximal growth problems.

From (HDS) and (HDS)′, we see that the properties of static technology as characterized by the Hamiltonian function will be basic to the dynamic analysis of competitive and optimizing economic dynamical systems. So far, as can be seen in (HDS) or (HDS)′, the development of consumption goods prices, $p(t)$ or p_t, and of primary factors, $l(t)$ or l_t, has been left as exogenous to the model.

The Hamiltonian dynamical system is said to be autonomous if H depends on time solely through q and k. The system would be autonomous if, for example, p and l are constant through time, i.e., $p(t) = \bar{p}$ and $l(t) = l$, or $p_t = \bar{p}$ and $l_t = l$. Autonomous Hamiltonian dynamical systems have special structure that has been exploited in the mathematics and physics literature. (For references, see [8, 10, 19]).

Let (q^*, k^*) be a rest point to the autonomous version of the Hamiltonian dynamical system so that

$$0 \in \partial_q H(\bar{p}, q^*, k^*, l),$$
$$0 \in \partial_k H(\bar{p}, q^*, k^*, l).$$

Of course, (q^*, k^*) is a rest point for the autonomous version of (HDS) if and only if it is a rest point for the parallel autonomous version of (HDS)′. Assume further that H is quadratic in (q, k) so that the (HDS) is a system of linear differential equations and (HDS)′ is a system of linear difference equations. In this case, a theorem of Poincaré tells us that if λ is a root to the characteristic equation for the autonomous version of the system (HDS) at (q^*, k^*) then $-\lambda$ is also a root. For the autonomous version of (HDS)′, the analogous theorem tells us that if λ is a characteristic root, so also is $1/\lambda$.

The Poincaré theorem, although extremely simple, is suggestive of deep fundamental results. If, for example, in the continuous-time case we could rule out λ's with zero real parts (Re $\lambda \neq 0$), then we would have established a saddlepoint result for the autonomous version of the linear Hamiltonian dynamical system: In (q, k)-phase space, the dimension of the manifold of (forward) solutions tending to (q^*, k^*) as $t \to +\infty$ is equal to the dimension of the manifold of (backward) solutions tending to (q^*, k^*) as $t \to -\infty$, and each manifold is equal in dimension to half the dimension of the phase space.

Since this "saddlepoint property" and related properties turn out to be of basic interest in dynamic economic analyses, a natural question is: Are there interesting restrictions that can be imposed on the Hamiltonian function (i.e., on technology) such that the general (nonlinear) autonomous Hamiltonian dynamical system will possess the saddlepoint property (or related properties)? Rockafellar [12], in a paper motivated by problems of economic growth (see [18]), established the saddlepoint property of the autonomous version of (HDS) for cases in which H is *strictly* concave in k and *strictly* convex in q. Rockafellar goes on to consider the (HDS) as derived from an intertemporal optimization model which can be interpreted as the problem of consumption–optimal growth with zero discounting. Here $c(t)$ is interpreted as the (scalar) current instantaneous utility at time t. Because of zero-discounting, the (scalar) price of "utility" $p(t)$ must be constant. $l(t)$ is also considered to be a constant scalar.

Global stability is established for the optimal trajectory: Given initial capital stocks $k(0) = k_0$, $\lim_{t\to\infty} k(t) = k^*$. Rockafellar's stability result had been foreshadowed, in one form or other, in the earlier literature on maximal growth and optimal growth. For continuous-time versions, see, e.g., [15, 16]. For discrete-time versions, see, e.g., [3, 9, 11].

Consider next the problem of consumption-optimal growth with positive (constant) rate of time discount, $\rho > 0$. Again, $c(t)$ or c_t must be interpreted as the (scalar) instantaneous utility at time t. Assume, for simplicity, a constant scalar fixed factor, $l(t) = 1$ or $l_t = 1$. Because of discounting, $-\dot{p}(t)/p(t) = \rho$ or $-(p_t - p_{t-1})/p_t = \rho$. In this case, (HDS) and (HDS)′ are no longer autonomous and further analysis is required. (HDS) can now be rewritten as

$$\begin{cases} \dot{k} \in \partial_Q H(Q, k), \\ \dot{Q} \in -\partial_k H(Q, k) + \rho Q \end{cases} \qquad \text{(PHDS)}$$

where $Q \equiv q/p$ is the vector of current capital-goods prices and $H(Q, k) \equiv H(1, q/p, k, 1)$ is the current value Hamiltonian. The above system can be thought of as a perturbation (by the term ρQ) of a Hamiltonian dynamical system. In discrete time, the perturbed Hamiltonian dynamical system is

$$\begin{cases} k_{t+1} \in k_t + \partial_Q H(Q_t, k_t), \\ Q_{t+1} \in Q_t - \partial_k H(Q_{t+1}, k_{t+1}) + \rho Q_t . \end{cases} \qquad \text{(PHDS)}'$$

Let (Q^*, k^*) be a steady state, or rest point to (PHDS) and (PHDS)′. Then

$$0 \in \partial_Q H(Q^*, k^*),$$
$$0 \in -\partial_k H(Q^*, k^*) + \rho Q^*.$$

One can ask the question: Do the properties of models based on (HDS) and (HDS)′, such as the stability of optimal economic growth, carry over to models based on the perturbed systems, (PHDS) and (PHDS)′, for the case of ρ positive? The simple answer is, of course, "no," as Kurz's local analysis [6] shows. Two approaches to modifying this question have been studied: (1) Papers such as that by Samuelson [16] and José Scheinkman's contribution in this volume take technology as fixed and investigate whether or not global stability properties of the optimal growth model are preserved as the discount rate ρ is changed from zero to a small positive number. (2) The approach taken by the contributions in this volume by Cass and Shell, Rockafellar, and Brock and Scheinkman can be thought of as generalizations of the Samuelson–Scheinkman approach. Conditions are sought on the geometry of the Hamiltonian function (i.e., on static technology) that suffice to preserve under (not necessarily small) perturbation the basic properties of the Hamiltonian dynamical system.

Scheinkman's paper treats local and global stability in a discrete-time, optimal growth model with an indirect, current, instantaneous utility function which is discounted at some small rate $\rho > 0$. His model therefore satisfies (PHDS)′ although he does not work directly with the Hamiltonian formalism. A critical assumption is that linearization of the Euler equations in the neighborhood of the "optimal steady state" possesses no characteristic roots with $|\lambda| = 1$. Using Radner's bounded-value loss technique, a turnpike property is established for $\rho > 0$ sufficiently small. Scheinkman's proof is in two steps. First, an ingenious argument shows that optimal paths "visit" neighborhoods of the "modified optimal steady state," (Q^*, k^*). Second, local stability follows directly from his assumption about characteristic roots.

The contribution by Cass and Shell treats optimal growth and decentralized or descriptive growth models in both continuous and discrete time as applications of Hamiltonian dynamics. They review the problem of optimal growth with zero discounting and show that a steepness condition on the Hamiltonian function (a condition weaker than Rockafellar's [12] strict convexity–concavity of the Hamiltonian) suffices to insure uniqueness of the steady-state capital vector, k^*. Furthermore, a uniform strengthening of the steepness condition insures global stability of k^* and can be thought of as a generalization of Radner's assumption of bounded value-loss. This approach relies crucially on the property that (q^*, k^*) is a saddlepoint of the Hamiltonian function $H(1, q, k, 1)$.

In the optimal-growth model with discount rate $\rho \neq 0$, the rest point (Q^*, k^*) is not a saddlepoint of $H(Q, k)$. Therefore, the Cass–Shell steepness condition must be modified by a term dependent on ρ. The modified steepness condition establishes the uniqueness of k^*. A uniform strengthening of the steepness condition insures global stability of k^* for the continuous-time, optimal growth model. Because of different effects of "interest compounding," a different steepness condition is required to establish stability in the discrete-time, optimal growth model.

While optimal growth with time-discounting yields a very simple perturbation of a Hamiltonian dynamical system, there are other perturbed Hamiltonian dynamical systems that arise in economic theory. Cass and Shell discuss the general problem of decentralized growth with instantaneously adjusted expectations about price changes. In its general form, the model allows for the interpretation of competitive growth with utility-maximizing agents. However, in such a model the form of the perturbation from Hamiltonian dynamics can be very subtle. The formal analysis in the Cass–Shell paper is thus restricted to the continuous-time version of the model in which demand for consumption is Marxian. It is shown that if in addition to the strengthened steepness assumption from zero-discount optimal growth, the Hamiltonian is such that a "capital-intensity" condition is satisfied, then decentralized growth in the Marxian decentralized model is stable.

In his first contribution to this volume, Rockafellar studies the basic mathematics of the problem which arises in continuous-time, optimal growth with positive discounting. His paper is thus a generalization of his earlier work [12], where his results follow from the strict convexity–concavity of the Hamiltonian function. In the present paper, he relies on a strengthened convexity–concavity: The Hamiltonian function, $H(Q, k)$, is assumed to be α-concave in k and β-convex in Q with $\alpha > 0, \beta > 0$, and $4\alpha\beta > \rho^2$. This convexity–concavity hypothesis is substantially stronger than the related Cass–Shell steepness condition (because Rockafellar's hypothesis ignores cross effects between Q and k), but Rockafellar goes substantially beyond the Cass–Shell paper in the range of results established for this model. In addition to local and global stability results, Rockafellar establishes the existence of solutions to optimizing programs satisfying (PHDS). He shows directly that the saddlepoint property holds when his convexity–concavity hypothesis is satisfied: The dimension of the manifold of solutions to (PHDS) tending to (Q^*, k^*) as $t \to +\infty$ is equal to the dimension of the manifold of solutions to (PHDS) tending to (Q^*, k^*) as $t \to -\infty$. Each manifold has dimension equal to half the dimension of the (Q, k) phase space. Rockafellar goes on to develop deep results on intertemporal duality, which have important bearing on the economic theory of the asymptotic behavior of $(Q(t), k(t))$.

Gaines' paper in this volume is a study of the existence of solutions to the full dynamical system arising in the time-discounted, optimal growth problem. He follows the Cass–Shell formulation and appends to (PHDS) appropriate nonnegativity conditions and boundary conditions, an initial capital–stock endowment condition and a transversality condition. The full system is thus

$$\begin{cases} \dot{k} \in \partial_Q H(Q, k), \\ \dot{Q} \in -\partial_k H(Q, k) + \rho Q, \\ Q(t) \geqslant 0, \\ k(t) \geqslant 0, \\ k(0) = k_0 \geqslant 0, \\ \lim_{t \to +\infty} Q(t)\, e^{-\rho t} k(t) = 0. \end{cases} \qquad \text{(FPHDS)}$$

Gaines assumes only that H is convex in Q and concave in k, a much weaker assumption than Rockafellar's curvature assumption, but his methods require exploitation of other structural features of the Hamiltonian function. In the spirit of the Cass–Shell paper, he assumes (1) the existence of a stationary point (Q^*, k^*), (2) limitation of feasible output by primary factors, (3) productivity of technology, (4) bounded rate of depreciation, and (5) free disposal in allocation. From these conditions, a priori bounds are developed for solutions to a finite-time problem associated with (FPHDS). The proof of existence relies on a continuation principle from the theory of fixed points for multivalued mappings in infinite-dimensional spaces.

Both the Cass–Shell and Rockafellar papers in this volume make heavy analytic use of monotonicity properties of the Lyapunov function $V = (Q - Q^*)(k - k^*)$ in studying optimal growth (as did, for example, the Samuelson paper [16]). The contribution to this volume by Brock and Scheinkman also investigates stability of optimal growth in continuous-time models and develops "local" properties of the Hamiltonian function that insure monotonicity of V or some alternative Lyapunov function, and thus insure stability of optimal growth for bounded trajectories. They assume throughout that the Hamiltonian function is twice-continuously differentiable. Much of their analysis is based on what they call the "curvature matrix"

$$C = \begin{bmatrix} H_{QQ} & (\rho/2)I \\ (\rho/2)I & -H_{kk} \end{bmatrix},$$

where H_{QQ} is the matrix of cross partial derivatives with respect to current prices and H_{kk} is the matrix of cross partial derivatives with respect to capital stocks.

Brock and Scheinkman show that if the quadratic $(\dot{Q}, \dot{k})\, C(\dot{Q}, \dot{k})$ is positive except when $(\dot{Q}, \dot{k}) = 0$, then every bounded trajectory satisfying (PHDS) converges to a rest point. Their result is obtained by using the Lyapunov function $\dot{Q}\dot{k}$. In their paper, it is also shown that if $(Q - Q^*)\dot{k} + (k - k^*)\dot{Q} = 0 \Rightarrow (Q - Q^*, k - k^*)C(Q - Q^*, k - k^*) > 0$ and if the matrix C evaluated at (Q^*, k^*) is positive definite, then every bounded trajectory satisfying (PHDS) converges to (Q^*, k^*). The last conditions imply the Cass–Shell steepness condition, and thus insure that the Lyapunov function $V = (Q - Q^*)(k - k^*)$ is monotonically increasing for trajectories from (PHDS).

By convexity–concavity of H, the matrices H_{QQ} and $-H_{kk}$ are positive semidefinite. Let α be the smallest eigenvalue of H_{QQ} and β be the smallest eigenvalue of $-H_{kk}$. For the special case of twice continuous differentiability, the Rockafellar curvature condition is that $4\alpha\beta > \rho^2$, which insures the positive definiteness of the matrix C. While the matrix C, when it is calculable, does provide a measure of "curvature" of H, it does not allow for the more general interpretation of Hamiltonian steepness since information about the interaction terms between the Q's and the k's, representable for the differentiable case by the matrix of cross partials H_{Qk}, are ignored in the C matrix.

In his second contribution to this volume (and our final paper), Rockafellar offers an (apparently weaker) alternative to the Cass–Shell steepness condition. With the Rockafellar growth condition, which is very much like the steepness condition, trajectories satisfying (PHDS) and the transversality condition $\lim_{t\to\infty} (Q(t) - Q^*)(k(t) - k^*) = 0$ are shown to converge to (Q^*, k^*).

The present volume has been in planning and preparation for more time than we would like to admit. Extensive (and often intense) communication among the authors was stimulated by two fruitful conferences—one at the Minary Center, Squam Lake, New Hampshire, the other at the University of Pennsylvania, Philadelphia. We are grateful to the Mathematical Social Science Board for its generous and understanding support of these opportunities for scholarly interaction.

While the topics treated here have mostly to do with the theory of economic growth, in particular, with existence and stability of macroeconomic growth, we hope you will be able to read between the lines and appreciate the potential usefulness to economic theory of what we call the Hamiltonian approach to economic dynamics.

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