UNITED NATIONS INDUSTRIAL DEVELOPMENT ORGANIZATION, VIENNA

INDUSTRIAL PLANNING AND PROGRAMMING SERIES NO. 3

PLANNING FOR ADVANCED SKILLS AND TECHNOLOGIES

Studies presented at the Ad Hoc Meeting of Experts on the Role of Advanced Skills and Technologies in Industrial Development, New York, 22—29 May 1967

UNITED NATIONS
New York, 1969

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TECHNOLOGICAL KNOWLEDGE AND ECONOMIC GROWTH

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EMPIRICAL LITERATURE on the sources of economic growth has convinced the contemporary economist of the important role played by "technical progress" in the process of economic development.[1] Only to have models of economic growth in which such an important factor is treated as exogenous is certainly unsatisfactory, and it was because of this that the models of Kaldor and particularly of Arrow were introduced.[2] Both these models may be thought of as positing that the production of new technical knowledge (invention) and its transmission (innovation) are social by-products of the production and adoption of new capital goods.

While it is no doubt true that technical change is related to gross investment both as a by-product of capital goods production and as a vehicle for embodying new techniques in new capital equipment, it is also true that the rate of production of technical knowledge can be increased by increasing the allocation of economic resources explicitly devoted to inventive activity. Elsewhere,[3] I have treated models in which the level of technical knowledge is increased by the explicit allocation of economic resources to inventive activity. My model is much like a heterogeneous capital-goods model except that for the enterprise economy the stock of technical knowledge enters as a pure public good of production. Such a specification is suggested by the very low cost of transmission of technical knowledge with respect to its production cost.[4] For the enterprise economy, therefore, the specification of the form of the production functions which is consistent with the usual competitive hypothesis is altogether different in the inventive activity model from the consistent specification in the ordinary heterogeneous capital-goods model.

As I see it, endogenous technical change models of the heterogeneous capital-goods type represent one of the promising first steps in this area. The currently available models of this type, however, suffer from two severe short-comings: *(a)* in none of them does uncertainty enter in an intrinsic fashion; *(b)* the manner in which technical knowledge enters production functions is specified in advance in these models, no economic choice being left between say labour-saving or capital-saving inventive activity.

There are probably two distinguishing qualities of the commodity "technical knowledge". The first is its lack of appropriability: it can be used by many economic units without altering its character. It is this quality which makes social intervention in the enterprise economy on behalf of inventive activity so desirable. This quality of technical knowledge and its implications for social policy are stressed in my two papers just mentioned. The second distinctive quality of technical knowledge is the riskiness of its production. This uncertainty is treated in my papers in a way that removes all the inherent difficulties by assuming that, given factor allocations to the inventive sector, the dispersion of aggregative benefits from invention about its mean is zero, so that under social planning no risk remains.

Recently there have been a number of contributions to the literature of economic growth employing a construct such as the invention pos-

[1] See R. W. SOLOW, "Technical Change and the Aggregate Production Function", *Review of Economics and Statistics*, Vol. 37, 1957; J. W. KENDRICK, *Productivity Trends in the United States*, Princeton, 1961; and E. F. DENNISON, *The Sources of Growth in the United States and the Alternatives Before Us*, Supplementary Paper No. 13, Committee for Economic Development, New York, 1962.

[2] N. KALDOR, "A Model of Economic Growth", *Economic Journal*, Dec. 1957, pp. 591–624; and K. J. ARROW, "The Economic Implications of Learning by Doing", *Review of Economic Studies*, June 1962, pp. 155–173.

[3] K. SHELL, "Toward a Theory of Inventive Activity and Capital Accumulation", *American Economic Review*, Vol. 56, May 1966, pp. 62–69; and "A Model of Inventive Activity and Capital Accumulation", in K. SHELL (ed.), *Essays on the Theory of Optimal Economic Growth*, M.I.T. Press, forthcoming.

[4] In a provocative unpublished paper, "A Theory of Invention in the Firm", W. D. NORDHAUS has extended the analysis to include the case in which, although "spill-overs" of technical information are present, the firm is able to internalize some of the returns from inventive activity.

sibility set (or the innovation possibility set).[5] In these models the invention possibility set is given at every instant of time and is independent of economic variables. From feasible points in the invention possibility set, firms or planners are free to choose optimal combinations of capital-augmenting and labour-augmenting technical progress.

As a prologue to the theory of endogenous technical change the next section deals with models of stylized enterprise economies and planning models in which there is exogenous technical change. In section 2 I treat three different models of education and technological change, drawing inspiration from the work of Uzawa, and Nelson and Phelps. The first model of education specifies that changes in labour-force efficiency are dependent upon the fraction of the labour force engaged in educational activity. The second model specifies that changes in labour force efficiency are dependent upon the gap between exogenously determined "available technology" and "technology in practice" and upon the "educational attainment" of the society. These two views are then integrated with the specification that changes in educational attainment are dependent upon the fraction of the labour force employed in the educational sector.

In section 3 I question the rigid assumption that technical change affects the production function in a pre-specified way that is not subject to economic calculation. I examine a planning model inspired by Samuelson and Nordhaus, in which the bias of technical change is open to choice by the planning authority. Certain turnpike properties are exhibited. The difficulties inherent in extending the analysis to enterprise economies are discussed.

1. Growth models with exogenous technical change

In a classic paper Robert Solow[6] studied the long-run dynamic behaviour of a simple economy in which the current rate of production of homogeneous output $Y(t)$ depends upon the current stock of physical capital $K(t)$ and the current size of the labour force $L(t)$ inelastically offered for employment,

$$Y(t) = \varphi[K(t), L(t); t], \tag{1.1}$$

[5] See W. D. Nordhaus, "The Optimal Rate and Direction of Technical Change", to appear in *Essays on the Theory of Optimal Economic Growth*, op.cit.

[6] R. M. Solow, "A Contribution to the Theory of Economic Growth", *Quarterly Journal of Economics*, Vol. 32, 1956, pp. 65–94 .

or in the particular case of Hicks-neutral technical change

$$Y(t) = A(t) F[K(t), L(t)], \tag{1.1'}$$

where $A(t)$ may be thought of as a measure of the current stock of technical knowledge. Solow assumes that there is no depreciation in physical capital, so that current output can be split into current consumption $C(t)$ plus current investment $Z(t)$, and that

$$Y(t) = C(t) + Z(t) = C(t) + \dot{K}(t). \tag{1.2}$$

If a constant fraction $0 < s < 1$ of output is saved and invested, then the differential equation of capital accumulation is

$$\dot{K}(t) = sY(t). \tag{1.3}$$

Solow further assumes that the labour force grows at the constant relative rate n so that

$$\dot{L}(t) = nL(t). \tag{1.4}$$

If lower-case letters denote quantities per worker then the system (1.1)–(1.4) can be rewritten as

$$y(t) = A(t) f[k(t)] \equiv A(t) F[k(t), 1], \tag{1.1*}$$

$$y(t) = c(t) + z(t) = c(t) + \dot{k}(t) + nk, \tag{1.2*}$$

$$\dot{k}(t) = sA(t) f[k(t)] - nk, \tag{1.3*}$$

if F is positively homogeneous of degree one (constant-returns-to-scale). First assuming the absence of technological change, $A(t) = \bar{A} > Q$, constant, it is easily shown that the Inada conditions

$$\begin{cases} f(k) > 0, f'(k) > 0, f''(k) < 0, 0 < k < \infty \\ f(0) = 0, f(\infty) = \infty \\ f'(0) = \infty, f'(\infty) = 0 \end{cases} \tag{1.5}$$

ensure that the economy (1.1)–(1.5) develops in such a way as to tend to the unique long-run balanced capital-labour ratio independent of initial conditions. If capital depreciates at the given exponential rate μ, and gross saving is a constant fraction of gross income,

$$\dot{k} = sAf(k) - \lambda k, \tag{1.6}$$

where $\lambda \equiv \mu + n$. Or if net saving is a constant fraction of net income then

$$\dot{k} = sAf(k) - \lambda' k, \tag{1.7}$$

where $\lambda' \equiv n/s + \mu$. Again (1.5) ensures that both the system (1.6) and the system (1.7) are globally stable in the sense of Lyapunov when s and A are given quantities independent of time.

For the special case that the exogenously given relative rate of increase of technical knowledge is constant

$$\dot{A} = \varrho A, \tag{1.8}$$

and the production function given in (1.1′) is Cobb-Douglas in capital and labour, so that

$$y = e^{\varrho t} k^{\alpha},\ 0 < \alpha < 1, \tag{1.9}$$

then long-run growth tends to a balanced capital labour ratio measured in efficiency units (K/AL) when net (gross) saving is a fixed fraction of net (gross) income.[7]

Next consider the problem of planning economic development in a centrally-directed economy where Hicks-neutral technical change proceeds at a constant given rate ϱ. Assume that technology and labour-force growth are as given in (1.1)—(1.5) except that at every instant the savings fraction $s(t) \in [0, 1]$ is subject to control by the planning board. As an exercise, I consider the problem of maximizing the integral of discounted per worker consumption over a given (finite or infinite planning) period.[8]

The problem is to maximize the functional

$$\int_0^T c(t)\, e^{-\delta t}\, dt \tag{1.10}$$

subject to the constraints:

$$\dot{k}(t) = s(t)\, y(t) - \lambda k(t) \tag{1.11}$$

$$y(t) = e^{\varrho t} f(k(t)) \tag{1.12}$$

$$0 \leq s(t) \leq 1 \tag{1.13}$$

$$k(0) = k_0 \text{ and } k(T) \geq k_T \tag{1.14}$$

where δ, $\lambda \equiv n + \mu$, k_0, k_T are given constants and $s(t)$ is some measurable control (or policy) variable to be chosen. Units of measurement have been chosen such that $A(0) = 1$ and therefore $A(t) = e^{\varrho t}$. k_0 is the historically given capital-labour ratio, k_T is the "target" capital-labour ratio, while $T > 0$ is the length of the planning period.

The above problem is solved by employing the "maximum principle" expounded in *The Mathematical Theory of Optimal Processes.*[9] Introduce the Hamiltonian form

$$e^{-\delta t}\{(1-s)\, e^{\varrho t} f(k) + q\, [s e^{\varrho t} f(k) - \lambda k]\} = e^{-\delta t}\{[(1-s) + qs]\, e^{\varrho t} f(k) - q\lambda k\}.$$

The application of theorem 3 of the work mentioned yields the result that if a programme $[k(t), s(t);\ 0 \leq t \leq T]$ is optimal, then there exists a continuous function $q(t)$ such that

$$\dot{k}(t) = s(t)\, e^{\varrho t} f(k(t)) - \lambda k(t) \tag{1.15}$$

with initial condition $k(0) = k_0$,

$$\dot{q}(t) = (\delta + \lambda)\, q(t) - [(1 - s(t)) + q(t)\, s(t)]\, e^{\varrho t} f'(k(t)), \tag{1.16}$$

$s(t)$ maximizes

$$[1 - s(t) + q(t)\, s(t)] \text{ subject to } 0 \leq s(t) \leq 1 \tag{1.17}$$

and s is a piece-wise continuous function of t,

$$e^{-\delta T} q(T)\, [k(T) - k_T] = 0. \tag{1.18}$$

For convenience set

$$\gamma = \max_{0 \leq s \leq 1} [(1-s) + qs] = \max(1, q).$$

Notice that $q(t)$ has the interpretation of the social demand price of a unit of investment in terms of currently foregone unit of consumption. Therefore, differential equation (1.16) may be interpreted as the requirement of perfect foresight. In a competitive economy, for example, the change in the price of a unit of capital should compensate a *rentier* for loss due to depreciation and for "abstinence", net of any rewards from the employment of that unit of capital. Transversality condition (1.18) states that at the target date either the target requirement (1.14) must hold with equality or the target demand price of investment must be zero.

Next it is required to study the singular solutions of differential equations (1.15) and (1.16). Notice that $\dot{q} = 0$ if and only if

$$q = \frac{\gamma e^{\varrho t} f'(k)}{\delta + \lambda}. \tag{1.19}$$

(1.19) reduces to

$$e^{\varrho t} f'(k_t) = \delta + \lambda \qquad \text{for case } q \geq 1, \text{ and} \tag{1.20}$$

$$q_t = \frac{e^{\varrho t} f'(k_t)}{\delta + \lambda} \qquad \text{for case } q \leq 1. \tag{1.21}$$

If the production functions satisfy (1.5) it is well known that for any instant of time, equation (1.20) is uniquely solvable in k_t. Call the solution to (1.20) k_t^*. Determination of k_t^* is shown in figure I. $\tilde{k}_t$ is the maximum sustainable capital-labour ratio when technology is held fixed.

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[7] This is natural because in the Cobb-Douglas case Hicks-neutral technical change is also Harrod-neutral and thus technical change can be thought of as labour augmenting. See H. UZAWA, "Neutral Inventions and the Stability of Growth Equilibrium", *Review of Economic Studies*, Vol. 28, No. 2, pp. 117—124.

[8] This simple example is based upon the more complicated analysis appearing in K. SHELL, "Optimal Programmes of Capital Accumulation for an Economy in which there is Exogenous Technical Change", in *Essasy on the Theory of Optimal Economic Growth*, op.cit.

[9] L. S. PONTRYAGIN, V. G. BOLTYANSKII, R. V. GAMKRILEDZE, and E. F. MISCHENKO, *The Mathematical Theory of Optimal Processes*, New York and London: Interscience Publishers, 1962. *(Theorem 3 appears on p. 63.)*

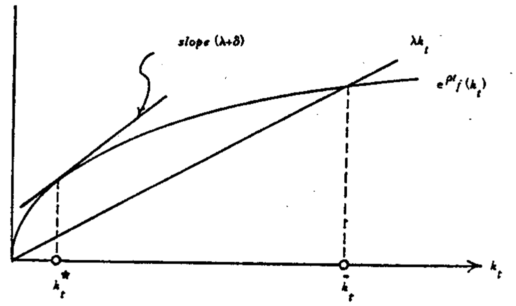

Figure I. Determination of k_t^* and $\tilde{k}_t$

It is shown that for fixed t, equation (1.19) describes a continuous curve in the (k, q) plane with a kink at $(k = k^*, q = 1)$. Differentiating (1.21) yields

$$\left.\frac{dq}{dk}\right|_{\dot{q}=0} = \frac{e^{\varrho t} f''(k)}{\delta + \lambda} < 0, \text{ for } q < 1. \qquad (1.22)$$

First we study the case of no technical change ($\varrho = 0$). The appropriate phase diagram is given in figure II. Condition (1.17) implies that for optimality it is necessary that

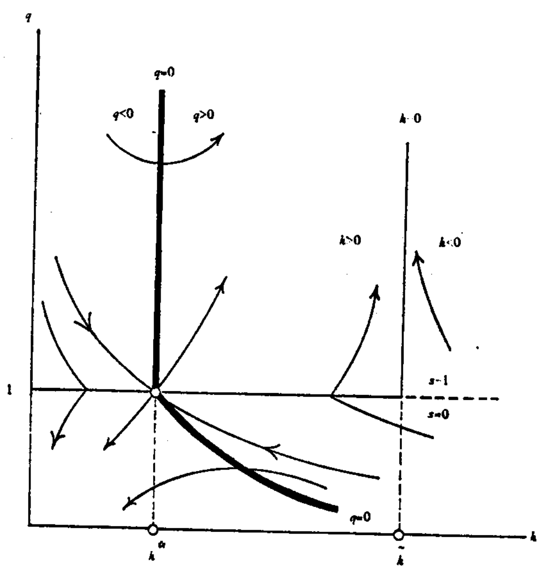

Figure II. Phase diagram for case $\varrho = 0$

$$\left\{\begin{array}{ll} s(q) = 1 & \text{when } q > 1 \\ 0 \leq s(q) \leq 1 & \text{when } q = 1 \\ s(q) = 0 & \text{when } q < 1 \end{array}\right\} \qquad (1.23)$$

Then on any given trajectory not passing through the point $(k^*, 1)$, k can be written as a continuous function of q.[10] In fact a trajectory $[k(t), q(t); t]$ not passing through $(k^*, 1)$ is uniquely determined by the specification of initial conditions $[k(t_0), q(t_0); t_0]$.

Assume for purposes of exposition that the initial capital-labour ratio is the balanced capital-labour ratio k^*, i.e. $k(0) = k^*$. Assume that the planning period is infinite, $T = \infty$, and that the target capital-labour ratio is left free. Then a programme of capital accumulation satisfying the necessary conditions is that of fixing $q(t) = 1$ for $0 \leq t \leq \infty$ and maintaining the balanced capital labour ratio $k(t) = k^*$ for $0 \leq t \leq \infty$.

For the case $\delta = 0$, the above programme

$$\left(k = k^*, \quad s = \frac{\lambda k^*}{f(k^*)}\ ; \quad 0 \leq t \leq \infty\right)$$

is what both Phelps and Robinson[11] have dubbed the golden rule of capital accumulation. For $\delta \neq 0$, this may be called the modified golden rule of capital accumulation.[12]

If $k(0) \neq k^*$, the planning board would assign initial price q_0 such that the point (k_0, q_0) lies on a trajectory that passes through $(k^*, 1)$. Let $0 \leq t^* < \infty$ be the time required for such a programme to achieve $(k^*, 1)$. Then the optimal programme is

$$\left(k = k^*, \quad s = \frac{\lambda k^*}{f(k^*)}\ ; \quad t^* \leq t \leq \infty\right).$$

The initial savings ratio is zero or one, depending upon whether the initial capital-labour ratio is greater or less than k^*.

The analysis is easily modified to handle the general case where $k(T) \geq k_T \geq 0$ and $T \leq \infty$. The

[10] By assigning the value $s(q) = 1$, the *RHS* of differential equations (1.15) and (1.16) are seen to be continuously differentiable functions of their arguments, k, q, and t, on the domain defined by $k > 0$, $q \geq 1$, $t \geq 0$. Further, by assigning the value $s(q) = 0$, the *RHS* of (1.15) and (1.16) are seen to be continuously differentiable functions of k, q, and t on the domain defined by $k > 0$, $q \leq 1$, $t \geq 0$. Thus, when the control $s(t)$ is appropriately assigned, the system (1.15) and (1.16) is shown to be trivially Lipschitzian over the respective domains of definition. By classic theorems of ordinary differential equations (see L. S. PONTRYAGIN, *Ordinary Differential Equations*, Reading, Addison-Wesley Publishing Company, 1962, pp. 159—167) we have that for a system satisfying (1.15)—(1.17) and (1.23) that specification of the parameters $(k(t_0), q(t_0); (t_0)$ uniquely determines the entire trajectory for trajectories not passing through the locus of points defined by $\{(k, q, t) \mid k = k^*(t), q = 1, t \geq 0\}$. In fact, the solutions to the system (1.15)—(1.17) vary continuously when the initial parameters $(k(t_0), q(t_0); t_0)$ are allowed to vary (see ibid., pp. 192—199).

[11] E. PHELPS, "The Golden Rule of Accumulation: A Fable for Growthmen", *American Economic Review*, Vol. 51, 1961; and J. ROBINSON, "A Neoclassical Theorem", *Review of Economic Studies*, Vol. 29, 1962.

[12] Or perhaps, "the adulterated golden rule". For $\varrho = 0$ and $T = \infty$, it is required that $\delta > 0$ in order that the value of the definite integral (1.10) be finite for all feasible programmes. For $T < \infty$, the requirement that δ be positive is too strong. Even for the case with non-zero technical change, if $\delta > f'(k_t) - \lambda$ for $t \geq 0$, then $k_t > k_t^*$. T. Koopmans, *On the Concept of Optimal Economic Growth*, Cowles Foundation (CF-30918), 1963, argues that if the ethical principle that all men are to be treated equally (independent of the size of their generation or its "timing") is held, then δ should be chosen equal to $(-n) < 0$, for the case of positive population increase. As long as $T < \infty$, our analysis is congenial to this interpretation.

initial point (k_0, q_0) is chosen on a trajectory leading to the point $(k^*, 1)$, if feasibility permits. The Pontryagin programme

$$\left(k = k^*, \quad s = \frac{\lambda k^*}{f(k^*)}\;; \quad t^* \leq t \leq t^{**}\right)$$

is followed, where t^{**} is the time at which the backward trajectory of the system (1.15)—(1.16) starting at $(k = k_T; t = T)$ passes through $(k = k^*, q = 1)$. If, however, $q(t) < 0$ for all backward trajectories to $(k^*, 1)$ starting at k_T, then t^{**} is defined to be the time at which a backward trajectory starting at time T and demand price $q(T) = 0$ intersects the point $(k^*, 1)$. Figure III illustrates a programme satisfying Pontryagin's necessary conditions.

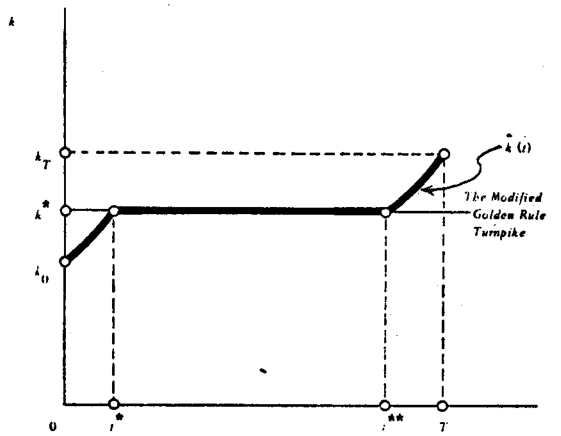

Figure III. $\hat{k}(t)$*, the Pontryagin path for case* $\varrho = 0$

Important assumptions are implicit in the construction of figure III. First, it is assumed that it is feasible for the economy with initial endowment $k(0) = k_0$ to achieve the target k_T in the specified time T. Even stronger, figure III assumes that in fact

$$T > t^{**} > t^* > 0. \tag{1.24}$$

If it is feasible to achieve the target during the planning period but (1.24) fails to hold, then the Pontryagin path is the appropriate envelope of a forward trajectory from (k_0, q_0) to $(k^*, 1)$ and the backward trajectory from (k_T, q_T) to $(k^*, 1)$. In the degenerate case in which only one feasible path exists, the Pontryagin path is, of course, a programme either of zero savings or of zero consumption. Since optimal programmes do not permit the demand price of investment to become negative, if no trajectory is found with $k(T) = k_T$ and $q(T) \geq 0$, then the Pontryagin problem will yield $q(T) = 0$ and $k(T) > k_T$.

Some observations are in order here. The linearity of the objective function (1.10) implies a kink in the graph of the stationary solutions to equation (1.16). Extending the argument presented in footnote [10], the backward solutions to the point $(k^*, 1)$ are unique. In general, however, q_0 will not be uniquely determined by (k_0, k_T, T). For the degenerate Pontryagin paths that are everywhere specialized to production of the same good, there is a family of trajectories satisfying (1.15)—(1.18). None the less, the Pontryagin programme of capital accumulation

$$[\hat{k}(t);\, 0 \leq t \leq T]$$

is uniquely determined by (1.15)—(1.18), if a feasible programme exists.

To summarize the Turnpike Property: For the case of neoclassical production without technical change, following the Pontryagin programme of capital accumulation requires the planning board to adopt the modified golden rule of capital accumulation for all but a finite amount of time. As the length of the planning period increases, the fraction of time spent on a programme not satisfying the modified golden rule approaches zero.[13]

Next, examine the case with positive technical progress $(\varrho > 0)$. Notice that if ϱ is non-zero, differential equations (1.15) and (1.16) are non-autonomous and thus the appropriate phase diagram must be drawn in three-dimensional space, (k, q, t). Time differentiation of equation (1.20) yields

$$\dot{k}_t^* = \frac{-\varrho\,(\delta + \lambda)\, e^{-\varrho t}}{f''(k_t^*)} \gtreqless 0 \text{ as } \varrho \gtreqless 0. \tag{1.25}$$

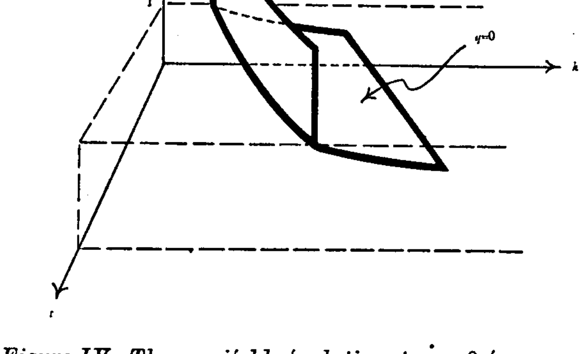

Figure IV. The manifold of solutions to $\dot{q} = 0$ *for case* $\varrho > 0$

[13] If $\delta \leq f'(\tilde{k}) - \lambda$, then the Pontryagin programme $[\hat{k}(t), 0 \leq t \leq T]$ is arbitrarily close to the ratio $\tilde{k}$ for all but a finite amount of time.

In general, stationary solutions to the differential equation

$$\dot{q}(t) = (\delta + \lambda)\, q(t) - \gamma e^{\varrho t} f'(k_t)$$

are shown to lie on a manifold embedded in (k, q, t) space. The manifold of solutions to $\dot{q} = 0$ is illustrated for ϱ positive in figure IV. The recollection that, given t, equation (1.20) has the unique solution k_t^*, suggests a programme satisfying the necessary conditions (1.15)—(1.18). Consider for ease of exposition the case when the initial condition is $k(0) = k_0^*$ and the target requirement is $k(T) = k_T^*$. A programme of capital accumulation that follows the modified golden rule turnpike is illustrated in figure V. This programme, though clearly satisfying the necessary conditions (1.15), (1.16) and (1.18), does not guarantee condition (1.17). In other words, it is not guaranteed that a programme of capital accumulation lying on the turnpike of figure V will have for $0 \leq t \leq T$, a feasible savings ratio $0 \leq s_t \leq 1$.

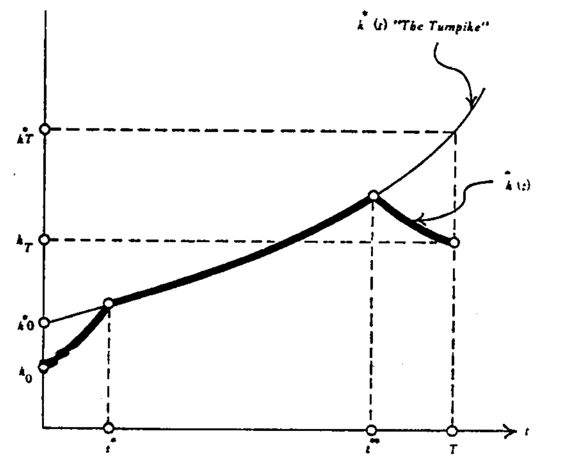

Figure V. The Turnpike when $\varrho > 0$: the Pontryagin programme of capital accumulation $\hat{k}(t)$ is shown by heavy curve

If $k_t = k_t^*$, equation (1.15) becomes

$$\dot{k}_t = s_t e^{\varrho t} f(k_t) - \lambda k_t^*. \qquad (1.26)$$

The problem is to find $s_t = s_t^*$ such that when $k_t = k_t^*$, $\dot{k}_t = \dot{k}_t^*$.

Equating $\dot{k}_t$ to $\dot{k}_t^*$ yields

$$s^* e^{\varrho t} f(k_t^*) - \lambda k_t^* = \frac{-\varrho(\delta+\lambda)\, e^{-\varrho t}}{f''(k_t^*)}$$

from (1.25) and (1.26). Or re-writing

$$s_t^* = \frac{\lambda k_t^*}{e^{\varrho t} f(k_t^*)} - \frac{\varrho(\delta+\lambda)}{e^{2\varrho t} f(k_t^*) f''(k_t^*)} > 0 \text{ for } \varrho > 0. \quad (1.27)$$

This is the common-sense result that to achieve a programme of positive capital accumulation requires a positive savings fraction. However, (1.27) does not guarantee that $s_t^* \leq 1$ for $\varrho > 0$. To see this, consider the case where the production function is linear-logarithmic in capital and labour, $y_t = e^{\varrho t} k_t^a$. Let $0 < a < 1$ so that a is capital's share of output in a competitive economy and the production function is Cobb-Douglas. For the Cobb-Douglas case

$$k_t^* = \left[\frac{a\, e^{\varrho t}}{\delta + \lambda}\right]^{1/b},$$

and

$$\dot{k}_t^* = \frac{\varrho}{b}\left[\frac{a\, e^{\varrho t}}{\delta + \lambda}\right]^{1/b},$$

where b is defined by $b = 1 - a$. For the Cobb-Douglas case, therefore,

$$s^* = \frac{a\, m}{b(\delta + \lambda)},$$

where m is defined by $m = \lambda b + \varrho$. For the Cobb-Douglas case s^* is independent of time and greater than zero, but whether s^* is less than, equal to, or greater than one depends upon the values of the parameters a, ϱ, λ, δ.[14]

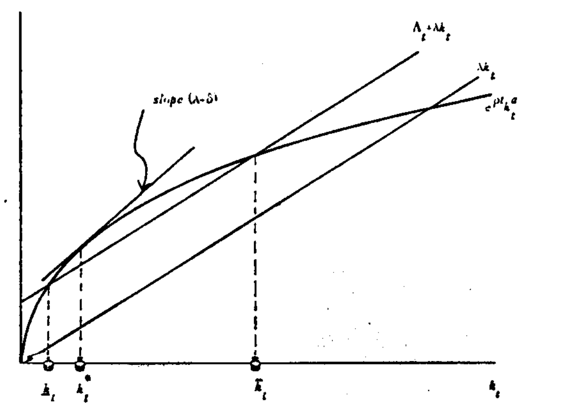

Figure VI. Determination of k_t and $\bar{k}_t$ for the Cobb-Douglas case

This can be illustrated by figure VI. It is of interest to know the sign of the expression

$$(\dot{k}_t)_{s=1} - \dot{k}_t^*, \qquad (1.28)$$

that is, to know when it is possible for the economy to grow as fast as the turnpike path. For the Cobb-Douglas case (1.28) reduces to

$$e^{\varrho t} k_t^a - [\lambda k_t + A_t],$$

where $A_t = \frac{\varrho}{b} k_t^*$. Figure VI illustrates the case where $0 < s^* < 1$. In that case setting expression

[14] Consider the "familiar economy" where $a = 0.30$, $\lambda = n + \mu = 0.10$, and $\varrho = 0.03$. If the planning board's rate of discount $\delta = 0.05$, then $s^* = 2/7 < 1$. Hence, if the "familiar economy" achieves the capital-labour ratio $k^*(t)$ at time t, then it can maintain the turnpike capital-labour ratio. It is not surprising that s^* is independent of t for Cobb-Douglas functions. Since technical change is labour-augmenting in this case, to remain on the turnpike it is required that the capital-labour ratio measured in efficiency units be held constant.

(1.28) equal to zero and holding t fixed yields an equation with exactly two positive solutions k_t and $\bar{k}_t$. If $k_t < \underline{k}_t$ the maximum (current!) growth rate for the economy is less than $\dot{k}_t^*$.

It is important to establish that if $k_0 < k^*(0)$, then $0 < t^* < \infty$. For the special Cobb-Douglas case with $0 < s^* < 1$, t^* can be calculated and shown to be finite. For the special case, (1.15) is integrated to yield

$$k = e^{-\lambda t}\left\{k_0^b - \frac{bs}{m} + \frac{bs}{m} e^{mt}\right\}^{1/b}, \qquad (1.29)$$

when $k_0 > 0$. When $k_0 \leq k^*(0)$, t^* is the root to the equation $(k)_{s=1} = k^*$. That is, solve

$$\left[\frac{b}{m} - k_0^b\right] e^{-\lambda b t} = \left[\frac{b}{m} - \frac{a}{\delta+\lambda}\right] e^{\varrho t}$$

to yield

$$t^* = \frac{1}{m}\log\left[\frac{1 - \frac{m k_0^b}{b}}{1 - s^*}\right], \text{ when } k_0 \leq k^*(0). \qquad (1.30)$$

Since $(1 - s^*)$ is assumed to be positive, t^* will be real if and only if

$$\frac{m k_0^b}{b} \leq 1.$$

But t^* will be non-negative if and only if

$$\frac{m k_0^b}{b} \leq s^* < 1. \qquad (1.31)$$

(1.31) can be re-written as

$$k_0 \leq \left[\frac{a}{\delta+\lambda}\right]^{1/b} = k^*(0). \qquad (1.32)$$

Since the logarithm is a single-valued function, (1.32) says that $t^* = 0$ when $k_0 = k^*(0)$ and that if $k_0 < k^*(0)$, then $0 < t^* < \infty$ is the unique solution to (1.30). Of course, if $k_0 > k^*(0)$, (1.30) has no non-negative solutions.

Returning to the case of general neoclassical production, an example of a Pontryagin programme of capital accumulation is presented in figure V. In drawing this figure it is implicitly assumed that $0 < t^* < t^{**} < T$ and therefore that $s^*(t) \leq 1$ for $t^* < t < t^{**}$. It is further assumed that $\hat{q}(T) \geq 0$ when $\hat{k}(T) = k_T$.

The general case where s^* changes with time presents a sophisticated mathematical difficulty. If the number of switches from $s^* < 1$ to $s^* > 1$ and vice versa is sufficiently large it may be impossible to find a piece-wise continuous control $\hat{s}(t)$ satisfying (1.15)—(1.18). If no such control exists, then no maximum to (1.10) exists.[15]

[15] If the class of admissible controls $\{0 \leq s(t) \leq 1;\ 0 \leq t \leq T\}$ is restricted to be piece-wise continuous, then a maximum to (1.10) exists if and only if the number of such switches in $[0, T]$ is finite. Therefore if s^* is an analytic function of t, then a maximum to (1.10) exists.

In the previous sections programmes satisfying necessary conditions (1.15)—(1.18) are referred to as Pontryagin programmes. It remains to show that the necessary conditions are also sufficient, and that such programmes are indeed optimal.[16]

Let $\{\hat{c}(t), \hat{z}(t), \hat{k}(t), \hat{q}(t), \ldots\}$ be a programme satisfying conditions (1.15)—(1.18). Let $\{c(t), z(t), k(t), q(t), \ldots\}$ be any feasible programme, i.e. any programme satisfying (1.11)—(1.14). It is required to show

$$\int_0^T (\hat{c} - c)\, e^{-\delta t}\, dt \geq 0. \qquad (1.33)$$

The LHS of (1.33) can be rewritten in the form

$$\int_0^T e^{-\delta t}\, dt\, \{(\hat{c} - c) + \hat{\gamma}[(e^{\varrho t} f(\hat{k}) - \hat{z} - \hat{c}) - (e^{\varrho t} f(k) - z - c)] + \hat{q}\,[(\hat{z} - \lambda\hat{k} - \dot{\hat{k}}) - (z - \lambda k - \dot{k})]\},$$

which reduces to

$$\int_0^T e^{-\delta t}\, dt\, \{(1 - \hat{\gamma})(\hat{c} - c) + (\hat{q} - \hat{\gamma})(\hat{z} - z) + \hat{\gamma} e^{\varrho t}[f(\hat{k}) - f(k)] + \hat{q}\,[\lambda(k - \hat{k}) + (\dot{k} - \dot{\hat{k}})]\}. \qquad (1.34)$$

Notice that

$$(1 - \hat{\gamma})(\hat{c} - c) \geq 0 \text{ and } (\hat{q} - \hat{\gamma})(\hat{z} - z) \geq 0.$$

Therefore (1.34) is not less than the following expression

$$\int_0^T e^{-\delta t}\, dt\, \{\hat{\gamma} e^{\varrho t}[f(\hat{k}) - f(k)] + \hat{q}\,[\lambda(k - \hat{k}) + (\dot{k} - \dot{\hat{k}})]\}. \qquad (1.35)$$

But since $f(\cdot)$ is a concave function, (1.35) is not smaller than

$$\int_0^T e^{-\delta t}\, dt\, \{\hat{\gamma} e^{\varrho t}[(\hat{k} - k) f'(\hat{k})] + \hat{q}\,[\lambda(k - \hat{k}) + (\dot{k} - \dot{\hat{k}})]\}.$$

By collecting terms the expression above yields

$$\int_0^T \hat{q} e^{-\delta t}(\dot{k} - \dot{\hat{k}})\, dt + \int_0^T e^{-\delta t}\, dt\, (\hat{k} - k)\{\hat{\gamma} e^{\varrho t} f'(\hat{k}) - \hat{q}\lambda\}. \qquad (1.36)$$

[16] It is essential to impose some measurability requirement upon the set of admissible controls $\{0 \leq s(t) \leq 1,\ 0 \leq t \leq T\}$. If, as implied by (1.17), attention is restricted to those controls which are piece-wise continuous, then integration performed in (1.10) and (1.33)—(1.38) is to be interpreted in the sense of Stieltjes. On the other hand, if attention is restricted to Lebesgue measurable controls, then the integration in (1.10) and (1.33)—(1.38) is to be interpreted in the sense of Lebesgue.

Integrating the first term in (1.36) by parts yields

$$\hat{q}(T)\,e^{-\delta T}\{k(T)-\hat{k}(T)\}-\hat{q}_0\{k(0)-\hat{k}(0)\}- -\int_0^T (k-\hat{k})(\dot{\hat{q}}-\delta\hat{q})\,e^{-\delta t}\,dt. \qquad (1.37)$$

Transversality condition (1.18) says that the first term in (1.37) is non-negative. Since every feasible path must satisfy the given initial condition k_0, the second term in (1.37) is identically zero. Hence

$$\int_0^T \hat{q}\,e^{-\delta t}(\dot{k}-\dot{\hat{k}})\,dt \geq -\int_0^T (k-\hat{k})(\dot{\hat{q}}-\delta\hat{q})\,e^{-\delta t}\,dt. \qquad (1.38)$$

Hence (1.36) is not smaller than

$$\int_0^T e^{-\delta t}dt[(\hat{k}-k)\{\hat{\gamma}e^{\varrho t}f'(\hat{k})-\lambda\hat{q}\}+(\hat{k}-k)(\dot{\hat{q}}-\delta\hat{q})]= =\int_0^T e^{-\delta t}dt\,(\hat{k}-k)\{\dot{\hat{q}}-(\delta+\lambda)\hat{q}+\hat{\gamma}e^{\varrho t}f'(\hat{k})\}$$

which by (1.16) and (1.17) is identically zero. Hence optimality requirement (1.33) is established. In fact, if $k \neq \hat{k}$ on some interval then inequation (1.33) is strict.

2. Models of economic growth and education

In the previous section some of the simplest descriptive and planning models of economic growth with exogenous technical change have been treated. The shortcoming of such models is that one of the quantitatively most important ingredients of growth is left unexplained and thus is ostensibly beyond the control of policy makers. In this section I shall critically examine two models of growth in which technical progress is the endogenous result of improving labour force quality.

The first model is that of Hirofumi Uzawa.[17] He postulates that current productive output $Y(t)$ depends upon the current level of the capital stock $K(t)$, the current allocation of workers to production $L_P(t)$, and the current efficiency $A(t)$ of the labour force:

$$Y(t)=F[K(t),A(t)L_P(t)]. \qquad (2.1)$$

It is merely for convenience of analysis that the improvements in labour quality appear in (2.1) in a (Harrod neutral) labour-augmenting manner. There is, of course, no *a priori* reason to identify improvements in labour quality due to education with labour augmenting technical progress.

Next assume that improvement in labour force quality depends upon the ratio of educators to the labour force

$$\dot{A}/A=\varphi(L_E/L), \qquad (2.2)$$

where L_E is the number of educators and $L \geq L_P + + L_E$ is the size of the total labour force. It is assumed that the higher the proportion of the labour force employed in education the higher the improvement in labour force efficiency.

$$\varphi'(L_E/L)\geq 0,\ \varphi''(L_E/L)\leq 0 \text{ for } (L_E/L)\in[0,1]. \qquad (2.3)$$

as before

$$\left\{\begin{aligned} \dot{L}&=nL,\\ Z+C&=Y,\\ \dot{K}&=Z-\mu K.\end{aligned}\right\} \qquad (2.4)$$

The planning board inherits at time zero stocks $K(0)$, $L(0)$, and $A(0)$. As before define:

$$y=Y/L,\ k=K/L,\ u=L_P/L,\ s=Z/Y. \qquad (2.5)$$

The optimal accumulation problem (over the infinite planning period) is to maximize the functional

$$\int_0^\infty (1-s)\,y\,e^{-\delta t}\,dt, \text{ where } \delta>0, \qquad (2.6)$$

subject to:

$$\dot{k}=sy-\lambda k, \qquad (2.7)$$

$$\dot{A}=A\varphi(1-u), \qquad (2.8)$$

$$y=Auf\left(\frac{k}{Au}\right), \qquad (2.9)$$

$$s\in[0,1],\ u\in[0,1], \qquad (2.10)$$

where δ, $\lambda \equiv n+\mu$, $k(0)=(K(0)/L(0))$ are given constants and u and s are piece-wise continuous controllers.

The Hamiltonian for this problem is

$$e^{-\delta t}\left\{(1-s)Auf\left(\frac{k}{Au}\right)+q\left[sAuf\left(\frac{k}{Au}\right)-\lambda k\right]+ +vA\varphi(1-u)\right\}. \qquad (2.11)$$

Thus the optimal programme must be such that

$$\dot{v}=(\delta-\varphi)\,v-\gamma u\left[f\left(\frac{k}{Au}\right)-\frac{k}{Au}f'\left(\frac{k}{Au}\right)\right], \qquad (2.12)$$

$$\gamma=\max(1,q), \qquad (2.13)$$

$$\lim_{t\to\infty} q\,e^{-\delta t}=0=\lim_{t\to\infty} v\,e^{-\delta t}. \qquad (2.14)$$

Uzawa shows that (2.7)—(2.13) imply that the

[17] H. Uzawa, "Optimal Technical Change in an Aggregative Model of Economic Growth", *International Economic Review*, Vol. 6, No. 1, Jan. 1965, pp. 18—31. Although the formal model discussed here is that of Uzawa, the economic interpretations of it are not necessarily his.

unique optimal trajectory tends to the balanced state given by the starred variables which solve:

$$\left\{\begin{aligned} &\varphi(1-u^*)+u^*\varphi'(1-u^*)=\delta,\\ &f'\left(\frac{x^*}{u^*}\right)=\delta+\lambda,\\ &f\left(\frac{x^*}{u^*}\right)-\frac{x^*}{u^*}f'\left(\frac{x^*}{u^*}\right)=v^*\varphi'(1-u^*),\\ &\frac{s^*f\left(\frac{x^*}{u^*}\right)}{\frac{x^*}{u^*}}=\lambda+\varphi(1-u^*), \end{aligned}\right\} \quad (2.15)$$

where $x^* \equiv \frac{k^*}{A^*}$.

Now that the formal educational planning model has been examined, let us re-examine its basic premises. Apart from the usual difficulties that come from the one sector production formulation (2.1) and the "vulgar" maximand (2.6) the Uzawa model above does not allow for the substitution of capital for labour in the education sector. Perhaps this is appropriate for planning in the context of underdeveloped societies; it is certainly a very limiting assumption for planning educational effort in the so-called advanced economies. Further, the above formulation assumes that although transmission of knowledge (education) to the labour force is costly, there is no way available to produce new technological knowledge.[18]

The second model of education and economic growth that I will treat in this section is due to Nelson and Phelps.[19] In their model the role of education is thought to be primarily that of facilitating the flow of technological information. My interpretation of education in the Uzawa model is the process of transmitting already known technological information to the labour force. The distinction between this view and that of Nelson-Phelps is rather subtle. Nelson-Phelps thinks of education as the process of training the productive actors in the economy to receive and "decode" the technological information that is being transmitted by other sectors in the economy.

In the Nelson-Phelps model production is given as in (2.1) except that the entire labour force is engaged in production, i.e. $L_P = L$, or

$$Y = F(K, AL). \quad (2.1^*)$$

The notion of the theoretical level of technology at time t, $R(t)$, plays an important role in the model. If all "best practice" technological knowledge were available to all economic agents then $R = A$. When this is not the case $A < R$.

In its most interesting formulation the Nelson-Phelps model has

$$\dot{A} = (R - A)\,\psi(h), \quad (2.16)$$

where h is a measure of the educational attainment of society and $\psi(0) = 0$, $\psi'(h) > 0$ for $h > 0$. According to hypothesis (2.16) the rate of increase of the technology in practice is an increasing function of educational attainment and is proportional to the "gap" $(R - A)$.

Following the Schumpeterian hypothesis that inventions do not depend upon other economic variables it can be assumed that R grows at the constant relative rate e >0,

$$\dot{R} = \varrho R. \quad (2.17)$$

If h is a positive constant, then the system (2.1*), (2.16) and (2.17) has the solution

$$A = \left(A_0 - \frac{\psi}{\psi+\varrho} R_0\right) e^{-\psi t} + \frac{\psi}{\psi+\varrho} R_0\, e^{\varrho t}. \quad (2.18)$$

Therefore the long-run equilibrium path of technology in practice $A^*(t)$ is given by

$$A^*(t) = \left(\frac{\psi(h)}{\psi(h)+\varrho}\right) R_0\, e^{\varrho t}, \quad (2.19)$$

so that the gap between A and R tends to a long-run constant for constant h.

In the Nelson-Phelps model not only is "technical progressiveness of the economy" (the time path of R) left outside the model but (even more important) the determination of the level of technical education h (or its time path) is left as a datum.[20] In what follows I construct an educational planning model that requires scarce resources to be devoted to the educational sector in order to increase educational attainment h. The model may be thought of as a synthesis of the Nelson-Phelps and the Uzawa models.

Assume that productive output[21] per worker can be written as

$$y = A\,u\,f\left(\frac{x}{u}\right), \quad (2.20)$$

where $u \in [0, 1]$ is the fraction of the labour force to be employed in production, A is the "digested"

[18] Production of new technological knowledge in enterprise and planned economies is the theme of two papers by the present author, K. SHELL, "Toward a Theory of Inventive Activity and Capital Accumulation", *American Economic Review*, May 1966, pp. 62–69; "A Model of Inventive Activity and Capital Accumulation", *Essays on the Theory of Optimal Economic Growth*, op. cit.

[19] R. R. NELSON and E. S. PHELPS, "Investment in Humans, Technological Diffusion and Economic Growth", *American Economic Review*, May 1966, pp. 69–75.

[20] In an unpublished Cowles Foundation discussion paper Phelps solves for the golden rule level of educational attainment.

[21] Assuming, of course, that there is no curvature to the production possibility frontier in consumption-investment space.

stock of technical knowledge, and x is the capital-labour ratio in labour efficiency units. Assume that labour force growth is zero so that we can choose units such that $L=1$.

Assume further that increase in the educational attainment of the labour force is an increasing function of the fraction of the labour force engaged in education.

$$\dot{h} = \varphi\,[(1-u)\,L] \tag{2.21}$$

with $\varphi'>0$, $\varphi''<0$, and $\varphi\,(0)<0$. Capital accumulation follows

$$\dot{K} = sF\,(K, AuL) - \mu K, \tag{2.22}$$

where $\mu>0$. The law (2.16) governing the growth of technology in practice can be rewritten as

$$\frac{\dot{A}}{A} = \left[\frac{R-A}{A}\right]\psi\,(h). \tag{2.23}$$

Suppose that the planning board desires to maximize the functional

$$\int_0^\infty U\left[(1-s)\,Auf\left(\frac{x}{u}\right)\right]\mathrm{e}^{-\delta t}\,\mathrm{d}t. \tag{2.24}$$

(2.24) is merely the discounted integral of utility of *per capita* consumption, since x is defined by $x=K/AL$. $\delta>0$ is the planners' subjective rate of time discount. Suppose further that $U'>0$, $U''<0$, and $U'\,[0]=\infty$ so that consumption will always be positive along an optimal programme. It is convenient to consider the Hamiltonian H defined by

$$He^{\delta t} = U\left[(1-s)\,A\,u\,f\left(\frac{x}{u}\right)\right] + q\left\{sA\,u\,f\left(\frac{x}{u}\right) - \mu\,k\right\} + \\ + v\,\{\psi(h)\,(R_0 e^{\varrho t} - A)\} + w\varphi\,[(1-u)]. \tag{2.25}$$

Choosing utility as the *numéraire*, q is the social demand price of investment, v is the social demand price of technology in practice, and w is the demand price of educational attainment as measured by h. H is therefore the present social value of net national product at time t. Maximization of GNP with respect to the controllers $s\,(t)\,\varepsilon\,[0, 1]$ and $u\,(t)\,\varepsilon\,[0, 1]$ yields

$$U' \geq q, \text{ with equality when } s>0, \tag{2.26}$$

and

$$U' \geq w\varphi, \text{ with equality when } u<1. \tag{2.27}$$

A programme of development maximizes the integral (2.24) subject to the technological constraints (2.4), (2.20)—(2.23) if there exist continuous prices $q\,(t)$, $v\,(t)$ and $w\,(t)$ such that

$$\dot{q} = (\delta+\mu)\,q - f'\left(\frac{x}{u}\right)U' \tag{2.28}$$

$$\dot{v} = (\delta+\psi)\,v - u\left\{f\left(\frac{x}{u}\right) - \frac{x}{u}f'\left(\frac{x}{u}\right)\right\}U' \tag{2.29}$$

$$\dot{w} = \delta w - v\psi'\,(h)\,[R-A] \tag{2.30}$$

while (2.4), (2.20)—(2.23), and (2.26)—(2.27) hold.

Condition (2.26) says that when investment in physical capital is positive, the marginal utility of consumption must equal the social demand price for investment. Condition (2.27) says that when educational effort is positive, the marginal utility of consumption must be equal to the social demand valuation of the marginal contribution to increase in educational attainment due to added educational effort.

Differential equation (2.28) says that the demand valuation of physical capital must change in a way so as just to compensate for losses due to depreciating the "waiting" net of the social demand value of physical capital's marginal product. Differential equation (2.29) says that the demand valuation of technical knowledge in practice must change so as just to compensate for losses due to narrowing the gap between available technology and technology in practice plus "waiting" net of the value of the marginal product of available technical knowledge. The path of the shadow valuation $v\,(t)$ is inconsequential in the sense that all allocation decisions are independent of the value assigned to $v\,(t)$. This is because, given h, R, and A, the value of $\dot{A}$ is uniquely determined.

On the other hand, as evidenced by condition (2.27), $w\,(t)$ plays an essential role in allocation. Differential equation (2.30) says that the social demand valuation of education attainment must change in order to just compensate for "waiting" less the value of the marginal contribution to increasing technological knowledge in practice.

To the above system we must append the boundary conditions

$$x\,(0)=x_0,\ R\,(0)=R_0,\ A\,(0)=A_0,\ h\,(0)=h_0. \tag{2.31}$$

That is, that the initial values of stock variables are given to the planning board. In addition we must append the necessary terminal transversality conditions

$$\lim_{t\to\infty} q\mathrm{e}^{-\delta t} = \lim_{t\to\infty} v\mathrm{e}^{-\delta t} = \lim_{t\to\infty} w\mathrm{e}^{-\delta t} = 0. \tag{2.32}$$

That is, we require for optimality that the present value of all stocks tend to zero.

Next we observe that the necessary system of differential equations possesses a unique stationary solution provided production satisfies the Inada conditions.

$$f\,(x)>0,\ f'\,(x)>0,\ f''\,(x)<0 \text{ for } 0<x<\infty$$

$$f\,(0)=0,\ f'\,(0)=\infty,$$

$$f\,(\infty)=\infty\ \ f'\,(\infty)=0.$$

Remembering that $L=1$, let $u^\dagger$ be the unique stationary to (2.21), i.e.

$$\varphi(1-u^\dagger)=0. \tag{2.33}$$

For $u=u^\dagger$ there is one and only one value x that yields a stationary to (2.28) with $U'=q$. Define $x^\dagger$ by

$$f'\left(\frac{x^\dagger}{u^\dagger}\right)=\delta+\mu. \tag{2.34}$$

Given $u^\dagger$ and $x^\dagger$, then $h^\dagger$ is the unique value for which (2.29) is stationary when $U'=v$,

$$\delta+\psi(h^\dagger)=u^\dagger\left\{f\left(\frac{x^\dagger}{u^\dagger}\right)-\frac{x^\dagger}{u^\dagger}f'\left(\frac{x^\dagger}{u^\dagger}\right)\right\}. \tag{2.35}$$

Again for $q=v=w=U'$, and given $x^\dagger$, $u^\dagger$, and $h^\dagger$, we can solve for $[R-A]^\dagger$ as the unique stationary to (2.30), i.e.

$$[R-A]^\dagger=\frac{\delta}{\psi'(h^\dagger)}>0. \tag{2.36}$$

Thus for the above economy the optimal trajectory is such that it tends to the capital-labour ratio $x^\dagger$ (in efficiency units) independent of initial endowments. In long-run optimal growth the gap $[R-A]$ between the theoretical technology and the available technology remains constant while the capital-labour ratio in natural units declines.

3. THEORIES OF THE INDUCED BIAS OF TECHNOLOGICAL CHANGE

In section 1, naive models of Hicks-neutral exogenous technological change were treated, and in section 2, models of education in which the induced technological change was specified to be labour-augmenting (and thus Harrod-neutral).[22] I have posited elsewhere[23] that technological change induced by inventive activity is of the Hicks-neutral form.

This specification of how the volume of inventions, the education of labour, or the learning-by-doing of entrepreneurs affects production functions is a rigid carry-over from capital theory. Certainly if there is an economic choice as to how much technological change a society should seek, there must be an economic choice amongst different types of technological change. A planner or an entrepreneur must be faced with a choice between "labour-saving" and "capital-saving" technological change.[24] This choice must be crucial both in explaining the direction of progress in enterprise economies and in planning research and development, educational policy, etc. in centrally directed economies.

Recently there have been a number of contributions to the theory of economic growth that are addressed to this point.[25] Underlying these models is a construction like the invention possibility frontier (IPF). If output is given by

$$Y=F(BK,AL) \tag{3.1}$$

then society (or, as some authors have claimed, the entrepreneurs) faces an IPF characterized by g (.) where

$$\frac{\dot B}{B}=g\left(\frac{\dot A}{A}\right) \tag{3.2}$$

with $g'<0$ and $g''<0$. This means that between labour-augmenting and capital-augmenting technical change a trade-off exists independent of other economic variables. In particular it is assumed that the rate of technical progress is independent of the economic resources devoted to invention and education. As before,

$$\dot K=sY-\mu K \tag{3.3}$$

with $\mu>0$. Define the ratios:

$$\begin{cases} y=Y/L,\\ k=K/L,\\ x=BK/AL.\end{cases} \tag{3.4}$$

Then (3.1) reduces to

$$y=Af(x) \tag{3.5}$$

[22] The main reason that certain specifications of the bias (or neutrality) of technological change are made is for their convenience in modelling. There is, for example, no *a priori* reason to expect that the technological progress resulting from increased education is labour augmenting. This may be the case, but it may also not be the case. Of course, if labour comes in "vintages" it may be natural to specify that educationally induced technological change is labour embodied.

[23] K. SHELL, "Towards a Theory of Inventive Activity and Capital Accumulation", *American Economic Review*, May 1966, pp. 62–69, and K. SHELL, "A Model of Inventive Activity and Capital Accumulation" in K. SHELL (ed.), *Essays on the Theory of Optimal Economic Growth*, op. cit.

[24] Such a choice between "labour-saving" and "capital-saving" change is crucial to the theories due to Hicks and Fellner of innovation in enterprise economies (J. R. HICKS, *The Theory of Wages*, Macmillan, 1932 [Ch. 6]; W. J. FELLNER, "Two Propositions in the Theory of Induced Innovations", *Economic Journal*, Vol. 71, June 1961).

[25] See C. KENNEDY, "Induced Bias in Innovation and the Theory of Distribution", *Economic Journal*, Sept. 1964; P. A. SAMUELSON, "A Theory of Induced Innovation along Kennedy-Weizsäcker Lines", *Review of Economics and Statistics*, Vol. 47, No. 4, Nov. 1965; E. M. DRANDAKIS and E. S. PHELPS, "A Model of Induced Invention, Growth and Distribution", Cowles Foundation Discussion Paper No. 186; W. D. NORDHAUS, "The Optimal Rate and Direction of Technological Change", in K. SHELL (ed.), *Essays on the Theory of Optimal Economic Growth*, op. cit. The treatment that follows has benefited from Nordhaus' paper and discussions with him.

and (3.3) reduces to

$$\dot{k} = sAf(x) - \lambda k \tag{3.6}$$

where $\lambda = \mu + n$, where $n = \dot{L}/L$ is the relative rate of growth of the labour force.

In treating such a model for the representative firm in a competitive economy, one is skating on thin ice. In the enterprise economy it seems that the most important feature of technical knowledge as a commodity is its inexpensive re-use. Therefore, in treating enterprise economies the fact has to be faced that most of the firm's technical knowledge will be "imported" at typically very low cost.

The planning problem is, however, much easier to specify. Assume that it is desired to maximize a functional of the form

$$\int_0^\infty U\,[(1-s)\,Af(x)]\,e^{-\delta t}\,dt \tag{3.7}$$

where $\delta > 0$ and $U' > 0$, $U'' < 0$, $U'(0) = \infty$. That is, it is desired to maximize the discounted integral of utilities of *per capita* consumption.

In order to apply the maximum principle, construct the Hamiltonian H which is defined by

$$He^{\delta t} = U\left[(1-s)Af\left(\frac{kB}{A}\right)\right] + q\left\{sAf\left(\frac{kB}{A}\right) - \lambda k\right\} + \\ + ve^{zt}g(\beta)B + w\beta A \tag{3.8}$$

where

$$\beta = \dot{A}/A. \tag{3.9}$$

If a feasible programme maximizes (3.7) then there must exist continuous prices $q(t)$, $v(t)$, and $w(t)$ such that:

$$\dot{q} = (\delta + \lambda)\,p_1 - Bf'(x)\,U' \tag{3.10}$$

$$\dot{v} = (\delta - z - g)\,v - ke^{-zt}f'(x)\,U' \tag{3.11}$$

$$\dot{w} = (\delta - \beta)\,w - \{f(x) - xf'(x)\}\,U', \tag{3.12}$$

where

$$U'\left[(1-s)Af\left(\frac{kB}{A}\right)\right] \geq q \tag{3.13}$$

with equality when $s > 0$. Following Nordhaus we introduce the shadow price of A in the form $ve^{(z-\delta)t}$. Maximization of *GNP* (3.8) implies that

$$vg'(\beta)\,Be^{zt} + wA = 0. \tag{3.14}$$

It is further required that the system satisfy the boundary conditions:

$$k(0) = k_0,\; B(0) = B_0,\; A(0) = A_0 \tag{3.15}$$

of historically given stocks and the appropriate transversality conditions

$$\lim_{t\to\infty} qe^{-\delta t} = \lim_{t\to\infty} e^{(z-\delta)t}\,v(t) = \lim_{t\to\infty} e^{-\delta t}\,w(t) = 0. \tag{3.16}$$

If we indicate the stationary value of x (etc.) by x^0 (etc.) then stationaries are found by solving

$$g'(\beta^0) = -\frac{1-\alpha^0}{\alpha^0}, \tag{3.17}$$

$$B^0 f'(x^0) = \delta + \lambda, \tag{3.18}$$

$$\beta^0 = z, \tag{3.19}$$

$$s^0 = \left(\frac{\lambda + z}{\lambda + \delta}\right)\alpha^0, \tag{3.20}$$

where α^0 is the equilibrium share of capital

$$\alpha^0 = \frac{x^0 f'(x^0)}{f(x^0)}. \tag{3.21}$$

If the elasticity of substitution

$$\sigma = \frac{-f'[f - xf']}{xff''}$$

is less than unity and if $\delta > z$ then a (forever) stationary programme is seen to satisfy all necessary and feasibility conditions.

In concluding this section, some comments are in order. We have sketched out some basic properties of the planner's optimal programme of capital accumulation and choice of the bias of technical change. We are thus left with only a partial analysis, since we have not investigated the possibility of simultaneous acceleration of technological progress by way of devoting economic resources to inventive activity. It is toward such an integration that future research in this area should proceed.