GAMES AND ECONOMIC BEHAVIOR 2, 362–377 (1990)

Liquid Markets and Competition*

JAMES PECK

Department of Managerial Economics and Decision Sciences, J. L. Kellogg Graduate School of Management, Northwestern University, 2001 Sheridan Road, Evanston, Illinois 60208-2009

AND

KARL SHELL

Department of Economics, Cornell University, 402 Uris Hall, Ithaca, New York 14853-7601

Received March 7, 1989

A market is *liquid* if no individual's actions have a big effect on the prices of goods traded in that market. Perfectly competitive markets are therefore perfectly liquid. It is well known that market liquidity can be achieved by increasing the number of traders so that individual trades are small compared to total trades. We show that even when there are only few traders, market liquidity can be achieved through large short sales in which net trades are small relative to gross trades. In particular, for a natural variant of the market game which permits unlimited short sales, we show that there is always a Nash equilibrium allocation arbitrarily close to a competitive equilibrium allocation. Of course, not all NE are near competitive. Only the large-short-sales NE are nearly liquid and hence close to CE. *Journal of Economic Literature* Classification Numbers: 021, 026, 610.
© 1990 Academic Press, Inc.

* We acknowledge the support from the National Science Foundation under Grant SES-8606944 and from the Center for Analytic Economics at Cornell University. This version of the paper was presented at the Game Theory Conference at Ohio State in July 1988 and is based on the lemmas for the single-round case in the earlier paper "Liquid Markets and Efficiency" presented at the Game Theory Conference at Ohio State in June 1987 and seminars at Carnegie Mellon, Northwestern, Penn, Cornell, UBC, UCLA, UCSD, Rice, and Texas A&M. We thank Yves Balasko, Aditya Goenka, Jean-Francois Mertens, and Raghu Sundaram for their comments. We also thank the editor, the associate editor, and the referees of *Games and Economic Behavior* for their intelligent and useful comments.

0899-8256/90 $3.00
Copyright © 1990 by Academic Press, Inc.
All rights of reproduction in any form reserved.

1. Introduction and Summary

We say that a market is *liquid* if the effect of any individual's trades on the prices in that market is small.[1] In this sense, perfectly competitive markets are perfectly liquid. Perfect competition—and hence, perfect market liquidity—can be justified as an approximation to the outcome of a game with many players. In the strategic market-game model of imperfect competition, as the number of players becomes large, markets become liquid because each individual's trades are small relative to total trades. Hence, interior Nash equilibrium allocations become nearly competitive equilibrium allocations as the economy is replicated many times.[2]

In contrast, when the number of players is small, Nash equilibria of the usual market game are typically inefficient.[3] The basic source of this inefficiency corresponds to a feature of actual markets: the more a (relatively big) individual trades on a given market, the more he turns the price against himself. That is, the basic source of inefficiency in the usual market game is the illiquidity of markets.

In this paper, we reexamine Nash equilibria of market games. We show that, even with few traders, markets can be organized so that there is always a Nash equilibrium allocation arbitrarily close to a competitive equilibrium allocation. We use the fact that thick (i.e., high-volume) markets are more liquid than thin (i.e., low-volume) markets. In particular, allowing traders to make large short sales implies the existence of Nash equilibria close to competitive equilibrium.[4]

[1] Hence we consider the market for IBM shares to be liquid under normal circumstances, even though the price of IBM shares might be volatile. On the other hand, we consider the market for country estates (especially ones bought or sold in a hurry) to be illiquid, whether or not the price of country estates is volatile.

[2] See Shapley and Shubik (1977), Postlewaite and Schmeidler (1978), Mas-Colell (1982), or Peck and Shell (1985), who show that the set of *interior* Nash equilibria converges to the set of competitive equilibria as the economy becomes large. (Competition can also be justified in cooperative game models with many players. The core of an exchange economy shrinks to the set of competitive equilibria as the economy becomes large. See Debreu and Scarf (1963), Aumann (1964), and Anderson (1978), but the analysis of market liquidity is more naturally based on noncooperative, rather than cooperative, economic models.)

[3] Cf., e.g., Dubey and Rogawski (1985), Aghion (1985), and Peck and Shell (1985, Proposition 2.20, pp. 15–16).

[4] Okuno and Schmeidler (1986) prove a similar result for a game in which players submit linear demand functions and everyone trades at the market clearing price. In their model, "market liquidity" occurs when the submitted demand schedules are sufficiently price elastic (i.e., when the submitted price coefficients for the linear demand functions are sufficiently large). They show that a Nash equilibrium allocation is approximately competitive if the price coefficients are large enough. Furthermore, for any choice of the price coefficients, there exists a NE that is consistent with those coefficients. Therefore, approximately competitive NE exist in the model of Okuno and Schmeidler. (Okuno is also known in the literature as Okuno-Fujiwara.)

The ability to sell short, an irrelevant institutional detail in the Walrasian economy, matters a great deal when competition is not assumed to be inherently perfect.[5] Short sales contribute to market liquidity. With short sales, net trades can be small relative to gross trades and, hence, relative to overall market volume. Individual net trades can be large relative to endowments but small relative to market volume, so prices are almost unaffected by individual net trades.

We show how markets can be liquid even though there are only a few traders, but we do not have a Panglossian theorem. We find that there are near-efficient Nash equilibria to the market game in which unlimited short sales are permitted for every trader, but there also exist in this version of the market game many thin-market Nash equilibria, which are typically inefficient. This follows from the assumption that short sales, while permitted in this game, are not required.

Our results do not come for free. Instead of the usual bankruptcy rule in which only individual bankrupts are punished by the referee, our referee shuts down the game (i.e., cancels all trades) if *anyone* exceeds his budget constraint. Thus, we replace the economy in which no short sales are permitted but in which individuals are only punished for their own bankruptcies with an economy in which short sales are permitted but in which the bankruptcy punishments are impersonal.

The formal game in the present paper is static. Dynamic models might better describe some of the bases for market liquidity. In particular, the possibility of retrading in dynamic models can substitute for short selling, so that complicated bankruptcy rules need not be invoked to generate liquid markets. Indeed, retrading as a basis for market liquidity was how we originally came to this subject.[6]

In Section 2, we introduce the one-round Market Game Γ with unrestricted short sales. Consumers are permitted to supply more than their endowments. They need only meet their overall budget contraints to avoid bankruptcy. The Nash equilibria are parameterized by offers. We show that there is a sequence of "thick-market" Nash equilibrium allocations which converges to a competitive equilibrium allocation. To establish this result, we show first that as offers become large, markets become liquid, and then we show that a perfectly liquid Nash equilibrium allocation is also a competititve equilibrium allocation.

[5] See Peck and Shell (1989) for an analysis of (and a catalogue of) some other institutional details that do not matter under perfect competition but do matter under imperfect competition.

[6] See Peck and Shell (1987). The present paper is essentially an account of Proposition (3.7) in the 1987 paper, which appeared there as part of the fuller dynamic analysis. Since there are so many interesting, but complex, issues which arise in the dynamic game, we believe it is best to record separately our static analysis before returning to the analysis of the dynamic game. Our hope is that the severe bankruptcy rule might not be required in the dynamic game with retrading.

In Section 3, our concluding remarks, we *sketch* the extension of our main result on market liquidity to the analysis of "liquidity traders." Liquidity traders are players who are (heavily) on both sides of the market. Even if ordinary traders cannot supply more than their endowment and must operate on only one side of the market, as long as there are at least two liquidity traders then there exist nearly competitive Nash equilibria. In Section 3, we also discuss the trade-off between the liquidity of markets and the simplicity of bankruptcy rules, and comment on a dynamic variant of the market game in which retrading can substitute for short sales.

2. The Market Game Γ: No Restrictions on Short Sales

We adopt the basic approach of Shapley and Shubik. See Shapley and Shubik (1977) and the references therein. We assume that each commodity is traded for money at a separate trading post. There can be two sources of inefficiency: (1) restrictions on credit possibly caused by a limited amount of commodity money, and (2) the oligopolistic power of each trader. We follow Postlewaite and Schmeidler (1978) in assuming the existence of costless credit (or, perhaps better, the existence of perfect inside, fiat money); see also Pazner and Schmeidler (undated) and Peck and Shell (1985). Hence we are able to focus on market illiquidity which is the result of oligopoly power. Our present departure is to place no upper limit on short sales. In particular, offers of commodities are permitted to exceed endowments of commodities. We state our formal assumptions in what follows.

There are $l + 1$ goods: l commodities (or consumption goods) and money. There are neither taxes nor transfers, so all money is "inside money," representing the private debt of the consumers. There are n consumers (or traders). Consumer h is endowed with a positive amount of commodity i, ω_h^i, for $i = 1, \ldots, l$ and $h = 1, \ldots, n$. If we denote by ω_h the endowment vector $(\omega_h^1, \ldots, \omega_h^i, \ldots, \omega_h^l)$, then we have $\omega_h \in \mathcal{R}_{++}^l$ for $h = 1, \ldots, n$.

There are l trading posts. For each commodity there is a single trading post on which the commodity is exchanged for money. Consumer h supplies a nonnegative quantity of commodity i, q_h^i, at trading post i. He also supplies a nonnegative quantity of money, b_h^i, at trading post i. We say that q_h^i is his *offer* (of commodity i) and that b_h^i is his (money) *bid* (for commodity i). Let $b_h = (b_h^1, \ldots, b_h^i, \ldots, b_h^l)$ and $q_h = (q_h^1, \ldots, q_h^i, \ldots, q_h^l)$ denote (respectively) his *bids* and his *offers*. The *strategy set* S_h of consumer h is then given by $S_h = \{(b_h, q_h) \in \mathcal{R}_+^{2l}\}$. Note that $q_h^i > \omega_h^i$ is permitted.

Each consumer faces a single overall budget constraint, which he must meet or else the referee will suspend trading: the consumer is required to finance his bids (for commodities) by his offers (of commodities). Hence, the *budget constraint* for consumer k is

$$\sum_{j=1}^{j=l} \left\{ \left(q_k^j \Big/ \sum_{h=1}^{h=n} q_h^j \right) \sum_{h=1}^{h=n} b_h^j \right\} \geq \sum_{j=1}^{j=l} b_k^j, \tag{2.1}$$

for $k = 1, \ldots, n$.

Let x_h^i denote the consumption of commodity i by consumer h, and let $x_h = (x_h^1, \ldots, x_h^i, \ldots, x_h^l)$ be his consumption vector. Assume that consumer k chooses the strategy $(b_k, q_k) \in \mathfrak{R}_+^{2l}$ for $k = 1, \ldots, n$; then the consumption of consumer h is given by

$$x_h^i = \omega_h^i - q_h^i + \left(b_h^i \Big/ \sum_{k=1}^{k=n} b_k^i \right) \sum_{k=1}^{k=n} q_k^i \quad \begin{array}{l} \text{if (2.1) is satisfied} \\ \text{for } \textit{each } k = 1, \ldots, n \end{array}$$

$$\text{and} \tag{2.2}$$

$$x_h^i = \omega_h^i \quad \begin{array}{l} \text{if (2.1) is not satisfied} \\ \text{for } \textit{some } k = 1, \ldots, n \end{array}$$

for $i = 1, \ldots, l$ and $h = 1, \ldots, n$. If some consumer violates Budget Constraint (2.1), then the result is autarky.

In the game with no short sales, the referee can credibly punish every bankrupt while ensuring that every other trader gets what "he bargained for." This is not possible in the game that allows for short selling. The referee might not be able to cover the short sales of a bankrupt player. Stronger bankruptcy rules are necessary. We assume that the referee shuts down the game if any player violates his budget constraint. If any player is bankrupt, each player is allocated his endowment.

An alternative approach would be to assume that the referee has a large enough stock of commodities to guarantee all contracts. He would then need to punish only bankrupts (as in the game without short sales) and he would guarantee all contracts. In their model in which consumers submit linear demand functions, Okuno and Schmeidler (1986) avoid the possibility of individually infeasible trades by expanding the consumption set to permit negative consumption. In this case, impersonal bankruptcy penalties are not necessary. In all of the games mentioned, there is no bankruptcy in any nontrival Nash equilibrium.

In our model, the consumption set of consumer h is the nonnegative

orthant $\{x_h | x_h \in \mathcal{R}^l_+\}$. His utility function, u_h, is strictly increasing, smooth, and strictly concave on the strictly positive orthant $\mathcal{R}^l_{++}$. Also, the closure in $\mathcal{R}^l$ of each indifference surface from $\mathcal{R}^l_{++}$ is contained in $\mathcal{R}^l_{++}$. (This last assumption allows us to avoid some complicated boundary solutions.) The boundary of the consumption set, $(\mathcal{R}^l_+ \backslash \mathcal{R}^l_{++})$, is also the indifference surface of least utility, so that (a) if we have $x_h \in (\mathcal{R}^l_+ \backslash \mathcal{R}^l_{++})$ and $y_h \in (\mathcal{R}^l_+ \backslash \mathcal{R}^l_{++})$, then we also have $u_h(x_h) = u_h(y_h) = u_h(0)$, and (b) if we have $x_h \in (\mathcal{R}^l_+ \backslash \mathcal{R}^l_{++})$ and $y_h \in \mathcal{R}^l_{++}$, then we also have $u_h(y_h) > u_h(x_h) = u_h(0)$.

It is well known that autarkic NE can be trivially exhibited in this type of market game. They also represent an extreme example of a general phenomenon which is readily explained by this model. Strategic (noncooperative) market games do not exhibit the same degree of coordination of plans which is assumed to be present in competitive economies. In strategic market games, supplies can be limited by an insufficiency of aggregate demands. Here supplies are zero because demands are zero. The zero supplies in turn justify the zero demands. The circle is closed.

Since $\Sigma_k q^i_k$ is measured in terms of commodity i it serves as a good measure of "market thickness." When $\Sigma_k q^i_k$ is zero, market i is closed. When $\Sigma_k q^i_k$ is small, we are tempted to say that market i is thin. In the game with no short sales, we are tempted to say that market i is thick if $\Sigma_k q^i_k$ is large, i.e., on the order of $\Sigma_k \omega^i_k$. In the present paper, there is no limit to market thickness: $\Sigma_k q^i_k$ can exceed $\Sigma_k \omega^i_k$.

What beliefs about market "thickness" are self-justifying? We have already seen that if consumers believe a market to be closed, their beliefs are justified.[7] In order to further pursue the analysis of market thickness, we next consider offer-constrained market games. We begin with definitions of the strategy sets for these games.

DEFINITION 2.3. Fix $q_h = \bar{q}_h \in \mathcal{R}^l_{++}$. Let $S_h(\bar{q}_h) = \{(b_h, q_h) | b_h \in \mathcal{R}^l_+$ and $q_h = \bar{q}_h\}$ be the *offer-constrained strategy set* for consumer h. Let $S(\bar{q}) = S_1(\bar{q}_1) \times \cdots \times S_h(\bar{q}_h) \times \cdots \times S_n(\bar{q}_n)$, where $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_h, \ldots, \bar{q}_n) \in \mathcal{R}^{nl}_{++}$.

We next define the offer-constrained game and the corresponding NE strategies and allocations.

[7] We can use the replicated market game to construct an example in which beliefs about market thickness are *not* self-justifying. Suppose, for the moment, that there is a Nash equilibrium in which consumers choose to supply a fixed small fraction of their endowments. For sufficiently many replications, any of these Nash equilibria must be nearly competitive. See Mas-Colell (1982, Section 2) and Peck and Shell (1985, Section 4). If offers are small, then net sales are necessarily small. We can easily specify offers to be small enough to prevent the resulting allocation from being near any competitive equilibrium. Hence, these beliefs about market thickness are not self-justifying in the replicated economy.

DEFINITION 2.4. The *offer-constrained market game* $\Gamma(\bar{q})$ is the same as the market game Γ except that the strategy set S is replaced by the offer-constrained strategy set $S(\bar{q})$ (Definition (2.3)).

PROPOSITION 2.5. *Let* $q = (q_1^1, \ldots, q_h^i, \ldots, q_n^l) \in \mathfrak{R}_{++}^{ln}$ be a vector of offers. There is a positive scalar ζ such that, if

$$q_h^i = \bar{q}_h^i > \omega_h^i - \zeta$$

for $i = 1, \ldots, l$ *and* $h = 1, \ldots, n$, *there there is a vector of bids* $b = (b_1^1, \ldots, b_h^i, \ldots, b_n^l) \in \mathfrak{R}_{++}^{ln}$ *which is an interior Nash equilibrium*[8] *to the offer-constrained game* $\Gamma(\bar{q})$, *where* $\bar{q} = (\bar{q}_1^1, \ldots, \bar{q}_h^i, \ldots, \bar{q}_n^l)$. *Furthermore, the b and q constructed by this method also constitute an interior Nash equilibrium to the unconstrained market game* Γ.

Proof. See Propositions (2.16) and (2.23) in Peck and Shell (1985) and Propositions (2.11) and (2.12) in Peck *et al.* (1989). As stated, the proofs of these propositions are for the case in which we have $\bar{q}_h^i \leq \omega_h^i$ for $i = 1, \ldots, l$ and $h = 1, \ldots, n$. They extend, however, without effort to the more general case, $\bar{q}_h^i \in [0, +\infty)$, which includes the possibility of (seemingly forced) short sales. ■

We shall now analyze the effect on the market game of large short sales. We focus on the offer-constrained market game $\Gamma(\nu\omega)$, where ν is a positive integer. For simplicity this game will be denoted simply as Γ^ν. Hence Γ^1 is the market game in which offers must be equal to endowments.

For $\nu = 2, 3, \ldots$, short sales are required in Γ^ν. Consumers supply more than their endowment but simultaneously buy back much of what they have offered, with the promise that they will make good on their net trades. Between the time the offers are made and the final resolution of the market, consumers are highly leveraged. We next show that the Nash equilibrium allocations of Γ^ν tend to some competitive allocation as the integer ν becomes large.

PROPOSITION 2.6. *Consider the sequence of games,* $\{\Gamma^\nu\}$. *There is a sequence of Nash equilibria for these games and a corresponding sequence of Nash equilibrium allocations* $\{x^\nu\}$, *where* $x^\nu \in \mathfrak{R}_{++}^{nl}$, *with the property that as the integer* ν *becomes large, the NE allocation* x^ν *tends to* x^*, *a competitive equilibrium allocation.*

Proof. Consider a particular consumer, Mr. h. Offers are given by $q_k = \nu\omega_k$ for $k = 1, \ldots, n$. Given the bids of the other consumers, it is straightforward to calculate the budget set for consumer h. It is given by the inequalities $x_h^i \leqq \sum_k \omega_k^i$ for $i = 1, \ldots, l$ and

[8] A Nash equilibrium (strategy) is *interior* if it is strictly positive.

$$\sum_{i=1}^{l}\left[\left(\sum_{k\neq h} b_k^i\right)\left(\frac{\omega_h^i - x_h^i}{\dfrac{\omega_h^i}{\nu} - \dfrac{x_h^i}{\nu} + \sum_{k\neq h}\omega_k^i}\right)\right] \geqq 0. \tag{2.7}$$

Define the vector $y_h \in \mathfrak{R}_{++}^l$ by

$$y_h = \left(\frac{1}{\nu}, \sum_{k\neq h} b_k^1, \ldots, \sum_{k\neq h} b_k^i, \ldots, \sum_{k\neq h} b_k^l\right). \tag{2.8}$$

The first component of y_h is no greater than unity, since we have $\nu \geq 1$. All other components of y_h are finite (since bids are nonnegative and can be normalized), and they are bounded from below (see Proposition (2.5) and the proof of Proposition (2.16) in Peck and Shell (1985), and the proof of Proposition (2.12) in Peck *et al.* (1989)). Denote the budget correspondence of consumer h by $B_h(y_h) \subset \mathfrak{R}_+^l$. Clearly, $B_h(y_h)$ is compact-valued.

CLAIM 1. *The budget correspondence $B_h(y_h)$ is upper hemicontinuous.*

Proof of Claim 1. Suppose $B_h(y_h)$ is not upper hemicontinuous. Then there exists a sequence $(y_h)^m$ tending to the limit $(y_h)^0$ and allocations $(x_h)^m \in B_h((y_h)^m)$ such that the sequence $(x_h)^m$ tends to the limit z_h, where $z_h \notin B_h((y_h)^0)$. Therefore, we have

$$\sum_{i=1}^{l}\left(\sum_{k\neq h} b_k^i\right)^0\left(\frac{\omega_h^i - z_h^i}{\dfrac{\omega_h^i}{\nu} - \dfrac{x_h^i}{\nu} + \sum_{k\neq h}\omega_k^i}\right) < \varepsilon \tag{2.9}$$

for some $\varepsilon > 0$. But since the left side of inequality (2.7) is continuous in y_h and x_h, and inequality (2.7) holds for each $(y_h)^m$ and $(x_h)^m$, it must hold in the limit as well. This contradicts inequality (2.9), so Claim 1 is proved. ■

CLAIM 2. *The budget correspondence $B_h(y_h)$ is lower hemicontinuous.*

Proof of Claim 2. We must show that for every sequence $(y_h)^m \to (y_h)^0$ and every $(x_h)^0 \in B_h((y_h)^0)$, there exists a sequence $(x_h)^m$ such that

$$(x_h)^m \in B_h((y_h)^m) \qquad \text{and} \qquad (x_h)^m \to (x_h)^0.$$

Let $(y_h)^m \to (y_h)^0$ and pick $(x_h)^0 \in B_h((y_h)^0)$. Construct the sequence $(x_h)^m$ according to

$$(x_h^i)^m = (x_h^i)^0 - A\left|\frac{(y_h^i)^0}{(y_h^i)^m} - 1\right| - B\left|\frac{1}{\nu^0} - \frac{(y_h^i)^0}{(y_h^i)^m \nu^m}\right|, \tag{2.10}$$

where $|\cdot|$ is the absolute value operator and the scalars A and B will be chosen later (independent of m). Clearly we have $(x_h)^m \to (x_h)^0$, so we must show $(x_h)^m \in B_h((y_h)^m)$ for large enough m. By construction, we have

$$\sum_{i=1}^{l} \left(\sum_{k\neq h} b_k^i\right)^m \left[\frac{\omega_h^i - (x_h^i)^m}{\dfrac{\omega_h^i - (x_h^i)^m}{\nu^m} + \sum_{k\neq h} \omega_k^i}\right]$$

$$= \sum_{i=1}^{l} (y_h^i)^m \left[\frac{s^i + A\left|\dfrac{(y_h^i)^0}{(y_h^i)^m} - 1\right| + B\left|\dfrac{1}{\nu^0} - \dfrac{(y_h^i)^0}{(y_h^i)^m \nu^m}\right|}{\dfrac{A}{\nu^m}\left|\dfrac{(y_h^i)^0}{(y_h^i)^m} - 1\right| + \dfrac{B}{\nu^m}\left|\dfrac{1}{\nu^0} - \dfrac{(y_h^i)^0}{(y_h^i)^m \nu^m}\right| + t^i}\right] \tag{2.11}$$

where s^i and t^i are defined by $s^i = \omega_h^i - (x_h^i)^0$ and $t^i = (s^i/\nu^m) + \sum_{k\neq h} \omega_k^i$. Since $(x_h)^0 \in B_h((y_h)^0)$ holds, we also have

$$\sum_{i=1}^{l} (y_h^i)^0 \left[\frac{s^i}{s^i/\nu^0 + \sum_{k\neq h} \omega_k^i}\right] \geqq 0. \tag{2.12}$$

From inequality (2.12), it follows that the left side of Eq. (2.11) is greater than or equal to

$$\sum_{i=1}^{l} \left\{\frac{(y_h^i)^m \left(\sum_{k\neq h} \omega_k^i\right)\left[A\left|\dfrac{(y_h^i)^0}{(y_h^i)^m} - 1\right| + B\left|\dfrac{1}{\nu^0} - \dfrac{(y_h^i)^0}{(y_h^i)^m \nu^m}\right|\right]}{\left[\dfrac{A}{\nu^m}\left|\dfrac{(y_h^i)^0}{(y_h^i)^m} - 1\right| + \dfrac{B}{\nu^m}\left|\dfrac{1}{\nu^0} - \dfrac{(y_h^i)^0}{(y_h^i)^m \nu^m} + t^i\right]\right. t^i}\right\}$$

$$+ \sum_{i=1}^{l} \left\{\frac{s^i\left[\left(\sum_{k\neq h} \omega_k^i\right)((y_h^i)^m - (y_h^i)^0) + s^i\left(\dfrac{(y_h^i)^m}{\nu^0} - \dfrac{(y_h^i)^0}{\nu^m}\right)\right]}{\left(s^i/\nu^0 + \sum_{k\neq h} \omega_k^i\right) t^i}\right\}. \tag{2.13}$$

For large enough m, we know that the first term in expression (2.13) is greater than or equal to

$$1/2 \sum_{i=1}^{l} \left\{ \frac{(y_h^i)^m (\Sigma_{k \neq h}\, \omega_k^i) \left[A \left| \frac{(y_h^i)^0}{(y_h^i)^m} - 1 \right| + B \left| \frac{1}{\nu^0} - \frac{(y_h^i)^0}{(y_h^i)^m \nu^m} \right| \right]}{\left[\frac{\omega_h^i - (x_h^i)^0}{\nu^0} + \Sigma_{k \neq h}\, \omega_k^i \right] t^i} \right\}.$$

Therefore, combining the terms in expression (2.13), we know that the left side of Eq. (2.11) is greater than or equal to zero whenever we have, for $i = 1, 2, \ldots, l$,

$$\begin{aligned} &\left(\sum_{k \neq h} \omega_k^i\right) \left[\frac{A}{2} \left| (y_h^i)^0 - (y_h^i)^m \right| + s^i((y_h^i)^m - (y_h^i)^0) \right] \\ &+ \left(\sum_{k \neq h} \omega_k^i\right) (y_h^i)^m \frac{B}{2} \left| \frac{1}{\nu^0} - \frac{(y_h^i)^0}{(y_h^i)^m \nu^m} \right| \\ &+ (s^i)^2 (y_h^i)^m \left[\frac{1}{\nu^0} - \frac{(y_h^i)^0}{(y_h^i)^m \nu^m} \right] \geqq 0. \end{aligned} \tag{2.14}$$

Since $|\omega_h^i - (x_h^i)^0| \leqq \Sigma_{k \neq h}\, \omega_k^i$ must hold for all i, inequality (2.14) holds when we have, for $i = 1, 2, \ldots, l$,

$$A \geqq 2 \sum_{k \neq h} \omega_k^i \quad \text{and} \quad B \geqq 2 \sum_{k \neq h} \omega_k^i. \tag{2.15}$$

Select A and B according to condition (2.15) and so that the left side of Eq. (2.11) is nonnegative for large enough m. Therefore, we have $(x_h)^m \in B_h((y_h)^m)$. ■

Since $B_h(y_h)$ is both upper hemicontinuous and lower hemicontinuous, it is continuous. It is easily shown that $B_h(y_h)$ is compact-valued and convex-valued. Given the assumptions on the utility functions u_h, by the Maximum Theorem, $\phi_h(y_h)$, the utility maximizing allocation for player h given y_h, is a continuous function.

We know that an interior Nash equilibrium exists for the offer-constrained game Γ^ν for $\nu = 1, 2, \ldots$ (see Proposition (2.5)). Bids, normalized to sum to unity, are strictly positive for this NE (or else consumers would not be maximizing utility). It follows that bids and (constrained) offers that constitute a NE for Γ^ν also constitute a NE for Γ by Proposition (2.5).

Consider a convergent sequence of NE for $\{\Gamma^\nu\}$, each normalized so that

$$\sum_{k=1}^{n} b_k^l \Big/ \sum_{k=1}^{n} \omega_k^l = 1$$

holds. For $h = 1, \ldots, n$, we have

$$(y_h)^\nu \to \left(0, \left(\sum_{k\neq h} b_k^1\right)^0, \ldots, \left(\sum_{k\neq h} b_k^i\right)^0, \ldots, \left(\sum_{k\neq h} b_k^l\right)^0\right)$$

and

$$(x_h)^\nu \to \phi_h((y_h)^0),$$

where $(x_h)^\nu$ is consumer h's NE allocation and $\phi_h((y_h)^0)$ is the solution to

$$\text{maximize } u_h(x_h)$$

$$\text{subject to } \sum_{i=1}^{l} \frac{(\Sigma_{k\neq h} b_k^i)^0}{\Sigma_{k\neq h} \omega_k^i} (\omega_h^i - x_h^i) \geqq 0.$$

We know that a convergent sequence $\{(y_h)^\nu\}$ exists because NE bids are contained in a set which is compact in the product topology. ∎

CLAIM 3. *There exists a vector* $P = (P^1, \ldots, P^i, \ldots, P^l) \in \mathfrak{R}_{++}^l$ *such that*

$$\frac{\Sigma_{k\neq h} (b_k^i)^0}{\Sigma_{k\neq h} \omega_h^i} = P^i$$

for $i = 1, \ldots, l$, *and* $h = 1, \ldots, n$.

Proof. The NE allocation must satisfy

$$\frac{x_h^i}{\nu} - \frac{\omega_h^i}{\nu} - \left[\frac{b_h^i \Sigma_k \omega_k^i}{\Sigma_k b_k^i} - \omega_h^i\right] = 0 \tag{2.16}$$

for all h, i, and ν. The left side of Eq. (2.16) can be expressed as a continuous function of x_h^i, $1/\nu$, and $\{\Sigma_{k\neq h} b_k^i\}_{h=1}^n$, so Eq. (2.16) must hold in the limit as well. Keeping in mind that b_h^i and $\Sigma_k b_k^i$ are continuous functions of $\{\Sigma_{k\neq h} b_k^i\}_{h=1}^n$ (and therefore converge), we have

$$\frac{(b_h^i)^0}{\Sigma_k (b_k^i)^0} = \frac{\omega_h^i}{\Sigma_k \omega_k^i},$$

which implies

$$\frac{(\Sigma_{k\neq h}\, b_k^i)^0}{\Sigma_k\, \omega_k^i} = \frac{\Sigma_{k\neq h}\, \omega_k^i}{\Sigma_k\, \omega_k^i}.$$

Therefore, we have

$$\frac{(\Sigma_{k\neq h}\, b_k^i)^0}{\Sigma_{k\neq h}\, \omega_k^i} = \frac{\Sigma_k\, (b_k^i)^0}{\Sigma_k\, \omega_k^i}. \tag{2.17}$$

Let the right side of Eq. (2.17), which is the same for all consumers, be denoted as P^i. This completes the proof of Claim 3. ■

We have shown that NE allocations of Γ^ν are converging to the solution of

$$\begin{aligned} &\text{maximize } u_h(x_h) \\ &\text{subject to } P \cdot (\omega_h - x_h) \geqq 0 \end{aligned}$$

for $h = 1, \ldots, n$. Since materials balance holds along the sequence, it holds at the limit. Thus, P is a competitive equilibrium price and the corresponding competitive equilibrium allocation is given by

$$\lim_{\nu\to\infty} (x_h)^\nu \qquad \text{for } h = 1, \ldots, n,$$

which completes the proof of Proposition (2.6). ■

COROLLARY 2.18. *For the market game* Γ (*which allows for unlimited short sales*), *there are Nash equilibria which are nearly competitive.*

Proof. The corollary follows from Proposition (2.6) and the fact that a Nash equilibrium to Γ^ν is also a Nash equilibrium to Γ by Proposition (2.5). ■

Remark 2.19. The strategy of the proof for Proposition (2.6) mirrors our opening remarks about market liquidity. Consider Fig. 1, which depicts a two-commodity example, $l = 2$, of the game Γ^ν. Consumer h is a net supplier of commodity one in equilibrium, and the frontier of his budget set, $B_h(y_h)$, is strictly concave to the origin. The slope of the budget frontier at the endowment point, ω_h, is greater than the slope of the (average) price line, S_L, which is greater than S_B, the slope of the budget frontier at the NE allocation. That is, consumer h can sell the first unit at a relatively high price, but the more he sells the lower is the price.

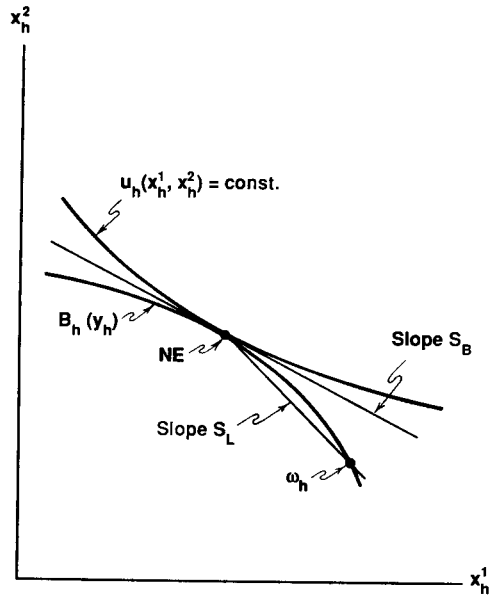

FIGURE 1

The curvature of the budget frontier, $B(y_h)$, indicates the imperfectness of the liquidity provided by the market. To prove Proposition (2.6), we showed that each consumer's budget frontier becomes "flatter" as ν becomes large, converging to a standard budget hyperplane in the limit. We then showed that these individual budget hyperplanes are colinear to each other, making the limit of NE allocations a competitive equilibrium allocation. Since a NE allocation for Γ^ν is also a NE allocation for Γ, it follows that for the market game Γ *with no restriction on short sales* some (exact) NE allocations are arbitrarily close to a CE allocation.

Remark 2.20. Okuno and Schmeidler (1986) analyze a game in which consumers submit linear excess demand schedules of the form $(b_h^i - a_h^i P^i)$. Their model, like ours, exhibits a continuum of equilibria, some of which are nearly competitive. The similarity between our model and theirs extends well beneath the surface. We can rewrite the first Allocation Rule in system (2.2) as

$$x_h^i - \omega_h^i = \frac{b_h^i}{P^i} - q_h^i, \tag{2.21}$$

where the market clearing price P^i equals $\Sigma_k\, b_k^i / \Sigma_k\, q_k^i$. From Eq. (2.21), we see that the submission of bids and offers can also be interpreted as the

submission of an *excess* demand schedule. Instead of submitting linear demand schedules, consumers here submit *excess* demand curves with unitary price elasticity (which are then translated by the amount q_h^i).[9]

3. Summary and Concluding Remarks

(a) We have shown that market liquidity can be achieved even in small economies. Even with few players, the institution of short selling is a source of market liquidity. As short selling is increased, budget frontiers become flatter. Therefore, there are always *some* Nash equilibria arbitrarily close to competitive equilibrium in the market game with no restrictions on short selling.

(b) Market liquidity is possible in small economies because, through short selling, individual net trades can be small relative to overall gross trades. Hence the effects of any individual's actions on market prices can be small. This liquidity can be provided by the short sales of ordinary consumers (as shown in Section 2), but it should be noted that the proof of our main result, Proposition (2.6), relies only on the fact that there are two consumers who offer large multiples of their endowments. In fact, it can easily be shown that there are nearly competitive Nash equilibria in which all but two consumers operate on just one side of each market and only those two consumers are permitted to make short sales. Hence liquidity can be achieved even if there are only a few "liquidity traders" who buy and sell "many times in a given day," while other traders place a single order to buy or sell, constrained by their endowments. If the liquidity traders gain utility or profit based upon the magnitude of their trades, then *all* Nash equilibria would tend to competitive equilibria as short-selling restrictions are relaxed. See Peck and Shell (1988, Sect. 4) for some details.

Although the choice of whether or not to become a liquidity trader is not formally modeled, one can tell a story in which liquidity traders receive small payments in proportion to the size of their offers. The right to this income might be auctioned against commodities. This could lead to an endogenous theory of which traders become the liquidity traders. In any case, it is reasonable to suppose that liquidity traders care about the size of their participation (as, e.g., an indicator of "reputation") in addition to caring about their final allocations. The endogenous theory of liquidity traders is worth further attention.

(c) The formal game analyzed in this paper is static. A dynamic model might better describe the institutions of short selling and liquidity trading.

[9] We thank Jean-Francois Mertens for this interpretation.

Moreover, the process of retrading plays in dynamic games a role similar to that of the process of short selling in static games. Retrading is a source of market liquidity. We have begun to explore the implications of retrade in a truly dynamic model (see Peck and Shell, 1987). We expect to report further results in a sequel.

(d) The more "liquid" is the static game the more unrealistic is our strong (interpersonal) bankruptcy rule. It is not our purpose to provide a model of bankruptcy. However, it is of interest from the mechanism-design point of view to note the trade-off between liquidity of markets and ease of enforcement of market rules. In the usual market game, the game without short sales, people are punished for going bankrupt, but this punishment rule can be enforced while still delivering what is promised to the nonbankrupt players. When players are allowed to supply more than their endowments, there is no guarantee that the referee can ensure deliveries to nonbankrupts without his own (large) inventory of commodities. The referee must keep such an inventory, consumers must take some risk of not being fully paid, or the rules of trade must be more complicated. Retrading in dynamic games without intertrade consumption provides liquidity without the institution of short sales. Retrading tends to do away with the necessity of impersonal bankruptcy procedures, but new problems are introduced because of the potential complexity of equilibrium strategies in dynamic games.

References

AGHION, P. (1985). "On the Generic Inefficiency of Differentiable Market Games," *J. Econ. Theory* **37**, 126–146.

ANDERSON, R. M. (1978). "An Elementary Core Equivalence Theorem," *Econometrica* **46**, 1483–1487.

AUMANN, R. J. (1964). "Markets with a Continuum of Traders," *Econometrica* **32**, 39–50.

DEBREU, G., AND SCARF, H. (1963). "A Limit Theorem on the Core of an Economy," *Int. Econ. Rev.* **4**, 235–246.

DUBEY, P., AND ROGAWSKI, J. D. (1985). "Inefficiency of Nash Equilibria in Strategic Market Games," Faculty Working Paper No. 1104, BEBR, University of Illinois, Urbana-Champaign, January.

MAS-COLELL, A. (1982). "The Cournotian Foundations of Walrasian Equilibrium Theory: An Exposition of Recent Theory," in *Advances in Economic Theory* (W. Hildenbrand, Ed.), pp. 183–224. London/New York: Cambridge Univ. Press.

OKUNO, M., AND SCHMEIDLER, D. (1986). "An Allocation Rule Using Linear Excess Demand Functions," in *Contributions to Mathematical Economics: In Honor of Gerard Debreu* (W. Hildenbrand and A. Mas-Colell, Eds.), pp. 361–385. Amsterdam: North-Holland.

PAZNER, E., AND SCHMEIDLER, D. "Non-Walrasian Nash Equilibria, in Arrow-Debreu Economies," Undated mimeograph, Urbana, University of Illinois.

PECK, J., AND SHELL, K. (1985). "Market Uncertainty: Sunspot Equilibria in Imperfectly Competitive Economies," CARESS Working Paper No. 85-21, University of Pennsylvania, July.

PECK, J., AND SHELL, K. (1987). "Liquid Markets and Efficiency," mimeo. Northwestern University, April; abstracted version to appear in *Game Theory and Applications: Proceedings of the 1987 International Conference, Ohio State University, Columbus, Ohio* (T. Ichiishi, A. Neyman, and Y. Tauman, Eds.). Academic Press, in press.

PECK, J., AND SHELL, K. (1988). "Liquid Markets and Competition," Center for Analytic Economics Working Paper No. 88-25RRR, Cornell University, October.

PECK, J., AND SHELL, K. (1989). "On the Nonequivalence of the Arrow Securities Game and the Contingent Claims Game," in *Economic Complexity: Sunspots, Bubbles, Chaos and Nonlinearity* (W. Barnett, J. Geweke, and K. Shell, Eds.), pp. 61–85. London/New York: Cambridge Univ. Press.

PECK, J., SHELL, K., AND SPEAR, S. E. (1989). "The Market Game: Existence and Structure of Equilibrium," Center for Analytic Economics Working Paper No. 89-15RR, Cornell University, November.

POSTLEWAITE, A., AND SCHMEIDLER, D. (1978). "Approximate Efficiency of Non-Walrasian Nash Equilibria," *Econometrica* **46**(1), 127–135.

SHAPLEY, L., AND SHUBIK, M. (1977). "Trade Using One Commodity as a Means of Payment," *J. Polit. Econ.* **85**, 937–968.