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The market game: Existence and structure of equilibrium*

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We analyze the canonical market game. There are ℓ commodities, a single inside money, ℓ markets in which commodities are exchanged for inside money, and n consumers. Each consumer's strategy is the nonnegative vector of his commodity offers and his money bids. Given endowments and sufficiently large offers, the set of interior Nash equilibrium strategies is finite and non-empty. Hence the set of interior Nash equilibria in strategy space is parametrized by the ℓn-dimensional vector of offers. In allocation space the manifold of Nash equilibrium allocations has *generic* dimension $\ell n-\ell$, which is also the dimension of the set of feasible allocations.

1. Introduction and summary

For models of perfect competition, there are extensive and systematic studies of the existence of equilibrium and of the structure of the equilibrium set. See the references in Debreu (1959), Arrow and Hahn (1971), Mas-Colell (1985), and Balasko (1987). For models of imperfect competition, analyses of the existence and the structure of equilibrium are less extensive and less

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systematic. Our purpose in this paper is to analyze Nash equilibrium in a strategic market game of the type originally proposed by Shapley and Shubik (1977). The strategic market game is probably the most fully developed general-equilibrium model of imperfect competition. It has been successfully applied to the analysis of the noncooperative foundations of competitive analysis [Mas-Colell (1982)], rational expectations in dynamic economies [Dubey, Geanakoplos, and Shubik (1987)], extrinsic uncertainty in imperfectly competitive economies [Peck and Shell (1985, 1991)], short selling as a basis of 'competitive' behavior [Peck and Shell (1990)], and other monetary and nonmonetary phenomena.

In section 2, we define the market game Γ. It is a very simple pure-exchange economy in which the mathematical inessentials and asymmetries are absent; see Postlewaite and Schmeidler (1978, 1981) and Peck and Shell (1985). There is a trading post for each commodity, on which consumers place offers of the commodity and bids of inside money. Consumers are constrained not to bid more money than they receive from their sales of commodities. A single money is used at all the posts, so each individual faces a single budget constraint. We show that the interior first-order conditions of the consumer's maximization problem are necessary and sufficient for an interior optimum.

In order to better understand the market game Γ, we define the offer-constrained game $\Gamma(\bar{q})$. In the latter game, consumers are constrained to send a fixed vector of commodity offers, $\bar{q}$, to the market. We show that any Nash equilibrium to $\Gamma(\bar{q})$ in which bids are strictly positive is also a Nash equilibrium to the unconstrained market game Γ. We also show that when (constrained) offers are close enough to endowments, the equilibria of $\Gamma(\bar{q})$ will have positive bids and will therefore also be equilibria of the market game Γ. Hence our proof in section 4 of the existence of an interior equilibrium to $\Gamma(\bar{q})$ for every $\bar{q}$ near the endowment vector is also an existence proof for Γ. Moreover, the ability to arbitrarily fix offers and still find an equilibrium is the first step for our indeterminacy results: we have shown that the Nash equilibrium *strategies* to the market game Γ are parametrized by the vector of offers. In section 3, we establish that the set of bids, offers, and endowments defining an interior Nash equilibrium to Γ is a manifold of dimension $2\ell n$, where ℓ is the number of commodities and n is the number of consumers. For a given economy with fixed endowments, the set of equilibrium bids and offers is generically of dimension ℓn, reflecting the ℓn degrees of freedom available to fix offers and still find an equilibrium.

The intuition behind this indeterminacy is simple. Given the bids and offers of others, the consumer chooses an ℓ-dimensional vector of bids and an ℓ-dimensional vector of offers to maximize utility subject to his budget constraint. The resulting interior first-order conditions define for each consumer $2\ell+1$ equations in $2\ell+1$ unknowns (ℓ offers, ℓ bids, and the

Lagrange multiplier associated with the budget constraint), but ℓ of the equations are redundant because a small change in a bid has the same effect (but with reversed sign) as a small change in the corresponding offer. (That is, there are no 'transactions costs' in the market game Γ and hence the money price of any commodity is equal to the inverse of that commodity's price of money.) The budget constraint makes one unknown and one equation superfluous. Hence we have ℓ independent equations in 2ℓ unknowns. Therefore the dimension of indeterminacy in strategy space is at least ℓ per consumer, or at least ℓn overall. We establish in section 3 that for generic endowments there is an ℓn-dimensional set of Nash equilibria; this result also follows from the paper by Dubey and Rogawski (1990).

A Nash equilibrium is said to be *interior* if all offers and bids are positive, in which case all markets are open because gross trades are being made on each market. Hence, if the Nash equilibrium is interior, the first-order conditions of individual utility maximization hold with equality. Our proof of the existence of interior equilibrium is in section 4. It is based on the one given in Peck and Shell (1985, section 2). See also Pazner and Schmeidler (undated). In the present version, the presentation is somewhat simplified by using degree theory instead of Brouwer fixed-point theory. We show that the natural projection, which projects the manifold of interior equilibria to the range of endowments and large offers, is proper. In establishing properness, we show that the equilibrium bids are strictly positive and bounded away from zero. At a Pareto optimal endowment, the vector of equilibrium bids is unique (up to a normalization), given the offers. Since these bids, endowments, and offers are regular, then at all regular offer–endowment pairs, the number of equilibria is odd. It follows that the equilibrium mapping is *onto*, so for any endowment vector we can find an interior equilibrium consistent with any vector of large offers.

In section 5, we analyze the set of Nash equilibrium allocations for the market game Γ. We show that the set of equilibrium allocations has generic dimension ℓn-ℓ in allocation space, $\mathbb{R}^{\ell n}_{++}$. This is the 'maximal' dimension, since it is the same as the dimension of the set of all feasible allocations. It is not surprising that the result in allocation space is merely *generic*. We know, for example, that if endowments are Pareto optimal, then the Nash equilibrium allocation is unique.

Our results in section 5 are consistent with other papers on the structure of the set of Nash equilibria in market games. Shapley (1976) analyzes a similar game but with only 2 commodities and 2 consumers. His geometric argument shows that if endowments are not Pareto optimal, then there is a 2-dimensional set of equilibrium allocations. Yao (1987a, b) extends Shapley's result to the case with ℓ commodities and 2 consumers. He shows that the generic dimension of the set of equilibrium allocations is ℓ. In her Cornell dissertation, Ramani (1989) considers ℓ commodities and n consumers in the

Peck–Shell (1985) game, but she restricts offers to be proportional to endowments. She finds that generically there is a continuum of Nash equilibrium allocations indexed by the coefficient of proportionality.

What should we conclude from the indeterminacy of pure-strategy Nash equilibrium in market games? We believe that this indeterminacy represents a realistic result for general-equilibrium models of imperfect competition. It captures the idea that outcomes can be affected by the 'optimism' or 'pessimism' of the economic actors. If everyone believes markets will be thick, then all traders believe they can supply large quantities without adversely affecting the price very much. When everyone acts according to these beliefs, markets are thick and traders' optimistic beliefs are self-justifying. However, there is a multi-dimensional set of self-justifying beliefs in which some or all markets are less than perfectly thick. There could be a role for the government or other institution to coordinate plans, or to manage 'effective demand'. On the other hand, the indeterminacy may not always be as large as the dimension argument suggests. The generic dimension of indeterminacy is indeed large, but the 'radius of indeterminacy' for large economies is small, since the set of interior pure-strategy NE allocations *does* shrink to the (finite) set of competitive equilibrium allocations as the economy is replicated. The indeterminacy is related to (but *not* equivalent to) the existence of sunspot equilibria in imperfectly competitive economies; see Peck and Shell (1985, 1991). The indeterminacy also provides a basis for a theory of market liquidity in small economies, driven in part by the possibility of wash sales and short sales; see Peck and Shell (1989, 1990).

2. The market game Γ

We consider an exchange economy with $\ell+1$ goods: ℓ commodities (or consumption goods) and money. There is no active government fiscal policy, so all money is 'inside money', representing the private debt of the consumers. There are n consumers (or traders). Consumer h is endowed with a positive amount of commodity i, ω_h^i, for $i=1,\ldots,\ell$. We denote by ω_h the endowment vector $(\omega_h^1,\ldots,\omega_h^i,\ldots,\omega_h^\ell)$, so we have $\omega_h \in \mathbb{R}^\ell_{++}$ for $h=1,\ldots,n$.

There are ℓ trading posts. For each commodity, there is a single trading post on which the commodity is exchanged for money. Consumer h supplies a nonnegative quantity of commodity i, q_h^i, at trading post i. He also supplies a nonnegative quantity of money, b_h^i, at trading post i. We say that q_h^i is his *offer* (of commodity i) and that b_h^i is his (money) *bid* (for commodity i). Let $q_h=(q_h^1,\ldots,q_h^i,\ldots,q_h^\ell)$ and $b_h=(b_h^1,\ldots,b_h^i,\ldots,b_h^\ell)$ denote (respectively) his *offers* and his *bids*. Offers must be made in terms of the physical com-

modities. Hence, offers cannot exceed endowments, i.e., we have $q_h^i \leqq \omega_h^i$ for $i=1,\ldots,\ell$.[1] The *strategy set* S_h of consumer h is then given by

$$S_h = \{(b_h, q_h) \in \mathbb{R}_+^{2\ell} \mid q_h \leqq \omega_h\}.$$

The trading process is simple. The total amount of commodity i which is offered, $Q^i = \sum_{k=1}^n q_k^i$, is allocated to nonbankrupt consumers in proportion to their shares of the bids for commodity i. Consumer h's share of the bids at post i is (b_h^i/B^i), where B^i is defined by $B^i = \sum_{k=1}^n b_k^i$. Thus the *gross* receipts of commodity i for consumer h are $(b_h^i Q^i/B^i)$ for $i=1,\ldots,\ell$ and $h=1,\ldots,n$. If all bids at post i are zero, the ratio (b_h^i/B^i) is equal to 0/0 and would appear to be indeterminate. It is assumed, however, that if there are no bids on post i, all offers on this post are 'lost', i.e., no commodity is delivered. Thus we take the fraction (b_h^i/B^i) to be zero if there are no positive bids at post i. At trading post i, the money from bids, B^i, is allocated to consumers in proportion to their offers of commodity i. Consumer h's share of the offers at post i is (q_h^i/Q^i). Thus the gross money receipts on post i for consumer h are $(q_h^i B^i/Q^i)$ for $i=1,\ldots,\ell$ and $h=1,\ldots,n$. If all offers of commodity i are zero, the ratio (q_h^i/Q^i) is equal to 0/0. It is assumed that if there are no offers on post i, all money bids on the post are 'lost'. Thus we take the fraction (q_h^i/Q^i) to be zero if there are no positive offers on post i.

Let $Q \in \mathbb{R}_+^\ell$ be the vector of aggregate offers and $B \in \mathbb{R}_+^\ell$ be the vector of aggregate bids; hence we have $Q=(Q^1,\ldots,Q^i,\ldots,Q^\ell)$ and $B=(B^1,\ldots,B^i,\ldots,B^\ell)$, where $Q^i = \sum_{k=1}^n q_k^i$ and $B^i = \sum_{k=1}^n b_k^i$ for $i=1,\ldots,\ell$. Let $Q_h^i \in \mathbb{R}_+$ be the sum of the offers of commodity i by every consumer save h, and $B_h^i \in \mathbb{R}_+$ be the sum of the bids for commodity i by every consumer save h; hence we have $Q_h^i = \sum_{k \neq h} q_k^i$ and $B_h^i = \sum_{k \neq h} b_k^i$ for $i=1,\ldots,\ell$. Define $Q_h \in \mathbb{R}_+^\ell$ and $B^h \in \mathbb{R}_+^\ell$ by $Q_h = (Q_h^1,\ldots,Q_h^i,\ldots,Q_h^\ell)$ and $B_h = (B_h^1,\ldots,B_h^i,\ldots,B_h^\ell)$.

Consumers do not face liquidity constraints, i.e., constraints which restrict their debt issuance on any given market or proper subset of markets. Each consumer does face a single overall budget constraint, which he must meet or be punished. He is required to finance his bids (for commodities) by his offers (of commodities). The *budget constraint* for consumer h is

$$\sum_{j=1}^{j=\ell} (B^j q_h^i/Q^j) \geqq \sum_{j=1}^{j=\ell} b_h^j \tag{2.1}$$

[1]This assumption does not play a role in the mathematical analysis of the structure of interior Nash equilibrium. If we drop the restriction $q_h \leqq \omega_h$ in the definition of S_h, we would have to strengthen the bankruptcy rule in order to maintain the credibility of the referee; see Peck and Shell (1990). The allocation rule (2.2) and the first-order equality conditions are unaffected. With an appropriately strengthened bankruptcy rule, our existence and structure results extend to cases in which we have $q_h^i > \omega_h^i$ for some (or all) h and i.

for $h=1,\ldots,n$. The right-hand side of inequality (2.1) is the sum of the dollars delivered by h in the form of bids to the trading posts. The left-hand side is the sum of the dollars delivered to h from the trading posts in payment for his commodity offers. The consumer is bankrupt if he issues more money than he collects.

Let x_h^i denote the consumption of commodity i by consumer h, and let $x_h=(x_h^1,\ldots,x_h^i,\ldots,x_h^\ell)$ be his consumption vector. Assume that consumer k chooses the strategy $(b_k,q_k)\in\mathbb{R}_+^{2\ell}$ for $k=1,\ldots,n$; then the consumption of consumer h is given by

$$\begin{aligned} x_h^i &= \omega_h^i - q_h^i + (b_h^i/B^i)Q^i \quad \text{if (2.1) is satisfied,} \\ x_h^i &= 0 \qquad\qquad\qquad\qquad\quad \text{if (2.1) is not satisfied} \end{aligned} \tag{2.2}$$

for $i=1,\ldots,\ell$ and $h=1,\ldots,n$. Failure to meet budget constraint (2.1) leads to confiscation of all of the consumer's goods.[2]

The consumption set of consumer h is the nonnegative orthant $\{x_h \mid x_h\in\mathbb{R}_+^\ell\}$. His utility function, u_h, is strictly increasing, smooth, and strictly concave on the strictly positive orthant $\mathbb{R}_{++}^\ell$. Also, the closure in $\mathbb{R}_+^\ell$ of each indifference surface from $\mathbb{R}_{++}^\ell$ is contained in $\mathbb{R}_{++}^\ell$.[3] The boundary of the consumption set, $(\mathbb{R}_+^\ell\setminus\mathbb{R}_{++}^\ell)$, is also the indifference surface of least utility.

We have specified the *strategy sets* S_h, the *outcomes* x_h [through eq. (2.2.)], and the *payoffs* $u_h(x_h)$ for the market game Γ. We adopt the standard concept of *Nash equilibrium* (NE).

Clearly the trivial strategy in which all offers and bids are zero is a NE for Γ. Since in this case we have $Q^i=0$ for $i=1,\ldots,\ell$, gross trading on each market is zero. This is the case in which all markets are (endogenously) closed. The focus of the present paper is on the case of *interior* Nash equilibria, NE in which all offers and bids are positive. In this case, we have $Q^i>0$ for $i=1,\ldots,\ell$. This is a case in which all markets are (endogenously) open, since there are positive gross trades on each market.

Since Q^i is measured in units of commodity i, it serves as a measure of the 'thickness' of market i. When Q^i is zero, we say that market i is closed; otherwise, we say that market i is open. When Q^i is small relative to $\sum_k \omega_k^i$, we are tempted to say that market i is thin. For large values of Q^i (on the order of $\sum_k \omega_k^i$) we are tempted to say that market i is thick.

What beliefs about market 'thickness' are self-justifying? We have already seen that if all consumers believe that all markets are closed, their beliefs are justified. We shall establish that uniform beliefs in sufficiently thick markets

[2]The referee for this game is credible: For all *feasible* strategies; the allocation defined by eq. (2.2) is feasible.

[3]This assumption rules out boundary equilibria.

are also self-justifying. In order to analyze market thickness, we must consider the offer-constrained market game $\Gamma(\bar{q})$. We then go on to relate the NE of $\Gamma(\bar{q})$ to the NE of Γ.

Definition 2.3. Fix $q_h = \bar{q}_h \in \mathbb{R}^{\ell}_{++}$, where we have $0 < \bar{q}^i_h \leqq \omega^i_h$ for $i = 1, \ldots, \ell$. Let $S_h(\bar{q}_h) = \{(b_h, q_h) | b_h \in \mathbb{R}^{\ell}_{+}$ and $q_h = \bar{q}_h\}$ be the *offer-constrained strategy set* for consumer h. Let $S(\bar{q}) = S_1(\bar{q}_1) \times \cdots \times S_h(\bar{q}_h) \times \cdots \times S_n(\bar{q}_n)$, where $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_h, \ldots, \bar{q}_n) \in \mathbb{R}^{n\ell}_{++}$. The *offer-constrained (market) game* $\Gamma(\bar{q})$ is constructed from the market game Γ by replacing the strategy set $S \subset \mathbb{R}^{2\ell n}_{+}$ with the offer-constrained strategy set $S(\bar{q}) \subset \mathbb{R}^{\ell n}_{+}$.

Four propositions on interior Nash equilibria follow. Propositions (2.4) and (2.9) give important properties of the interior NE of Γ. Propositions (2.11) and (2.12) provide a connection between the interior NE of $\Gamma(\bar{q})$ and the interior NE of Γ.

Proposition 2.4. Consider the individual utility-maximization problem (which arises in the analysis of the market game Γ):

$$\text{maximize } u_h(x_h),$$

$$\text{subject to } \quad (2.1), (2.2), \text{ and } (b_h, q_h) \in S_h.$$

Given the strictly positive bids and offers of consumers other than h, a strategy (b_h, q_h) is optimal for h if and only if the budget constraint (2.1) holds with equality and we have

$$[\partial u_h(x_h)/\partial x^i_h)/\partial u_h(x_h)/\partial x^j_h] = (Q^i_h/B^i_h)(B^i/Q^i)^2(B^j_h/Q^j_h)(Q^j/B^j)^2 \tag{2.5}$$

for $i = 1, \ldots, \ell$ and $j = 1, \ldots, \ell$.

Proof. By a straightforward manipulation of the budget constraint (2.1) along with the allocation rule (2.2), consumer h's problem becomes

$$\begin{aligned} &\text{maximize } u_h(x_h), \\ &\text{subject to } \sum_{i=1}^{\ell} (B^i_h(\omega^i_h - x^i_h)/(\omega^i_h - x^i_h + Q^i_h)) \geqq 0. \end{aligned} \tag{2.6}$$

Utility u_h is a strictly concave, smooth function of x_h, and the budget set (in allocation space) defined by the above constraint is a convex set, so we know

that there is a unique optimum, defined by the following necessary and sufficient conditions:

$$(\partial u_h(x_h)/\partial x_h^i)/(\partial u_h(x_h)/\partial x_h^j)$$
$$= B_h^i Q_h^i(\omega_h^j - x_h^j + Q_h^j)^2 / B_h^j Q_h^j(\omega_h^i - x_h^i + Q_h^i)^2 \tag{2.7}$$

for $i, j = 1, 2, \ldots, \ell$ with the constraint in (2.5) holding with equality. (The optimal consumption satisfies $x_h \in \mathbb{R}^\ell_{++}$ because we have $\omega_h \in \mathbb{R}^\ell_{++}$ and the closure of each indifference surface lies in $\mathbb{R}^\ell_{++}$.) Thus, a strategy (b_h, q_h) is optimal if and only if it gives rise to consumption satisfying (2.7).

The allocation rule (2.2) can be rewritten as

$$Q_h^i + \omega_h^i - x_h^i = (Q^i/B^i)B_h^i \tag{2.8}$$

for $i = 1, \ldots, \ell$. Whenever the left-hand side of eq. (2.8) appears in (2.7), replace it with the right-hand side of (2.8). Eq. (2.7) then reduces to eq. (2.5).

Therefore, (b_h, q_h) gives rise to consumption satisfying (2.7) if and only if eq. (2.5) is satisfied. □

Proposition 2.9. (*i*) *If the interior* NE *allocation* x *for the market game* Γ *is autarky, then the endowments* ω *are Pareto optimal for* Γ.
(*ii*) *If the endowments* ω *are Pareto optimal for the market game* Γ, *then there is a unique interior Nash equilibrium allocation which is autarky.*
(*iii*) *If the endowments* ω *are not Pareto optimal for the market game* Γ, *then we have* $u_h(x_h) \geqq u_h(\omega_h)$ *with at least one strict inequality for* $h = 1, \ldots, n$, where $x = (x_1, \ldots, x_h, \ldots, x_n)$ *is an interior* NE *allocation and* $\omega = (\omega_1, \ldots, \omega_h, \ldots, \omega_n)$ *is the vector of endowments.*

The proof presented here is based on that of Proposition (2.20) in Peck and Shell (1985). Proposition (2.9) was established for a variant of the present model by Dubey and Rogawski (1990). In fact, our result follows from theirs after transforming our market game Γ into a corresponding 'generalized game'.

Proof. (i) Let $x = (x_1, \ldots, x_h, \ldots, x_n)$ be an interior Nash equilibrium allocation which is autarkic, i.e., $x = (x_1, \ldots, x_h, \ldots, x_n)$ is equal to $\omega = (\omega_1, \ldots, \omega_h, \ldots, \omega_n)$. By Proposition (2.4), the interior first-order condition (2.7) for $i, j = 1, \ldots, \ell$ along with equality in budget constraint (2.6) is necessary and sufficient for consumer h's utility to be maximized given the strategies of the other consumers. Since $x_h = \omega_h$, we must have either $b_h^i/q_h^i = B^i/Q^i$ or $b_h^i = q_h^i = 0$ for $i = 1, \ldots, \ell$, because of eq. (2.2). Hence first-order condition (2.7) yields

$$\frac{\partial u_h(\omega_h)/\partial x_h^i}{\partial u_h(\omega_h)/\partial x_h^j}=\frac{B^iQ^j}{B^jQ^i} \tag{2.10}$$

for $i,j=1,\ldots,\ell$. Since the right-hand side of eq. (2.10) is independent of h, all consumers have the same marginal rates of substitution, establishing that the allocation is Pareto optimal.

(ii) A Nash equilibrium to Γ is clearly individually rational; i.e., if $x=(x_1,\ldots,x_h,\ldots,x_n)$ is a Nash equilibrium allocation, we have $u_h(x_h)\geqq u_h(\omega_h)$ for $h=1,\ldots,n$. It follows that, if ω is Pareto optimal, we have $u_h(x_h)=u_h(\omega_h)$ for $h=1,\ldots,n$. Since u_h is strictly increasing and concave on $\mathbb{R}^\ell_{++}$, if ω is Pareto optimal then the Nash equilibrium allocation must be autarkic, i.e., we must have $x=\omega$. Any strategy with positive offers and bids which satisfies $b_h^i/q_h^i=B^i/Q^i$ for $h=1,\ldots,n$ and $i=1,\ldots,\ell$ is an interior NE when ω is Pareto optimal.

(iii) Assume next that the endowment vector, $\omega=(\omega_1,\ldots,\omega_h,\ldots,\omega_n)$, is not Pareto optimal for Γ. Let $x=(x_1,\ldots,x_h,\ldots,x_n)$ be an interior Nash equilibrium allocation for Γ. We have established in part (i) of this proof that $x_h\neq\omega_h$ for at least one h and we know that $u_h(x_h)\geqq u_h(\omega_h)$ for all h. Since u_h is strictly increasing and strictly concave, and consumer h's budget set is a convex set, we know that the strict inequality $u_h(x_h)>u_h(\omega_h)$ holds for at least one h. □

Proposition 2.11. Let $\sigma=\{(b_h,q_h)\}_{h=1}^{h=n}$ be a NE strategy for the offer-constrained market game $\Gamma(\bar{q})$. If bids are strictly positive, i.e., if we have $b_h\in\mathbb{R}^\ell_{++}$ for $h=1,\ldots,n$, then σ is also a NE strategy for the market game Γ.

Proof. Consumer h's utility function, $u_h(\ldots,\omega_h^i-\bar{q}_h^i+b_h^iQ^i/B^i,\ldots)$, is concave in b_h, holding $b_{-h}=(b_1,\ldots,b_{h-1},b_{h+1},\ldots,b_n)\in\mathbb{R}^{\ell(n-1)}_{++}$ and $\bar{q}\in\mathbb{R}^{\ell n}_{++}$ constant. The budget constraint (2.1) is linear in own bids, so necessary and sufficient conditions for an interior optimum are that eq. (2.5) holds and budget constraint (2.1) holds with equality. Since σ is a NE for $\Gamma(\bar{q})$ and all bids are positive, all consumers are at an interior optimum. By Proposition (2.4), the strategy $(b_h,\bar{q}_h)$ is optimal for consumer h among unconstrained strategies in S_h. Therefore σ is a NE for the market game Γ. □

Proposition 2.12. Let $\sigma=\{(b_h,q_h)\}_{h=1}^{h=n}$ be a NE to the offer-constrained market game $\Gamma(\bar{q})$. There is a positive scalar ζ such that, if $q_h^i=\bar{q}_h^i>\omega_h^i-\zeta$ holds for $h=1,\ldots,n$ and $i=1,\ldots,\ell$, then σ is also a NE to the market game Γ.

Proof. Consider consumer h. Offers are fixed at $\bar{q}=(\bar{q}_1,\ldots,\bar{q}_h,\ldots,\bar{q}_n)\in\mathbb{R}^{\ell n}_{++}$. He takes the bids of others $b_{-h}=(b_1,\ldots,b_{h-1},b_{h+1},\ldots,b_n)\in\mathbb{R}^{\ell(n-1)}_{++}$ as fixed. He can guarantee his endowment by the bidding strategy given by

$$b_h^i = q_h^i B_h^i / Q_h^i \tag{2.13}$$

for $i=1,\ldots,\ell$ [cf. eq. (2.2)]. Hence we have $u_h(x_h(\sigma)) \geqq u_h(\omega_h)$. Thus we have that if $x=(x_1,\ldots,x_h,\ldots,x_n)$ is a NE allocation, then we know that $x_h \in \Pi_h \subset \mathbb{R}^\ell_{++}$, where

$$\Pi_h = \left\{ x_h \in \mathbb{R}^\ell_{++} \,\middle|\, u_h(x_h) \geqq u_h(\omega_h) \text{ and } x_h \leqq \sum_{k=1}^{n} \omega_k \right\}. \tag{2.14}$$

The set Π_h (the 'pie wedge') is convex, compact, and, by our assumption on the closure of indifference surfaces, bounded away from the axes. Hence there is a positive scalar ζ_h^i with the property that for each $x_h = (x_h^1,\ldots,x_h^i,\ldots,x_h^\ell) \in \Pi_h$, we have $x_h^i > \zeta_h^i$ for $i=1,\ldots,\ell$. Therefore for each NE allocation $x=(x_1,\ldots,x_h,\ldots,x_n)$, there is a positive scalar ζ such that $x_h^i > \zeta$ for $i=1,\ldots,\ell$ and $h=1,\ldots,n$.

Assume that $\bar{q}_h^i > \omega_h^i - \zeta$, where ζ is the positive scalar just constructed. Consumer h must make a positive bid on market i, $b_h^i > 0$, or else the consumption vector x_h would not belong to the set Π_h. Hence if $\sigma = \{(b_h, q_h)\}_{h=1}^{h=n}$ is a NE for $\Gamma(\bar{q})$, all bids must be positive, i.e., we have $b_h \in \mathbb{R}^\ell_{++}$ for $h=1,\ldots,n$. Therefore, by Proposition (2.11), σ is also a NE for the (unconstrained) market game Γ. □

Obviously not all beliefs about market thickness are self-justifying. If a relatively small offer, $\bar{q}_h^i$, is imposed on consumer h, he might have desired to sell more of commodity i given the bids and offers of others, i.e., the constraint $q_h^i \leqq \bar{q}_h^i$ would be binding. The constraint $b_h^i \geqq 0$ could then also become binding: he might want to bid a negative amount to make up for his meager offer, $\bar{q}_h^i$. On the other hand (as the previous proposition establishes), if all offers are sufficiently large, then they are self-justifying. Closed markets are self-justifying, thick markets are self-justifying, but some thin markets are not self-justifying.[4]

In section 3, we will study the structure of the set of interior Nash equilibria to $\Gamma(\bar{q})$ and hence to Γ.

[4]To see that some thin markets are not self-justifying, consider a replication of the market game Γ; see Mas-Colell (1982) and Peck and Shell (1985, section 4). As the number of replications becomes large, the set of interior Nash equilibrium allocations approaches the set of competitive equilibrium allocations. Assume that endowments are not Pareto optimal. Fix the vector of offers $\bar{q}$ at some strictly positive level but too small to support competitive trade. Namely, choose the positive scalar $\bar{q}_h^i$ so that $\bar{q}_h^i < \omega_h^i - (x_h^i)^*$ holds [where $(x_h^i)^*$ is the competitive allocation of commodity i to consumer h] for some i and h. This is possible since some consumer is a net supplier of some commodity in a competitive equilibrium as long as endowments are not Pareto optimal. Hence for a sufficiently large number of replications, $\bar{q}$ will not be a self-justifying vector of offers.

3. Structure of the set of interior Nash equilibria in strategy space

In this section, we shall show that the set E of interior Nash equilibria in the space of offers, normalized bids, and endowments is a manifold of dimension $2\ell n$. The set E is defined by

$$E=\{(b,q,\omega)\in\Delta\times\mathbb{R}^{2\ell n}\,|\,b \text{ is an interior NE of } \Gamma(q;\omega)\},$$

where Δ is the set of normalized bids $\{b\in\mathbb{R}^{n\ell}_{+}\,|\,\sum_{h,i}b^i_h=1\}$ and $\Gamma(q;\omega)$ is the offer-constrained market game with offers $q\in\mathbb{R}^{n\ell}_{+}$ and endowments $\omega\in\mathbb{R}^{n\ell}_{++}$.

Proposition 3.1. The equilibrium set E is a manifold of dimension $2\ell n$.

Proof. We define the mapping $\xi:\Delta\times\mathbb{R}^{2\ell n}\to\mathbb{R}^{\ell n-1}$ by

$$\xi(b,q,\omega)=b-(\iota^{\mathrm{T}}\tilde{b}(b,q,\omega))^{-1}\tilde{b}(b,q,\omega),$$

where $b\in\Delta$, $\tilde{b}(b,q,\omega)$ is the mapping of (non-normalized) best responses to the bids b given offers q and endowments ω, and $\iota^{\mathrm{T}}=[1,\ldots,1]$ is the ℓn-dimensional sum vector. The symbol T denotes transposition. Zeros of the mapping ξ constitute NE for the model, so to show that the equilibrium set E is a manifold, we need to show that ξ is transverse to zero. We show this by showing that the Jacobian matrix of derivatives with respect to ω and q has full rank $\ell n-1$.

We begin by calculating some derivatives. A routine differentiation of $\xi(b,q,\omega)$ with respect to q and ω yields

$$\begin{aligned}\mathrm{D}_{[q,\omega]}\xi&=-(\iota^{\mathrm{T}}\tilde{b})^{-1}(I-(\iota^{\mathrm{T}}\tilde{b})^{-1}\tilde{b}\iota^{\mathrm{T}})\mathrm{D}_{[q,\omega]}\tilde{b}\\&=-(\iota^{\mathrm{T}}\tilde{b})^{-1}P_{\tilde{b}}\mathrm{D}_{[q,\omega]}\tilde{b},\end{aligned}$$

where $P_{\tilde{b}}$ is the projection on the hyperplane $H_\iota=\{x\in\mathbb{R}^{\ell n}\,|\,\iota^{\mathrm{T}}x=0\}$ in the direction $\tilde{b}$. To establish the transversality result, then, we need to show that $P_{\tilde{b}}\mathrm{D}_{[q,\omega]}\tilde{b}$ has rank $\ell n-1$. The Jacobian $\mathrm{D}_{[q,\omega]}\tilde{b}$ takes the form

$$\begin{bmatrix} J_1 & E_{11} & 0 & E_{12} & \ldots & 0 & E_{1n}\\ 0 & E_{21} & J_2 & E_{22} & \ldots & 0 & E_{2n}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & E_{n1} & 0 & E_{n2} & \ldots & J_n & E_{nn}\end{bmatrix},$$

where $J_h=\mathrm{D}_{\omega_h}\tilde{b}_h$ and $E_{hk}=\mathrm{D}_{q_k}\tilde{b}_h$. We will show that this matrix has rank

$\ell n-1$ and that the kernel of $P_{\tilde{b}}$ is not contained in the span of the columns of J. It will then follow that $P_{\tilde{b}}$ maps J onto H_t.

To establish the rank result, we need to compute the component matrices of J. Applying the implicit function theorem to the equality first-order conditions (2.1) and (2.5) for consumer h (and suppressing consumer subscripts except where they are necessary) yields the system

$$\begin{bmatrix} A & w \\ w^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \mathrm{d}\tilde{b} \\ \mathrm{d}\lambda \end{bmatrix}+\begin{bmatrix} \mathrm{D}^2U\hat{Q}\hat{B}^{-2}\hat{B}_h & R & W \\ 0 & \gamma^{\mathrm{T}} & f^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathrm{d}\omega_h \\ \mathrm{d}q_h \\ \mathrm{d}q_k \end{bmatrix}=0, \tag{3.2}$$

where the various matrices and vectors involved in expression (3.2) are defined as follows:

$$U=(u_1,\ldots,u_h,\ldots,u_n),$$

$$A=\hat{Q}\hat{B}^{-2}\hat{B}_h\mathrm{D}^2U\hat{B}_h\hat{B}^{-2}\hat{Q}-2\hat{Q}\hat{B}_h\hat{B}^{-3}\mathrm{D}\hat{U},$$

$$w=\hat{Q}^{-1}Q_h,$$

$$R=-\hat{B}_h\hat{B}^{-1}\mathrm{D}^2U\hat{Q}\hat{B}_h\hat{B}^{-2}+2\lambda\hat{Q}_h\hat{Q}^{-2},$$

$$\gamma=\hat{B}\hat{Q}^{-2}Q_h,$$

$$W=\hat{B}^{-1}b_h\mathrm{D}^2U\hat{Q}\hat{B}_h\hat{B}^{-2}+\lambda\hat{Q}^{-2}(\hat{Q}_h-\hat{q}_h), \quad \text{and}$$

$$f=\hat{Q}^{-2}\hat{B}q_h,$$

where $\hat{\cdot}$ is the diagonal matrix formed from the vector $\cdot$ (i.e., $\hat{\cdot}=\mathrm{diag}\,\cdot$) and λ is the Langrangean multiplier on the budget constraint (2.1). Let the matrix K, the vector μ, and the scalar ε be defined by

$$\begin{bmatrix} A & w \\ w^{\mathrm{T}} & 0 \end{bmatrix}^{-1}=\begin{bmatrix} K & \mu \\ \mu^{\mathrm{T}} & \varepsilon \end{bmatrix}. \tag{3.3}$$

Then we have

$$\begin{bmatrix} \mathrm{D}\tilde{b} \\ \mathrm{D}\lambda \end{bmatrix}=-\begin{bmatrix} K & \mu \\ \mu^{\mathrm{T}} & \varepsilon \end{bmatrix}\begin{bmatrix} \mathrm{D}^2U\hat{Q}\hat{B}_h\hat{B}^{-2} & R & W \\ 0 & \gamma^{\mathrm{T}} & f^{\mathrm{T}} \end{bmatrix}.$$

Note that from the expressions for the derivatives $\mathrm{D}_{q_k}\tilde{b}_h$ for $k\neq h$, these

derivatives depend on other agents' offers only through the total offers Q and Q_h. It follows, then, that $D_{q_i}\tilde{b}_h = D_{q_k}\tilde{b}_h$ for all $i, k \neq h$. In terms of the components of the Jacobian matrix J, this implies that $E_{hi} = E_{hk}$ for all i, $k \neq h$. Hence we may denote these components simply by E_h.

We now perform some calculations on J to determine its rank. First, subtract the last block of columns of J from all blocks of columns having E entries. This yields a matrix.

$$J' = \begin{bmatrix} J_1 & E_{11}-E_1 & 0 & 0 & \dots & 0 & E_1 \\ 0 & 0 & J_2 & E_{22}-E_2 & \dots & 0 & E_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -E_{nn}+E_n & 0 & -E_{nn}+E_n & \dots & J_n & E_{nn} \end{bmatrix}.$$

For each h, the derivatives $J_h = D_{\omega_h}\tilde{b}_h$ are given by $J_h = -K_h D^2 u_h \hat{Q}\hat{B}_h \hat{B}^{-2}$. Since we are assuming that bids and offers are positive, the diagonal matrices $\hat{B}$, $\hat{Q}$, and $\hat{B}_h$ have full rank. Since utility functions are strictly concave, $D^2 u_h$ is negative definite and the matrix K_h is negative semi-definite. Hence J_h has rank $\ell - 1$ for each $h = 1, \dots, n$. Note also from the inversion relations (3.3) we have $w_h^{\mathrm{T}} K_h = 0$, and hence we have $w_h^{\mathrm{T}} J_h = 0$.

The matrices $E_{hh} - E_h$ are given by

$$E_{hh} - E_h = D_{q_h}\tilde{b}_h - D_{q_k}\tilde{b}_h = -K_h(R_h - W_h) - \mu_h(\gamma_h - f_h)^{\mathrm{T}}.$$

Pre-multiplying this expression by w_h (and using the fact that $w_h^{\mathrm{T}}\mu_h = 1$), we have $w_h^{\mathrm{T}}(E_{hh} - E_h) = (\gamma_h - f_h)^{\mathrm{T}} = \hat{B}\hat{Q}Q_h^{-2} + \hat{B}\hat{Q}q_h^{-2} = \hat{B}\hat{Q}Q^{-2}$. Hence we have

$$w_h^{\mathrm{T}}(E_{hh} - E_h) = \hat{B}\hat{Q}Q^{-2}.$$

Under our maintained hypothesis of strictly positive bids and offers, this vector is strictly positive. It follows then that the matrix $E_{hh} - E_h$ has at least one column which is linearly independent of the columns of J_h. Hence, the matrix $[J_h, E_{hh} - E_h]$ has rank ℓ for all $h = 1, \dots, n$.

Delete the last block of columns from J'. We claim that the resulting matrix has rank $\ell n - 1$. To prove this claim we obtained $\ell(n-1)$ linearly independent vectors from the first $n-1$ blocks of the columns containing the matrices $[J_h, E_{hh} - E_h]$, $h = 1, \dots, n-1$, and we obtain $\ell - 1$ additional linearly independent vectors from the block of columns containing J_n. We know that J' does not have full rank ℓn by a straightforward, although lengthy calculation, which shows that the vector $w^{\mathrm{T}} = [w_1^{\mathrm{T}}, \dots, w_n^{\mathrm{T}}]$ annihilates J'. Therefore, J' (and hence J) has a rank of exactly $\ell n - 1$.

To complete our transversality result, we need to show that the $(\ell n - 1)$-dimensional hyperplane spanned by the columns of J is mapped onto H_i by

the projection operator $P_{\tilde{b}}$. A sufficient condition for this to hold is that the kernel of the projection be linearly independent of the columns of J, since this implies that no linear combination of columns of J lies in the kernel and hence gets annihilated by $P_{\tilde{b}}$. The kernel of $P_{\tilde{b}}$ is the set of vectors colinear with $\tilde{b}$. So, suppose $\tilde{b}$ was in the span of the columns of J. It would then follow that $w^{\mathrm{T}}\tilde{b}=0$. But we have

$$w^{\mathrm{T}}\tilde{b}=\sum_{k=1}^{n} w_k^{\mathrm{T}}\tilde{b}_k \quad \text{and} \quad w_k^{\mathrm{T}}\tilde{b}_k=-\sum_{i=1}^{\ell}(\tilde{b}_k^i Q_k^i/Q^i).$$

Under our assumptions about bids and offers, the last expression is strictly negative for all $k=1,\ldots,n$, and hence we have $w^{\mathrm{T}}\tilde{b}<0$. Hence the columns of J are linearly independent of the kernel of the projection, and $P_{\tilde{b}}\mathrm{D}\tilde{b}$ has rank $\ell n-1$. We have shown that E is a manifold.

To show that the manifold E is non-empty, take $q=\omega$ and let ω be Pareto optimal. From Proposition (2.8), we know that there is a unique interior NE allocation, which in this case is autarky. Since consumers are putting all of their endowments on the market, they must make positive bids in order to buy back their endowments. Hence, the assumptions made above are satisfied, so that ξ is transverse to zero at any Pareto optimal Nash equilibrium, and the equilibrium manifold E is non-empty. Finally, notice that by the pre-image theorem [see, e.g., Guillemin and Pollack (1974, chapter 1, section 4, p. 21)] when ξ is transverse to the origin, $\xi^{-1}(0)=E$ has codimension $\ell n-1$ in $\mathbb{R}^{3\ell n-1}$. Hence the dimension of E is $2\ell n$. □

Using the Transversal Density Theorem [see, e.g., Hirsch (1976, p. 74)], we also know that for an open and dense set of endowments and offers, $\mathrm{D}_b\xi$ has full rank. When equilibrium bids exist, the implicit function theorem can be applied on a neighborhood of any regular (q,ω) pair to determine the equilibrium bids as functions $b(q,\omega)$ of endowments and offers. These functions are smooth and their derivatives are given by

$$\mathrm{D}_\omega b=-[\mathrm{D}_b\xi]^{-1}\mathrm{D}_\omega\xi \quad \text{and} \quad \mathrm{D}_q b=-[\mathrm{D}_b\xi]^{-1}\mathrm{D}_q\xi.$$

This last fact will become important in section 5, where we analyze the structure of the set of interior NE in allocation space.

Remark 3.4. Since equilibrium bids exist for any ω (see section 4), our transversality result guarantees that, for an open and dense set of endowments, the set of equilibrium strategies has the same codimension as E, $\ell n-1$. It follows that the set of equilibrium strategies has dimension ℓn.

Compare our result with Dubey and Rogawski (1990, Proposition 3). They consider a general class of smooth mechanisms and find that, when Nash equilibria exist, the set of equilibrium strategies is a manifold of generic codimension $\ell n - n$. Our codimension result differs from theirs, because our allocation rule (2.2) causes the mapping from strategies to allocations to be discontinuous, which violates their (smoothness) Assumption 1. Consider, however, a generalized game derived from the market game Γ in the obvious way, but in which a player's feasible strategies (given the strategies of others, σ_{-h}) are those bids and offers satisfying inequality (2.1).[5] The set of NE has not been changed, but the dimension of the strategy space is reduced to $2\ell n - n$. We can now apply Proposition 3 of Dubey and Rogawski (1990) to this generalized game, concluding that the set of Nash equilibrium strategies generically has codimension $\ell n - n$. Hence the dimension is generically $(2\ell n - n) - (\ell n - n)$, or ℓn, which is in accord with our results.

4. Existence of interior Nash equilibrium

In this section, we develop a simple proof of the existence of interior Nash equilibria using the structural results of the previous section. Our strategy for showing existence of equilibrium is to use the fact that if offers are sufficiently large, then all equilibria are interior [cf. Proposition (2.12)]. We replace the fixed-point argument of Peck and Shell (1985) with a similar degree-theory argument, which is also an adaptation of the program of Balasko (1987, especially chapters 4 and 5) to the market game setting.

Let L be a connected subset of offers yielding interior Nash equilibria and let Ω be the subset of endowments defined below. We define the natural projection $\chi: E \to L \times \Omega$ from the equilibrium manifold to $L \times \Omega$ as the restriction of the projection $\pi: \Delta \times L \times \Omega \to L \times \Omega$ to E. We then show that χ is proper, and that if endowments are Pareto optimal, then for any q, the equilibrium bids are unique. This implies that the mod 2 degree of the mapping χ at any Pareto optimum is 1. Since χ is proper, the mod 2 degree of χ is homotopy invariant, so $\deg_2 \chi = 1$ for all $(q, \omega) \in L \times \Omega$. This in turn implies that the number of equilibrium bids associated with any (q, ω) is (generically) odd and, since χ is onto, that an interior Nash equilibrium always exists.

Before giving the formal definition of (the subset of offers) L, we need the following assumption.

Assumption 4.1. Endowments lie in the set $\Omega \subset \mathbb{R}^{\ell n}_{++}$ defined by the con-

[5]We are grateful to Pradeep Dubey for showing us that our Proposition 3.1 follows from Proposition 3 of Dubey and Rogawski (1990). Several calculations (and related Remarks) from our proof are used in sections 4 and 5.

dition $\Omega=\{\omega\in\mathbb{R}^{\ell n}_{++}\,|\,0<\underline{\omega}<\omega<\bar{\omega}<\infty\}$, where we have $\underline{\omega}\in\mathbb{R}^{\ell n}_{++}$, $\bar{\omega}\in\mathbb{R}^{\ell n}_{++}$, and $\underline{\omega}<\bar{\omega}$.

Definition 4.2. Let $\zeta(\omega)>0$ be defined by the condition that for all h and i, $x^i_h>\zeta(\omega)$ for all allocations x_h in the set Π_h [given in expression (2.14)]. Let the scalar ζ be defined by $\zeta=\inf_\Omega\zeta(\omega)/2=1/2\inf_\Omega\zeta(\omega)$. Such a scalar ζ exists and is bounded above zero under our assumptions on preferences. Let M be an arbitrary positive scalar. Next define L, the set of sufficiently large offers, by

$$L=\{q\in\mathbb{R}^{\ell n}_{++}\,|\,\omega^i_h+M>q^i_h>\omega^i_h-\zeta \text{ for } i=1,\dots,\ell,\ h=1,\dots,n, \text{ and } \omega\in\Omega\}. \tag{4.3}$$

We shall be looking for equilibria in which the offers are in the set L.[6] It follows from the proof of Proposition (2.12) that all such Nash equilibria are interior. We show next that they must also be uniformly bounded away from zero. This will yield the properness of the mapping χ described above.

By positivity of bids, we may renormalize bids on market 1 such that $B^1=\sum_{k=1}^n b^1_k=1$. Individual rationality for consumer h implies

$$b^1_h\geqq\frac{\zeta+q^1_h-\omega^1_h}{Q^1/B^1}. \tag{4.4}$$

It follows that, at any equilibrium with $q\in L$, there exists $\varepsilon>0$ such that $b^1_h>\varepsilon$ for all $h=1,\dots,n$ and for all ω in the closure of Ω. That is, this bound on equilibrium bids does not depend on ω. We now show that bids on all other markets are bounded as well, given the renormalization, $B^1=1$.

Lemma 4.5. There exist constants $\bar{\theta}$ and $\underline{\theta}$ such that for $i=2,\dots,\ell$ we have $0<\underline{\theta}\leqq B^i\leqq\bar{\theta}<\infty$ for any equilibrium of Γ with $q\in L$.

Proof. Fix $\omega\in\Omega$. Since equilibrium bids are individually rational, the equilibrium is interior, and hence the interior first-order conditions for best responses obtain. By the continuity of the u_h and the fact that sets Π_h [defined in (2.14)] are compact, it follows that there exist positive constants $\alpha(\omega)$ and $\beta(\omega)$ such that at any interior equilibrium we have

[6]We are grateful to the *JME* referee for pointing out that L must have no boundary in order for the degree theory approach to be applied. To show existence of NE for $q=\omega$, we therefore allow L to contain offers greater than endowments and assume that allocations are given by eqs. (2.2) even though $q^i_h>\omega^i_h$ for some (or all) h and i. If such offers were allowed in the game, Theorem (4.10) shows that they would be consistent with interior Nash equilibrium. Refer also to footnote 1 above.

$$\alpha(\omega) \leqq \frac{\partial u_h/\partial x_h^i}{\partial u_h/\partial x_h^j} \leqq \beta(\omega)$$

for $h=1,\ldots,n$ and $i,j=1,\ldots,\ell$. Using the first-order conditions to express marginal rates of substitution in terms of bids and offers, we get

$$\alpha(\omega) \leqq (Q_h^i/B_h^i)(B^i/Q^i)^2(B_h^i/Q_h^j)(B^j/Q^j)^2 \leqq \beta(\omega) \tag{4.6}$$

for $h=1,\ldots,n$ and $i,j=1,\ldots,\ell$. Let $j=1$. By Definition (4.2), the left-hand side of inequality (4.6) implies that

$$\alpha(\omega) \leqq \left(\frac{\sum_{k\neq h}(\omega_k^i+M)}{B_h^i}\right)\left(\frac{B^i}{\sum_k(\omega_k^i-\zeta)}\right)^2\left(\frac{1}{\sum_{k\neq h}(\omega_k^1-\zeta)}\right)(\textstyle\sum_k(\omega_k^1+M))^2$$

holds for $i=2,\ldots,\ell$ and all h. Collecting all terms not involving bids on the left-hand side of this inequality and denoting the resulting expression by $M_1^i(\omega)$, we have

$$M_1^i(\omega) \leqq (B^i)^2/B_h^i$$

for $i=1,\ldots,\ell$ and $k=1,\ldots,n$. For some consumer h', we have $(B_h^i/B^i) \geqq 1/2$. Hence we have

$$\frac{1}{2}M_1^i(\omega) \leqq \frac{B_{h'}^i}{B^i}M_1^i(\omega) \leqq B^i$$

for $i=1,\ldots,\ell$. Let $\underline{\theta}(\omega)$ be defined by $\underline{\theta}(\omega)=\min[1/2M_1^i(\omega)]$. Finally, by Assumption (4.1), all endowments are bounded away from zero so that $\omega_h^i-\zeta$ is also uniformly bounded away from zero for all h and i, and we can define $\underline{\theta}$ by

$$\underline{\theta}=\inf_{\Omega}\underline{\theta}(\omega)>0.$$

A similar set of estimates on the right-hand side of inequality (4.6) yields the corresponding value of the upper bound $\bar{\theta}$. □

We have shown that the aggregate bids B^i are bounded away from zero. We next establish that the individual bids are bounded away from zero.

Lemma 4.7. Consider the game $\Gamma(q;\omega)$ where $\omega\in\Omega$ and $q\in L$. Then the (interior) NE bids $b\in\Delta$ are bounded away from zero.

Proof. From expression (4.4), we have $b_h^i \geqq k_h^i(q,\omega)B_h^i$, where $k_h^i(q,\omega) = [q_h^i - (\omega_h^i - \zeta)]/(\omega_h^i + Q_h^i - \zeta) > 0$. Hence we have $[1 + k_h^i(q,\omega)]b_h^i > k_h^i(q,\omega)B^i$, which yields

$$b_h^i \geqq \left(\frac{k_h^i(q,\omega)}{[1+k_h^i(q,\omega)]}\right)B^i \geqq \left(\frac{k_h^i(q,\omega)}{[1+k_h^i(q,\omega)]}\right)\underline{\theta} > 0. \qquad \square$$

We are ready to begin the degree-theory proof to establish the existence of interior Nash equilibrium. [The reader is referred to Balasko (1987, chapters 4 and 5) for the mathematics behind our general method of proof. We adapt the Balasko analysis of the competitive model to the present market-game setting.]

Lemma 4.8. Define the natural projection $\chi: E \to L \times \Omega$ as the restriction of the orthogonal projection $\pi: \Delta \times L \times \Omega \to L \times \Omega$ to the equilibrium manifold E. Then the natural projection χ is proper.

Proof. Since Δ is compact, it is sufficient to show that if a sequence $\{q_\tau, \omega_\tau\} \in L \times \Omega$ of regular offers and endowments converges in $L \times \Omega$ to a regular endowment and offer pair (q,ω), and $\{b_\tau\}$ is an associated convergent sequence of regular, interior equilibrium bids, then the bid $b = \lim_{\tau\to\infty} b_\tau$ is an interior equilibrium bid associated with the endowment-offer pair $(q,\omega) = \lim_{\tau\to\infty}(q_\tau, \omega_\tau)$.

Note first that by the previous lemma, we may renormalize bids so that they lie in the unit simplex Δ. For regular equilibria, we know that the equilibrium bids are given locally as continuous functions $b(q,\omega)$. Hence we have $\lim_{\tau\to\infty} b_\tau = \lim_{\tau\to\infty} b(q_\tau,\omega_\tau) = b(q,\omega) = b$, so we know that b is in fact an equilibrium bid. That b must be an interior Nash equilibrium bid now follows immediately from Lemma (4.5). Hence the natural projection χ is proper. $\square$

Lemma 4.9. The mod 2 *degree of χ is* 1.

Proof. Since χ is proper, the mod 2 degree of χ is a homotopy invariant [see Balasko (1987, page 236)]. Hence it suffices to calculate the degree of χ at any regular interior equilibrium. Consider the case where endowments are Pareto optimal. Then, for any q, the equilibrium bids $b \in \Delta$ are uniquely determined by the conditions

$$(q_h^i/b_h^i) = (Q^i/B^i) \quad \text{for} \quad i = 1,\ldots,\ell \text{ and } h = 1,\ldots,n$$

and

$$\frac{\partial u_h(\omega_h)/\partial x_h^i}{\partial u_h(\omega_h)/\partial x_h^j}=\frac{B^iQ^j}{Q^iB^j} \quad \text{for} \quad h=1,\ldots,n \text{ and } i,j=1,\ldots,\ell.$$

If we can show that these equilibrium bids, offers, and endowments represent a regular value, it would follow that $\deg_2 \chi=1$, so that for all regular offer–endowment pairs, the number of equilibria is odd [see Balasko (1987, p. 234)]. To show that some Pareto optimal equilibrium is regular, we need to show that we have $\xi \pitchfork \{0\}$ when ω is Pareto optimal. We shall show this by calculating $D_q\xi$ and showing that it has full rank for some $\bar{q}\in L$. It will then follow that $D_q\xi$ has full rank on an open neighborhood, $N(\bar{q})$, of $\bar{q}$. Then by the transversal density theorem, for an open and dense subset $N_0(\bar{q})\subset N(\bar{q})$, we have $\xi_q \pitchfork \{0\}$. Hence for $q\in N_0(\bar{q})$, the Pareto optimal interior Nash equilibrium is regular.

To calculate the rank of the Jacobian $D_q\xi$, we use the fact that at any Pareto optimal equilibrium, all allocations are autarkic [Proposition (2.19)]. As in the previous section, the rank of $D_q\xi$ is determined by the rank of the Jacobian of consumer h's best responses, D_qb. We can calculate the rank of the Jacobian D_qb by using the fact that at an interior, Pareto optimal equilibrium the allocation rule yields the identity

$$\hat{B}q_h=\hat{Q}b_h$$

for $h=1,\ldots,n$. Differentiating this expression with respect to q then yields the Jacobian matrix

$$\begin{bmatrix} \hat{Q}^{-1}\hat{B} & \hat{Q}_1^{-1}\hat{b}_1 & \ldots & \hat{Q}_1^{-1}\hat{b}_1 \\ \hat{Q}_2^{-1}\hat{b}_2 & \hat{Q}^{-1}\hat{B} & \ldots & \hat{Q}_2^{-1}\hat{b}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{Q}_n^{-1}\hat{b}_n & \hat{Q}_n^{-1}\hat{b}_n & \ldots & \hat{Q}^{-1}\hat{B} \end{bmatrix}.$$

Since $\hat{Q}^{-1}\hat{B}=\hat{Q}_h^{-1}\hat{B}_h$ for all h at any interior, Pareto optimal Nash equilibrium, the determinant of D_qb takes the form

$$|D_qb|=\pm|\hat{Q}_1^{-1}\hat{B}_1|\ldots|\hat{Q}_n^{-1}\hat{B}_n| \begin{vmatrix} I & \ldots & \hat{B}_1^{-1}\hat{b}_1 \\ \vdots & \ddots & \vdots \\ \hat{B}_n^{-1}\hat{b}_n & \ldots & I \end{vmatrix}.$$

Since the equilibrium is interior, we have $|\hat{Q}_h^{-1}\hat{B}_h|\neq 0$, so the rank of D_qb turns on the rank of the large matrix above. Since $\hat{B}_h^{-1}\hat{b}_h=\hat{Q}_h^{-1}\hat{q}_h$ at any (interior) Pareto-optimal equilibrium, this matrix is equivalent to the matrix

$$\bar{Q}=\begin{bmatrix} \hat{Q}_1 & \cdots & \hat{q}_1 \\ \vdots & \ddots & \vdots \\ \hat{q}_n & \cdots & \hat{Q}_n \end{bmatrix}.$$

The matrix $\bar{Q}$ in turn is rank-equivalent to the matrix

$$\bar{Q}'=\begin{bmatrix} \hat{Q}_1-\hat{q}_1 & 0 & \cdots & \hat{q}_1 \\ 0 & \hat{Q}_2-\hat{q}_2 & \cdots & \hat{q}_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_n \end{bmatrix},$$

where $M_n=\hat{Q}_n+\sum_{k=1}^{n-1}[\hat{Q}_n-\hat{q}_n][\hat{Q}_k-\hat{q}_k]^{-1}\hat{q}_k$, assuming that the inverse matrices exist. We note that for generic values of q_h, $\hat{Q}_h-\hat{q}_h$ will, in fact, have full rank. If the matrix M_n has full rank, then $\bar{Q}'$ will have full rank and hence, $\mathrm{D}_q b$ will be of full rank. For M_n to be singular requires (since all of the matrixes involved are diagonal) that we have for some market i

$$Q_n^i+\sum_{k\neq n}\left(\frac{Q_n^i-q_n^i}{Q_k^i-q_k^i}\right)q_k^i=0.$$

If this expression holds, however, we can make a perturbation of q_n^i which yields something nonzero. This can be verified by calculating the derivative

$$\frac{\partial M_n^i}{\partial q_n^i}=-\sum_{k\neq n}\left[\frac{q_k^i}{(Q_k^i-q_k^i)^2}(Q_k^i-q_k^i+Q_n^i-q_n^i)\right].$$

Since we have $Q_k^i+Q_n^i-q_k^i-q_n^i=2\sum_{j\neq k,n}q_j^i>0$, this derivative is negative. Hence for generic q, $\mathrm{D}_q b$ has full rank. We let $\bar{q}$ be such that $\mathrm{D}_q b$ has full rank. By the argument above, we obtain a unique, regular Pareto optimal equilibrium at $\bar{q}$ with associated bids $\bar{b}$. □

Theorem 4.10. For any feasible $(q,\omega)\in L\times\Omega$, there exists an interior Nash equilibrium.

Proof. By Lemma (4.9), we know that $\deg_2\chi=1$. Hence the mapping χ is onto. To show this, suppose that for some (q,ω), $\chi^{-1}(q,\omega)=\varnothing$. Then (q,ω) is regular for χ, but the number of pre-image points is 0 which is even. This contradicts the fact that $\deg_2\chi=1$. Hence, $\chi^{-1}(q,\omega)\neq\varnothing$ for any pair of offers and endowments in $L\times\Omega$. □

We have shown that interior NE exist for $\omega \in \Omega \subset \mathbb{R}^{\ell n}_{++}$ and $q \in L \subset \mathbb{R}^{\ell n}_{++}$. The set of interior NE is parametrized by endowments ω and market thickness q. We turn next to the analysis of the structure of the set of interior NE in allocation space.

5. Structure of the set of interior Nash equilibria in allocation space

We wish to show that for an open and dense set of endowments, the set of equilibrium allocations has dimension $\ell n - \ell$. From the structure result of section 3, we know that, for a regular economy (q_0, ω_0), the equilibrium bids b can be expressed locally as differentiable functions of offers and endowments on a neighborhood of (q_0, ω_0). Let $L_0 \times \Omega_0$ denote such a neighborhood, and let $b(q, \omega)$ denote the equilibrium bid function, so we have $b: L_0 \times \Omega_0 \to \Delta$.

Substitute the mapping $b(q, \omega)$ into the allocation equation (2.2) to obtain the mapping $\rho(q, \omega)$. Define the mapping of equilibrium net trades $z(q, \omega)$ by $z(q, \omega) = \rho(q, \omega) - \omega$. If we are treating ω as fixed, the function mapping offers q into equilibrium net trades is denoted $z_\omega(q)$. We know that if endowments are Pareto optimal, $z_\omega(q)$ is identically zero; however, this is a degenerate case. For the case in which endowments are not Pareto optimal, we shall show that, except for a closed nowhere dense set of offers, $z_\omega(q)$ is locally a submersion and hence maps neighborhoods of any given offer q_0 onto neighborhoods of $z_\omega(q_0)$ in allocation space. The dimension of the set of equilibrium allocations is $\ell n - \ell$, which is the same as the dimension of the set of feasible allocations. Since the choice of (q_0, ω_0) was arbitrary, the set of equilibrium allocations has dimension $\ell n - \ell$ in the neighborhood of any regular economy.

Our result can be intuitively explained as follows. There are ℓn allocation equations from (2.2) that must hold with equality, of which ℓ are redundant. There are also $n-1$ independent budget equations from (2.1) that must hold with equality. First, we fix q and consider perturbations of ω. There are ℓn degrees of freedom in perturbing ω, and there are ℓ market clearing restrictions. However, there are $n-1$ additional restrictions due to the budget constraints. The reason is that perturbing ω also perturbs the bids, so that a particular consumer's bids may adjust so as to violate another consumer's budget constraint. Thus, by considering endowment perturbations, we find $\ell n - \ell - (n-1)$ dimensions for the set of equilibrium allocations. By considering perturbations of q, holding endowments fixed, we recover the $(n-1)$ dimensions that were lost earlier. This intuition then suggests that the set of equilibrium allocations has generic dimension $\ell n - \ell$.

The formal analysis involves the derivative matrix $\mathrm{D}z(q, \omega)$. The maximal rank of $\mathrm{D}z$ is $\ell n - \ell$. We will show that for any $(q, \omega) \in L_0 \times \Omega_0$, $\mathrm{D}z$ in fact has exactly rank $\ell n - \ell$. For this analysis, it will prove convenient to

normalize bids such that $b_n^\ell = 1$. With this normalization, we also let $\bar{P}$ denote the projection on the space of normalized bids.

We first calculate the derivative matrix $\mathrm{D}_q z_h$ for an arbitrary consumer h. Express consumer h's allocations as

$$x_h = \omega_h - q_h + \hat{B}^{-1}\hat{Q}b_h, \tag{5.1}$$

which yields

$$\hat{B}z_h = -\hat{B}q_h + \hat{Q}b_h, \tag{5.2}$$

where z_h is consumer h's net trade. Differentiating expression (5.2) with respect to q_h yields

$$\begin{aligned}\hat{B}\mathrm{D}_{q_h}z_h + \hat{z}_h\mathrm{D}_{q_h}B &= -\hat{B} - \hat{q}_h\mathrm{D}_{q_h}B + \hat{b}_h + \hat{Q}\mathrm{D}_{q_h}b_h \\ &= -\hat{B}_h + \hat{Q}_h\mathrm{D}_{q_h}b_h - \hat{q}_h\mathrm{D}_{q_h}B_h\end{aligned}$$

(where the last step follows from writing $\hat{Q} = \hat{Q}_h + \hat{q}_h$ and $\hat{B} = \hat{B}_h + \hat{b}_h$). This in turn yields

$$\hat{B}\mathrm{D}_{q_h}z_h = -\hat{B}_h + (\hat{Q}_h - \hat{z}_h)\mathrm{D}_{q_h}b_h - (\hat{z}_h + \hat{q}_h)\mathrm{D}_{q_h}B_h. \tag{5.3}$$

Since $\hat{z}_h + \hat{q}_h = \hat{B}^{-1}\hat{Q}\hat{b}_h$, it follows that we have

$$\hat{Q}_h - \hat{z}_h = \hat{Q}_h - (-\hat{q}_h + \hat{B}^{-1}\hat{Q}\hat{b}_h) = \hat{Q} - \hat{B}^{-1}\hat{Q}\hat{b}_h. \tag{5.4}$$

Substituting expression (5.4) into expression (5.3) yields

$$\hat{B}\mathrm{D}_{q_h}z_h = \hat{Q}[I - \hat{B}^{-1}\hat{b}_h]\mathrm{D}_{q_h}b_h - \hat{B}^{-1}\hat{Q}\hat{b}_h\mathrm{D}_{q_h}B_h - \hat{B}_h,$$

which in turn yields

$$\hat{B}\mathrm{D}_{q_h}z_h = \hat{B}^{-1}\hat{Q}[\hat{B}_h\mathrm{D}_{q_h}b_h - \hat{b}_h\mathrm{D}_{q_h}B_h] - \hat{B}_h. \tag{5.5}$$

By a similar calculation, one obtains the following expression involving the derivative $\mathrm{D}_{q_k}z_h$:

$$\hat{B}\mathrm{D}_{q_k}z_h = \hat{Q}\hat{B}^{-1}[\hat{B}_h\mathrm{D}_{q_k}b_h - \hat{b}_h\mathrm{D}_{q_k}B_h] + \hat{b}_h \tag{5.6}$$

for $k \neq h$. Now, let Ψ be the $\ell n \times \ell n$ diagonal matrix having $\ell \times \ell$ diagonal blocks consisting of $\hat{B}$. Let Φ be defined similarly, but with $\ell \times \ell$ diagonal blocks consisting of $\hat{Q}$. Let Ξ be the matrix defined by

$$\Xi = \begin{bmatrix} -\hat{B}_1 & \hat{b}_1 & . & \dots & . & \hat{b}_1 \\ \hat{b}_2 & -\hat{B}_2 & \hat{b}_2 & \dots & . & \hat{b}_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{b}_{n-1} & \hat{b}_{n-1} & \dots & \dots & -\hat{B}_{n-1} & \hat{b}_{n-1} \\ \hat{b}_n & \hat{b}_n & . & \dots & \hat{b}_n & -\hat{B}_n \end{bmatrix}$$

and let Ξ' be the matrix Ξ with its last column deleted. Let $D_q b$ denote the derivative matrix of equilibrium bids with respect to q. This derivative is obtained from the mapping $\xi(b, q, \omega)$ for regular equilibria using the implicit function theorem, which yields

$$D_q b = -[D_b \xi]^{-1} D_q \xi.$$

The derivative matrix $D_q z$ can now be expressed as

$$D_q z = -\Psi^{-1}(\Phi \Psi^{-1} \Xi' D_q b - \Xi).$$

By a similar calculation, we find that the endowment derivatives take the form

$$D_\omega z = -\Psi^{-2} \Phi \Xi' D_\omega b.$$

Hence we have

$$Dz = [D_\omega z, D_q z] = [-\Psi^{-2}\Phi' D_\omega b, \Psi^{-1}\Xi - \Psi^{-2}\Phi\Xi' D_q b]. \tag{5.7}$$

We wish to show that Dz has full rank, i.e., rank $Dz = \ell n - \ell$. Consider $D_\omega z$. From the implicit function theorem, we have $D_\omega b = -[D_b \xi]^{-1} D_\omega \xi$ and $D_\omega \xi = \bar{P} D_\omega \tilde{b}$, where $\bar{P}$ is the projection on the space of normalized bids and $D_\omega \tilde{b}$ is a block diagonal matrix with components $D_{\omega_h} \tilde{b}_h$ on the main diagonal, and, as in section 3, $\tilde{b}_h$ is consumer h's best response bid given all other consumers' bids. By applying the implicit function theorem to the first-order conditions for determining best responses, we have

$$D_{\omega_h} \tilde{b}_h = K_h D^2 u_h \hat{Q} \hat{B}_h \hat{B}^{-2}.$$

At any interior NE, $D_{\omega_h} \tilde{b}_h$ has rank $\ell - 1$.

Let us show that $D_\omega z$ has rank $\ell n - \ell - n + 1$. Let $G_h = [D_{\omega_h} \tilde{b}_h, w_h]$, where $D_{\omega_h} \tilde{b}_h$ is the matrix of derivatives of best responses taken with respect to the first $\ell - 1$ commodities, and the vector w_h is such that $w_h^{\mathrm{T}} K_h = 0$. For each h, the matrix G_h has rank ℓ. Now, let $G = \mathrm{diag}_h[G_h]$, and let $\bar{G}$ denote the matrix obtained from G by deleting the last row. The matrix $\bar{G}$ has rank

$\ell n-1$. Now, let C be defined by $C=\Psi^{-2}\Phi\Xi'[\mathrm{D}_b\xi]^{-1}$. Since matrix Ξ' has rank $\ell(n-1)$ while all the other matrices in this expression are of full rank, C has rank $\ell(n-1)$. It follows then that the matrix $C\bar{G}$ has rank $\ell(n-1)$ since $\bar{G}$ is of full rank. This matrix, in turn, takes the form $C\bar{G}=[C\bar{P}\mathrm{D}_\omega\tilde{b}, CY]$, where $Y=\mathrm{diag}_h[w_h]$ is $(\ell n-1)\times(n-1)$. Since Y has $n-1$ linearly independent columns, the rank of CY is $n-1$. Since CG has rank $\ell(n-1)$, it follows that the rank of $C\bar{P}\mathrm{D}_\omega\tilde{b}$ is $\ell(n-1)-(n-1)=\ell n-\ell-n+1$. Since we have $C\bar{P}\mathrm{D}_\omega\tilde{b}=\mathrm{D}_\omega z$, the assertion follows.

With rank $\mathrm{D}_\omega z=\ell n-\ell-(n-1)$, it remains to show that the matrix $\mathrm{D}_q z$ has at least $n-1$ linearly independent columns which are also independent of the columns of $\mathrm{D}_\omega z$ in order to show that $\mathrm{D}z$ has full rank $\ell n-\ell$. To do this, we will show first that if endowments are not Pareto optimal, then $\mathrm{D}_q z$ has at least rank $n-1$. We then show that no column of $\mathrm{D}_q z$ lies in the subspace of the nullity of $\mathrm{D}_\omega z$ determined by the budget constraints. This subspace can be defined as follows. With $\mathrm{D}_\omega z=-\Psi^{-2}\Phi\Xi'\mathrm{D}_\omega b$ we know that $\Upsilon^{\mathrm{T}}\mathrm{D}_\omega z=0$, where $\Upsilon^{\mathrm{T}}=[I,\ldots,I]$ is an $\ell\times\ell n$ matrix which sums the net trades on each market. One can easily verify that this result holds because $\Upsilon^{\mathrm{T}}\Psi^{-2}\Phi\Xi'=0$. The vectors in the matrix Υ correspond to the linear restrictions on derivatives imposed by the market-clearing conditions. These vectors span an ℓ dimensional subspace in the nullity of $\mathrm{D}_\omega z$, which is of dimension $\ell+n-1$. Since $n>1$, this means that there exist $n-1$ vectors v_h which annihilate the matrix $\mathrm{D}_\omega z$ and are linearly independent of the vectors of Υ. From the calculations above, it follows that these vectors can be chosen such that

$$v_h^{\mathrm{T}}\Psi^{-2}\Phi\Xi'[\mathrm{D}_b\xi]^{-1}\bar{P}=[0,\ldots,w_h^{\mathrm{T}},\ldots,0]=w_h^{\mathrm{T}} \quad \text{for} \quad h=1,\ldots,n-1,$$

since the vectors w_h (and only these vectors) individually annihilate the $\mathrm{D}_{\omega_h}z_h$ matrices. Note also that since we never consider consumer n, this expression can also be written (with a slight abuse of notation) as

$$v_h^{\mathrm{T}}\Psi^{-2}\Phi\Xi F^{-1}=w_h^{\mathrm{T}},$$

where

$$F=\begin{bmatrix}\mathrm{D}_b\xi & 0\\ 0 & 1\end{bmatrix}$$

is a full rank extension of $\mathrm{D}_b\xi$. For the vectors v_h, then we have

$$v_h^{\mathrm{T}}\mathrm{D}_q z=v_h^{\mathrm{T}}\Psi^{-1}\Xi+v_h^{\mathrm{T}}\Psi^{-2}\Phi\Xi F^{-1}\mathrm{D}_q\tilde{b}.$$

We now prove the following proposition, showing that q-perturbations span the null space of $\mathrm{D}_\omega z$.

Proposition 5.8. If endowments ω are not Pareto optimal for the market game Γ, then for an open and dense set of q's, there exist $n-1$ linearly independent columns of $\mathrm{D}_q z$ which are independent of $\mathrm{D}_\omega z$.

The proof of Proposition (5.8) is based on two lemmas which we prove below. In order to state and prove the lemmas, we will need the following facts. First, because the matrix $\Psi^{-1}\Phi$ is diagonal with identical $\ell \times \ell$ blocks on the main diagonal (i.e., the matrices $\hat{B}^{-1}\hat{Q}$), this matrix commutes with the matrix Ξ. Hence we can write

$$\mathrm{D}_q z = \Psi^{-1}\Xi[I - \Psi^{-1}\bar{\Phi}\mathrm{D}_q b],$$

where $\bar{\Phi}$ is defined to be Φ with its last row deleted. Let $\eta_h^{\mathrm{T}} = v_h^{\mathrm{T}}\Psi^{-1}\Xi$. Since $v_h^{\mathrm{T}}\Psi^{-2}\Phi\Xi = v_h^{\mathrm{T}}\Psi^{-1}\Xi\Psi^{-1}\Phi$ has been assumed to be non-zero, it follows that η_h^{T} is not in the nullity of Ξ. One can also show that since the vectors v_h are independent, the vectors η_h are also independent, since Ψ is full rank and Ξ is full rank on the complement of its null space. Therefore, to show that $v_h^{\mathrm{T}}\mathrm{D}_q z \neq 0$ on $n-1$ linearly independent columns of $\mathrm{D}_q z$, it is sufficient to show that $\eta_h^{\mathrm{T}}[I - \Psi^{-1}\bar{\Phi}\mathrm{D}_q b] \neq 0$ for $h = 1, \ldots, n-1$, and that $[I - \Psi^{-1}\bar{\Phi}\mathrm{D}_q b]$ has at least rank $n-1$.

Lemma 5.9. If ω is not Pareto optimal, then we have rank $[I - \Psi^{-1}\bar{\Phi}\mathrm{D}_q b] \geqq n-1$ for an open and dense set of offers $q \in L_0$.

Proof. Suppose the matrix $[I - \Psi^{-1}\bar{\Phi}\mathrm{D}_q b]$ has rank less than $n-1$ at q_0. If the set

$$L_0^* = \{q \in L_0 \,|\, \mathrm{rank}\,[I - \Psi^{-1}\bar{\Phi}\mathrm{D}_q b] < n-1\}$$

is locally isolated, then by perturbing q away from q_0 and redefining the set L_0 to exclude the set L_0^*, we get the desired result. So assume that L_0^* is not locally isolated. The assumption that $[I - \Psi^{-1}\bar{\Phi}\mathrm{D}_q b]$ has rank less than $n-1$ implies that every collection of $n-1$ or more rows of the matrix is linearly dependent. Fix some commodity index $i \in \{1, \ldots, \ell\}$ and consider the $(n-1) \times \ell n$ matrix

$$\bar{S} - \frac{Q^i}{B^i}\mathrm{D}_q \bar{b}^i,$$

where $\bar{S}=[I_{n-1},0_{\ell n-n+1}]$ and $\bar{b}^i$ is b^i with consumer n's bid deleted. The matrix $[\bar{S}-(Q^i/B^i)(\mathrm{D}_q\bar{b}^i)]$ selects rows of $[I-\Psi^{-1}\bar{\Phi}\mathrm{D}_q b]$ corresponding to market i for consumers $1,\ldots,n-1$. By assumption, this matrix is singular, so there exists a vector $\mathrm{d}q_0$ such that

$$[\bar{S}-(Q^i/B^i)\mathrm{D}_q\bar{b}^i]\,\mathrm{d}q_0=0. \tag{5.10}$$

Since the set L_0^* is not isolated, there is also a path $q(t)\in L_0^*$, $t\in(-\varepsilon,\varepsilon)$ such that $q(0)=q_0$, $(\mathrm{d}q(t)/\mathrm{d}t)|_{t=0}=\mathrm{d}q_0$, and for all $t\in(-\varepsilon,\varepsilon)$,

$$[\bar{S}-(Q^i(t)/B^i(t))\mathrm{D}_q\bar{b}^i(t)]\dot{q}(t)=0, \tag{5.11}$$

where $\dot{q}$ denotes the 'time' derivative, $\mathrm{d}q(t)/\mathrm{d}t$. Eq. (5.11) thus determines an ordinary differential equation whose solution functions $\bar{b}^i(t)$ satisfy eq. (5.10) at $t=0$.

To determine the solution to eq. (5.11), consider the case where, for fixed commodity index i, there exists an agent h such that for $t\in(-\varepsilon,\varepsilon)$ we have $z_h^i(q(t))=0$. From the allocation rule, it follows that we have

$$q_h^i/b_h^i=Q^i/B^i. \tag{5.12}$$

By substituting $Q_h^i+q_h^i$ and $B_h^i+b_h^i$ for Q^i and B^i on the right-hand side of eq. (5.12), one can easily show that $q_h^i/b_h^i=Q_h^i/B_h^i$. Differentiating the identity $q_h^iB^i=b_h^iQ^i$ in the direction $\mathrm{d}q_0$ at q_0 then yields

$$[B^ie_h^i+q_h^i\mathrm{D}_qB^i-b_h^ie^i-Q^i\mathrm{D}_qb_h^i]\,\mathrm{d}q_0=0, \tag{5.13}$$

where $e_h^i=[0,\ldots,1,\ldots,0]\in\mathbb{R}^{\ell n}$ and $e^i=\sum_h e_h^i$. The matrix in eq. (5.13) can be rewritten as

$$[-b_h^i,\ldots,B_h^i,\ldots,-b_h^i]\begin{bmatrix}e_1^i\\ \vdots\\ e_h^i\\ \vdots\\ e_n^i\end{bmatrix}+[q_h^i,\ldots,-Q_h^i,\ldots,q_h^i]\begin{bmatrix}\mathrm{D}_qb_1^i\\ \vdots\\ \mathrm{D}_qb_h^i\\ \vdots\\ \mathrm{D}_qb_n^i\end{bmatrix}.$$

Since we have $q_h^i=Q^ib_h^i/B^i$ and $Q_h^i=Q^iB_h^i/B^i$, it follows that expression (5.13) can be written as

$$[-b_h^i,\ldots,B_h^i,\ldots,-b_h^i]\,[S'-(Q^i/B^i)\mathrm{D}_qb^i]\,\mathrm{d}q_0=0, \tag{5.14}$$

where $S'=[I_n,0_{\ell n-n}]$. By the hypothesis of the lemma, the second matrix in

brackets in (5.14) is annihilated by the vector dq_0. Hence, by uniqueness of solutions to differential equations, the solution function $b(t)$ to eq. (5.11) is such that $z_h^i(q(t))=0$ along the path $q(t)$.

Note next that by taking $h=1,\ldots,n-1$ in eq. (5.14), we change only the first vector of bids in the equation. Hence for all h, we have $z_h^i(q(t))=0$. Finally, given our assumption that $[I-\Psi^{-1}\bar{\Phi}D_q b]$ has rank less than $n-1$, eq. (5.14) must also hold for $i=1,\ldots,\ell$. Hence, for all i and h, we have $z_h^i(q(t))=0$. Proposition (2.9) thus implies that ω is Pareto optimal, contrary to hypothesis. □

We show next that the columns of $D_q z$ are independent of those of $D_\omega z$.

Lemma 5.15. For $j=1,\ldots,n-1$ and, $k=1,\ldots,n$, we have $v_j^T D_{q_k} z \neq 0$.

Proof. We need to calculate

$$v_j^T D_q z = v_j^T \Psi^{-1}\Xi + v_j^T \Psi^{-1}\Xi\Psi^{-1}\Phi F^{-1}\bar{P}D_q\tilde{b}. \tag{5.16}$$

Since we have $v_j^T\Psi^{-1}\Xi\Psi^{-1}\Phi F^{-1}=w_j^T$, it follows that $v_j^T\Psi^{-1}\Xi=w_j^T F\Phi^{-1}\Psi=w_j^T D_{\bar{b}}\xi\bar{P}\Phi^{-1}\Psi\bar{P}$ holds, where the last equality follows from the fact that $w_{jn}^\ell=0$, given our normalization of the bids. From the definition (in section 3) of the mapping ξ, one can check that, at an equilibrium b, we have

$$D_{\bar{b}}\xi=\bar{P}[I-D_{\bar{b}}\tilde{b}].$$

The matrix $D_{\bar{b}}\xi$ is of the form

$$D_{\bar{b}}\xi=\bar{P}\begin{bmatrix} I & \ldots & -M_1 \\ \vdots & \ddots & \vdots \\ -M_n & \ldots & I \end{bmatrix},$$

where the matrices M_h are obtained from the first-order conditions by applying the implicit function theorem, and

$$M_h=-K_h[\hat{b}_h\hat{Q}\hat{B}^{-2}D^2u_h\hat{Q}\hat{B}_h\hat{B}^{-2}]-u_h q_h^T\hat{Q}^{-1}.$$

Note that for all h, $w_h^T M_h=-q_h^T\hat{Q}^{-1}$, so we have

$$v_j^T\Psi^{-1}\Xi=[q_j^T,\ldots,-Q_j^T,\ldots,q_j^T]\Phi^{-2}\Psi.$$

The second term in expression (5.16) takes the form $v_j^{\mathrm{T}}\Psi^{-1}\Xi\Psi^{-1}\Phi F^{-1}\bar{P}\mathrm{D}_q\tilde{b}=w_j^{\mathrm{T}}\mathrm{D}_q\tilde{b}=[w_j^{\mathrm{T}}E_j,\ldots,w_j^{\mathrm{T}}E_{jj},\ldots,w_j^{\mathrm{T}}E_j]$, where, as in section 3, $E_{hh}=\mathrm{D}_{q_h}\tilde{b}_h$ and $E_h=\mathrm{D}_{q_k}\tilde{b}_h$ for all $k\neq h$. From the calculations of section 3, we have $w_h^{\mathrm{T}}E_h=q_h^{\mathrm{T}}\hat{Q}^{-2}\hat{B}$ and $w_h^{\mathrm{T}}E_{hh}=-Q_h^{\mathrm{T}}\hat{Q}^{-2}\hat{B}$ for all h. With these calculations, we now have

$$\begin{aligned} v_j^{\mathrm{T}}\mathrm{D}_q z &= [q_j^{\mathrm{T}},\ldots,-Q_j^{\mathrm{T}},\ldots,q_j^{\mathrm{T}}]\Phi^{-2}\Psi+[q_j^{\mathrm{T}},\ldots,-Q_j^{\mathrm{T}},\ldots,q_j^{\mathrm{T}}]\Phi^{-2}\Psi \\ &= 2[q_j^{\mathrm{T}},\ldots,-Q_j^{\mathrm{T}},\ldots,q_j^{\mathrm{T}}]\Phi^{-2}\Psi. \end{aligned}$$

Since $\Phi^{-2}\Psi$ is a diagonal matrix and the preceding vector is not zero, it follows that, at an interior Nash equilibrium, no column of $\mathrm{D}_q z$ is orthogonal to any of the vectors v_j for $j=1,\ldots,n-1$. □

Proof of Proposition 5.8. The proposition now follows from the two lemmas. By Lemma (5.9), we know that if endowments are not Pareto optimal, then for generic q, $\mathrm{D}_q z$ has at least $n-1$ linearly independent columns. By Lemma (5.15), each of these columns is independent of those of $\mathrm{D}_\omega z$. □

Corollary 5.17. For an open and dense subset of $L_0\times\Omega_0$, the rank of $[\mathrm{D}_q z,\mathrm{D}_\omega z]$ is $\ell n-\ell$.

Proof. Obvious. □

Corollary 5.18. For an open and dense subset of Ω_0, the rank of $\mathrm{D}_q z$ is $\ell n-\ell$.

Proof. By Corollary (5.17), the mapping $z: L_0\times\Omega_0\to\mathbb{R}^{\ell n-\ell}$ is a submersion and hence is transverse to anything. By the transversal density theorem, then, the mapping $z_\omega: L_0\to\mathbb{R}^{\ell n-\ell}$ is transverse to almost any point in $\mathbb{R}^{\ell n-\ell}$, which implies that the derivative matrix has full rank at almost any allocation in the range of z_ω. Since the null space of $\mathrm{D}_\omega z$ is of dimension $n-1$ and the mapping z is a submersion, it follows that z_ω is a submersion for an open and dense subset of Ω_0. □

Corollary 5.19. For generic ω, we have $\dim[Im\, z_\omega(q)]=\ell n-\ell$.

Proof. Since z_ω is generically a submersion, it acts locally like a projection and hence maps L_0 onto some neighborhood in $\mathbb{R}^{\ell n-\ell}$ having full dimension $\ell n-\ell$. The only endowments for which the dimension is less than $\ell n-\ell$ are the Pareto optimal endowments and endowments in the set $\{\omega\in\Omega|(q,\omega)$ is a singular point for all $q\in L\}$. □

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