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18

Neoclassical Growth Models

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Beginning with the classical economists, continuing through Adam Smith, Irving Fisher, and John Maynard Keynes, and down to contemporary theorists, *intertemporal economics* has been a central subject for our discipline. As the study of how individuals and societies choose between present and future consumption, intertemporal economics incorporates such classic themes as saving, investment, and the accumulation of wealth.

During the early 1950s, the center stage of economic theory was occupied by general equilibrium. In this period the Arrow-Debreu-McKenzie advances came to flower.[1] The general equilibrium model is by definition microeconomic and was, in its original form, largely intratemporal. Interpretation of the theory did allow for "dated commodities," but few of the special features of intertemporal choice, technology, and markets were incorporated. The early 1950s was also a time of intense policy interest in the subject of economic growth. Part of this interest was due to the Cold War and the perceived economic competition between the Soviet Union and the United States; another (not unrelated) part was due to the increased concern for economically less developed countries.

By the mid-fifties the time was right for a resurgence of intellectual interest in the process of economic development. In 1956, the *Quarterly Journal of Economics* published Robert M. Solow's "A Contribu-

[1]Kenneth J. Arrow and Gerard Debreu, "Existence of an Equilibrium for a Competitive Economy," *Econometrica* 22 (July 1954): 265–290: Lionel McKenzie, "On Equilibrium in Graham's Model of World Trade and Other Competitive Systems," *Econometrica* 22 (April 1954): 147–161.

tion to the Theory of Economic Growth." The analysis is in the macroeconomic tradition, that is, highly aggregative. Four fundamental economic quantities are featured: consumption, investment, capital, and labor. Solow's direct and indirect debts to predecessors (including Knut Wicksell, Frank P. Ramsey, Sir Roy Harrod, Evsey Domar, and James Tobin) are clearly apparent and some of his themes were independently published in the same year by Trevor W. Swan.[2] Nonetheless, it is Solow's 1956 contribution that seems to have been pivotal in touching off a decade of intense study in what has been called "the neoclassical theory of economic growth." Not only was Solow's theory sufficiently clear and sufficiently simple to be widely accessible, but his exposition was also suggestive of many important elaborations that would eventually follow.

The One-Sector Technology

I begin with this very simple and special case because it is a good vehicle for expounding some of the basic ideas. Generalizations and complications follow soon enough.

At an instant t, it is assumed that the flow of homogeneous output, $Y(t)$, is produced by the cooperation of two factors: the currently employed stock of machines (or capital), $K(t)$, and the currently employed labor force, $L(t)$. Technologically efficient input-output combinations are described by the production function, $F(\cdot)$, so that $Y(t) = F(K(t), L(t), t)$. To simplify matters further, technological change is

[2]Robert M. Solow's seminal article is "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics* 70 (February 1956): 65–94. Trevor W. Swan's companion piece, "Economic Growth and Capital Accumulation," appears in the *Economic Record* 32 (November 1956): 334–336. Sir Roy Harrod's "An Essay in Dynamic Theory" is from *Economic Journal* 40 (March 1939): 14–33, with errata on p. 377 of the June 1939 issue. Evsey Domar's "Capital Expansion, Rate of Growth, and Employment" appears in *Econometrica* 14 (April 1946): 137–147. Solow presents a neat analysis of what he calls the "Harrod-Domar" model. While Solow accurately captures the essence of Domar's analysis, Harrod's model is rather different from Solow's exposition of it. James Tobin's contribution, "A Dynamic Aggregative Model," appears in the *Journal of Political Economy* 63 (April 1955): 103–115. All of the above works are anthologized in an attractive volume edited by Joseph E. Stiglitz and Hirofumi Uzawa, *Readings in the Modern Theory of Economic Growth* (Cambridge, MIT Press, 1969). Another useful collection is *Growth Economics*, ed. Amartya Sen (Harmondsworth, Middlesex: Penguin, 1970). A careful, complete textbook teaching the best state of the art as of 1970, is Edwin Burmeister and A. Rodney Dobell's *Mathematical Theories of Economic Growth* (New York: Macmillan, 1970). For those who do not want to go so far, P. A. Neher has a primer on growth, *Economic Growth and Development* (New York: Wiley, 1971). (I am rather fond of the phase diagram on the dust jacket of Neher's book; it shows a morphogenetic model of endogenous technical change).

assumed away, i.e., $\partial F/\partial t \equiv 0$, so that the production relation can be rewritten as

$$Y(t) = F(K(t), L(t)).$$

What is this homogeneous output, Y ? The answer that is often given is this: the homogeneous output can be used perfectly well for either consumption or investment. In particular, if $C(t)$ is consumption at time t, and $Z(t)$ is investment at time t, we have that $C(t) + Z(t) \leq Y(t)$. In this interpretation, Y is sometimes thought of as a fanciful good like the shmoos in Al Capp's cartoon strip. Actually, Y is only a "helping variable," and can easily be dropped from the description of the one-sector technology. Technological opportunities could be described by the set of outputs and inputs lying in the feasible technology set, T , given by

$$T = \{(C, Z, K, L): C \geq 0, Z \geq 0, K \geq 0, L \geq 0, C + Z \leq F(K, L)\}.$$

Time-dependence in the above and in the following is not ordinarily indicated explicitly. Read the above expression as follows: a feasible production plan is a combination of nonnegative outputs and inputs, $(C, Z, K, L) \geq 0$, satisfying the production relation, $C + Z \leq F(K, L)$. Therefore, holding inputs K and L fixed, consumption goods and investment goods trade one-for-one along the (technogically efficient) production possibility frontier.

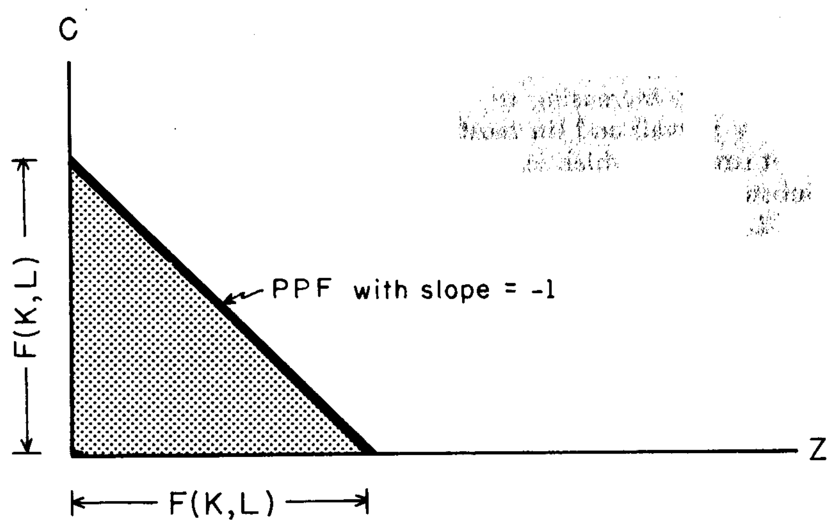

Figure 1.

Consider Figure 1, in which the heavy line segment is the production possibility frontier, or *PPF*. Because the *PPF* has a slope of minus unity throughout, an important feature of the model is that the supply price of consumption is always equal to the supply price of investment. If the market prices or, alternatively, the social demand prices of consumption and investment are not equal, the composition of output would be completely specialized to the good having the higher price.

This obviously very special technology has been seen behind the scenes in many macro-models including John R. Hicks' IS-LM account of the "Keynesian system."[3] The crucial feature is not that the absolute value of the PPF's slope is unity: what is important is that the slope is constant. Were that slope any other constant, a trivial transformation of the variables would return us to the one-sector world.

Constant returns to scale are assumed; the production function is then positively homogeneous of degree one so that, for any positive scalar θ, if $Y = F(K, L)$, then $\theta Y = F(\theta K, \theta L)$. For example, doubling the employment of each input yields a doubled output if technological efficiency is maintained. It is straightforward to generalize the analysis to incorporate decreasing returns. This can always be done by introducing fictitious fixed factors ("entrepreneurship"?) or real fixed factors (natural resources).[4] In the decreasing-returns case, steady states may not exist and may be less interesting when they do, but the absence of steady states should not be a drawback to serious analysis. Increasing returns, of course, cause deeper problems. In the case of increasing returns, optimal centralized development plans are much harder—although not impossible—to analyze. Most important is the fact that with increasing returns throughout, competition does not generally prevail and (in most cases) we are forced to abandon the market structure which economists understand best.

Substituting $\theta = 1/L$ in the definition of constant returns yields $Y/L = F(K/L, L/L)$. Denote by lower-case letters quantities in intensive (or per-worker) form, e.g., $y = Y/L$. Then rewrite as

$$y = F(k, 1) = f(k).$$

Output per worker is solely a function of capital per worker, reflecting a basic fact of constant returns.

[3]John R. Hicks, "Mr. Keynes and the 'Classics': A Suggested Interpretation," *Econometrica* 5 (April 1937): 147–159.

[4]See, e.g., David Cass' exposition of this old idea, "Duality: A Symmetric Approach from the Economist's Vantage Point," *Journal of Economic Theory* 7 (March 1974): 272–295.

The Continuous Production Function

If the production function is continuous and differentiable, then we can equate marginal products and derivatives. Marginal product of capital = $\partial Y/\partial K = f'(k)$. Marginal product of labor = $\partial Y/\partial L = f(k) - kf'(k)$. It is a basic theorem of duality theory that for a socialist economy marginal products represent social scarcity. We also know that for the competitive (profit maximizing) economy with no externalities, factors are rewarded by their marginal products, so that

$$r = f'(k) \quad \text{and} \quad w = f(k) - kf'(k),$$

where r and w are respectively the competitive rental and wage rates

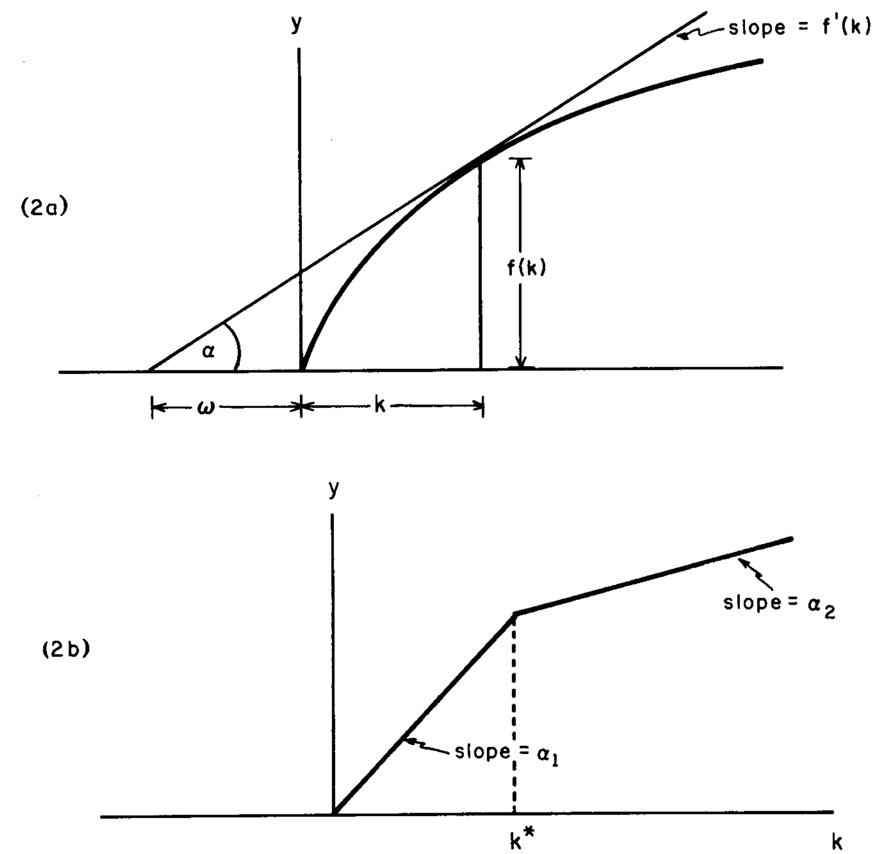

Figure 2.

in terms of output. Furthermore, r and w are also optimal (socialist) shadow rental and wage rates.

The relationship between capital intensity and factor rewards is described in Figure 2. Figure 2a shows a diminishing returns case: the second derivative, $f''(k)$, is negative throughout. The positively sloped line is tangent to the intensive production function at output per worker, $f(k)$. The tangent of the angle α is equal to $f(k)/(k + w) = f'(k)$. The intercept ω is also equal to the wage-rental ratio because $w/r = (f(k) - kf'(k))/f'(k)$.

The existence of such derivatives is not necessary for factor prices to be defined. In figure 2b, for $0 < k < k^*$ the marginal product of capital is α_1, and (because the intercept of tangents to the production function is at the origin) the marginal product of labor is zero. For $k > k^*$, $MP_K = \alpha_2$ and $MP_L = f(k) - \alpha_2 k$. For $k = k^*$, marginal products are defined but not unique, $\alpha_2 \leq MP_k \leq \alpha_1$ and $MP_L = f(k^*) - k^*MP_K$.

In the smooth, strictly concave case of Figure 2a in which $f''k) < 0$, we see that there is a one-to-one relationship between k and ω, with $d\omega/dk > 0$.

Economic Laws of Motion

The percentage increase in the labor force is assumed to be a positive constant, $\%\Delta L = n > 0$. Here n is the natural (or biological) rate of growth. In continuous time, this process is described by the simple differential equation $(dL\ /\ dt) = nL$ or if we use dots to denote time differentation (e.g., $\dot{L} \equiv dL/dt$), then $\dot{L}/L = n$.

Investment is defined to be accretion to the capital stock so if depreciation is absent (for simplicity), then $\dot{K} = Z$, where Z is net investment = gross investment. The percentage increase in capital intensity is the percentage increase in capital minus the percentage increase in the labor force, or

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = \frac{Z}{K} - n.$$

The growth equation can also be derived by logarithmically differentiating the identity, $k = K/L$, and it can be written as

$$\dot{k} = z - nk,$$

where $z = f(k) - c$ is investment per worker.

Steady States

In Figure 3, output per head, $f(k)$, is shown as a strictly concave function. The ray nk is also graphed. The intensive production function is assumed to be above the ray for small nonzero capital intensities and below the ray for very large capital intensities. Because $f(k) - nk = \dot{k} + c$, for a given capital intensity the difference between the curve and the ray shows the amount of "surplus" available for consumption or for increases in capital intensity. The point at which the ray and curve cross yields the maximum sustainable capital-labor ratio, $\tilde{k}$. Beyond $\tilde{k}$, even if all output is saved (and invested), investment per head will be less than that required to keep capital intensity constant in the face of population growth.

A *steady state* (or *balanced growth path*) is a path along which capital and labor grow at the same rate; that is, where $(\dot{K}/K) = (\dot{L}/L) = n$, or $\dot{k} = 0$. We see from Figure 3 that there is a range of possible steady-state capital-labor ratios, $0 \leq k \leq \tilde{k}$.

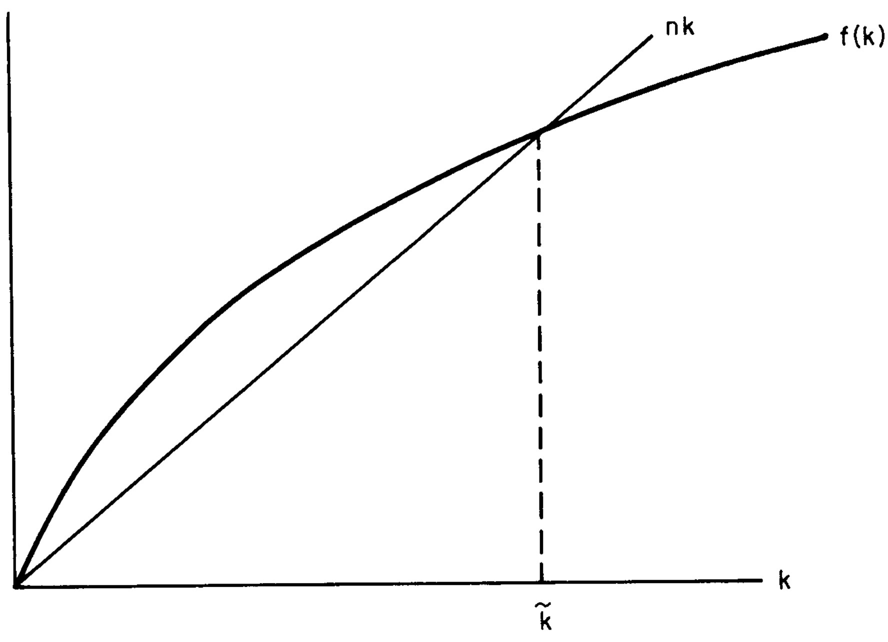

Figure 3.

Golden Rules

How do these steady states differ? Remember that $c = f(k) - nk - \dot{k}$. In steady states $\dot{k} = 0$, so we can write the steady-state supply of consumption per worker, c^S, as a function of capital per worker, i.e.,

$$c^S(k) = f(k) - nk.$$

In Figure 4, $c^S(k)$ is graphed.

The function rises from zero capital intensity, achieves a maximum at k^*, and is zero at $\bar{k}$. These properties can be verified from Figure 3 or directly, as $dc^S / dk = f'(k) - n$ and $d^2 c^S / dk^2 = f'(k) < 0$. c^S thus achieves a maximum at the capital-labor ratio k^*, where the marginal product of capital (or, as will be clear later, the interest rate) is equal to the biological growth rate. k^* is called the Golden Rule (GR) capital-labor ratio, discovered by Edmund S. Phelps and Joan Robinson, among others.[5] Notice that since $f'(k^*) = n$, $k^*f'(k^*) = nk^* = z^*$, so that at the GR if factors are rewarded by their marginal products—as would be the case under competition—then investment is exactly equal to aggregate rental income. (Of course, this consumption pattern obtains in the pure Marxian world, where "capitalists save all and workers consume all.")

k^* is an optimal capital-labor ratio in only a very restricted sense. The GR yields a greater consumption than any other steady state, but society may not choose to pursue a steady-state path; certainly it cannot elect its initial capital intensity, $k(0)$.

A society sacrifices current consumption in order to enlarge its capital with the aim of eventually enhancing future consumption. But notice, from Figures 3 and 4, that if the capital-labor ratio is forever bounded above k^*, then society has "oversaved," with insufficient withdrawals from Nature's bank. If at some point, a small unit of capital had been thrown away (or, if possible, consumed), then it

[5]There is a book on golden rules by the master, Edmund S. Phelps, *Golden Rules of Economic Growth* (New York: Norton, 1966), with complete references to the early literature. A careful proof of the Phelps-Koopmans inefficiency theorem and an exposition of optimal growth theory appear in my first Varenna lectures, "Applications of Pontryagin's Maximum Principle to Economics," in Kuhn and Szegö, eds. *Mathematical Systems Theory and Economics*, I (Berlin: Springer, 1969.) The Phelps-Koopmans theorem does not completely characterize inefficiency; this is done in a remarkable paper by David Cass, "On Capital Overaccumulation in the Aggregative, Neoclassical Model of Economic Growth: A Complete Characterization," *Journal of Economic Theory* 4 (April 1972): 200–223.

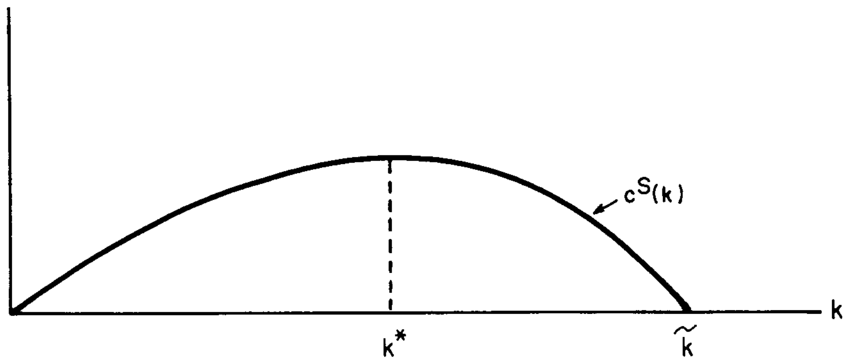

Figure 4.

would have been possible to follow a parallel accumulation path along which, from that date forward, consumption would have been strictly greater at each moment. This idea is the essence of the Phelps-Koopmans theorem: if there exist a time t_0 and a positive constant ϵ such that $k(t) \geq k^* + \epsilon$ for all $t \geq t_0$, then the program $\{k(t)\}$ is inefficient.

Phelps-Koopmans inefficiency stemming from overaccumulation is relevant only in infinite-horizon models. If the end of the world were known with certainty, an efficient accumulation program would involve running down capital stocks toward the terminal horizon.

Intertemporal Optimality

The GR analysis is useful in helping us to recognize overaccumulation, but we have yet to consider a criterion for intertemporal optimality. The optimal growth literature posits the maximand

$$\int_0^\infty U(c(t), t)\, dt,$$

where U is utility of consumption ($U' > 0$, $U'' \leq 0$). Frank P. Ramsey studied the case $\partial U/\partial t \equiv 0$, while David Cass, Tjalling C. Koopmans, and others have generalized the criterion to include the constant discount case in which $\partial U/\partial t = -\rho U$, where $\rho \geq 0$ is the constant social rate of time discount.[6] The consumption-optimal growth problem is constrained by technology and by initial endowments. This important constraint to initial endowments of capital and labor is what is lacking in the naive GR exercise. Solution of the Ramsey-Cass-Koopmans problems requires the calculus of variations and related techniques; although a full treatment lies beyond the scope of this essay, more about the characteristics of the solution appears later.

Descriptive Growth

So far, the emphasis has been on efficient and consumption-optimal growth. Solow's original problem was to study the behavior of growth in a decentralized economy in which choice between consumption and investment is determined by some aggregate behavioral relations, e.g., through the consumption function. The example I choose to present is that of accumulation in a model with government debt. It is largely the Phelps-Shell extension of the basic Solow (debtless) model.[7] Following Solow, assume the simplest possible Keynesian consumption function, i.e., consumption is a fixed fraction $(1 - s)$ of perceived income, consisting of rewards to privately owned factors and of government transfers (minus taxes).

If we make the heroic assumption that the central bank is able to keep the economy on a full-employment path with zero inflation, then private demand for consumption goods per worker is given by $c^D = (1 - s)[f(k) + \phi]$, where ϕ denotes net government transfers per head.

[6]The seminal work on optimal growth is Frank P. Ramsey's "A Mathematical Theory of Saving," *Economic Journal* 38 (December 1928): 543–559, reprinted in Stiglitz and Uzawa, *Readings*, 429–445. Also see David Cass, "Optimum Growth in an Aggregative Model of Capital Accumulation," *Review of Economic Studies* 32, (July 1965): 233–240, and Tjalling C. Koopmans, "On the Concept of Optimal Economic Growth," in *Semaine d'Etude sur le Rôle de l'Analyse Economètrique dans la Formulation de Plans de Dèveloppement* (Vatican City: Pontifical Academy of Sciences, 1965) 1: 225–287. A collection of essays on optimal growth appears in Karl Shell, ed., *Essays on Theory of Optimal Economic Growth* (Cambridge: Massachusetts Institute of Technology Press, 1967.

[7]The Phelps-Shell reference is "Public Debt, Taxation, and Capital Intensiveness," *Journal of Economic Theory* 1 (October 1969): 330–346. For an analysis of full-employment fiscal and monetary policy in a two-sector economy, see Duncan K. Foley and Miguel Sidrauski, *Monetary and Fiscal Policy in a Growing Economy* (New York: Macmillan, 1971). For other references to growth with the accumulation of paper assets, see James Tobin's pioneering "Money and Economic Growth," *Econometrica* (October 1965): 671–684, and Karl Shell, Miguel Sidrauski, and Joseph E. Stiglitz, "Capital Gains, Income, and Saving," *Review of Economic Studies* 36 (January 1969): 15–26.

If government expenditure is zero, then $\phi = \xi$, the per capita deficit, so that

$$c^D = (1 - s)[f(k) + \xi].$$

Government debt per head, x, therefore follows the simple dynamical law $\dot{x} = \xi - nx$ so that in balanced growth $x = \xi/n$. Therefore, steady-state demand for consumption per capita is

$$c^D = (1 - s)[f(k) + nx].$$

Steady-state equilibrium ($\dot{k} = 0 = \dot{x}$) is defined by the equality of c^D and c^S, or $(1 - s)[f(k) + nx] = f(k) - nk$, which can be rewritten as

$$c^D(y) = (1 - s)(y + nx) = y - nk(y) = c^S(y),$$

because capital per worker can be written as an increasing function of output per worker.

Figure 5 shows the determination of steady-state output per head and steady-state consumption per head. Consider first the debtless Solow case, in which the steady state is determined by the unique intersection of the $c^S(y)$ locus and the ray $(1 - s)y$. The simple Solow model thus possesses a unique nontrivial long-run capital intensity, $\hat{k}$, with $\hat{y} = f(\hat{k})$. The Solow steady state is efficient if the slope of the ray is sufficiently large, i.e., if $y^* > \hat{y}$.

If the government holds debt per head constant at some positive level, two steady-state values of output per capita may be possible. They are shown in Figure 5 as $\underline{y}$ and $\bar{y}$; they are determined by the intersection of the c^S locus and the c^D locus, which is a line parallel to the Solow ray and intercepting the vertical axis at a value of $(1 - s)nx$. If, on the other hand, the government runs a surplus with long term debt per head equal to a negative constant, then steady-state output per capita is uniquely determined. In this case, steady-state $y = y^\dagger$, the unique intersection of the c^S locus and the line parallel to the Solow ray but with negative vertical intercept equal to $(1 - s)nx < 0$.

Comparative Dynamics

Restricting attention to steady states, we notice some regularities for the Solow model (where $x = 0$): $\partial\hat{y}/\partial s$ and $\partial\hat{k}/\partial s$ are positive. In the long run, increased thriftiness leads to greater output per capita and

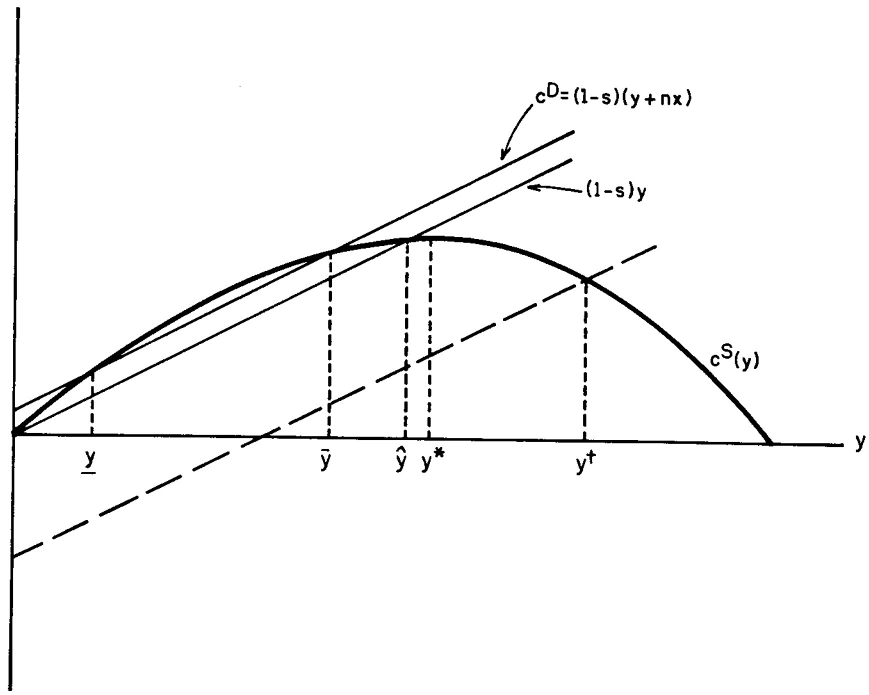

Figure 5.

greater capital intensity. However, $\partial\hat{c}/\partial s$ is greater than (or less than) zero depending on whether k is smaller than (or greater than) the Golden Rule value k^*.

How does debt affect the model? If we are in the surplus regime, things are simple. Decreasing the surplus decreases capital intensity and decreases output per head, i.e., $\partial k/\partial x < 0$ and $\partial y/\partial x < 0$.

In the deficit regime, life is more complicated. For "large" capital intensity cases, the "expected" results continue, $\partial y/\partial x$ and $\partial k/\partial x < 0$, but as can be seen by checking the effects on $\underline{y}$ of shifting c_D upwards, for "small" capital intensities, we get the "unexpected" result that $\partial y/\partial x > 0$ and $\partial k/\partial x > 0$.

This example shows that comparative statics results do not always carry over to comparative dynamics. Even though increasing the deficit promotes consumption at the expense of saving, the government intent upon permanently increasing capital intensity through fiscal policy may find, as it succeeds, that the public debt per head has been increased and that the necessary deficit per head has grown.

Stability

The interest in steady states is based on the claim that economies tend to settle into them with the passage of time if the basic economic environment has not been altered. For example, in the Solow model, where $\dot{k} = z - nk = sf(k) - nk$, it can be seen from inspection of Figure 6 that the equilibrium capital-labor ratio is stable. Because of the strict concavity of $f(\cdot)$, sign $\dot{k}$ = sign $(k - \hat{k})$ for positive capital-labor ratios. Thus, $\hat{k}$ is *globally stable*, because, for any nontrivial initial capital intensity the economy tends asymptotically to $\hat{k}$.

Stability is an important property of a growth model, but many interesting models possess steady states that are not stable. I leave it to the reader to drop the strict concavity assumption, $f''(\cdot) < 0$, and restudy the Solow model *à la* Figure 6 to exhibit cases in which steady states are (1) unstable, or (2) locally, but not globally, stable. Likewise, complication of the technology to allow for differing techniques of production in the consumption and investment goods sectors, or for alterations of saving-investment behavior, can generate models in which asymptotic behavior depends on initial conditions.

Stability and Government Debt

What about stability in the model with government debt? Assume *first* that deficit per head, ξ, is constant through time. Because $c = (1 - s)[f(k) + \xi]$ and $\dot{k} = f(k) - c - nk$, we have

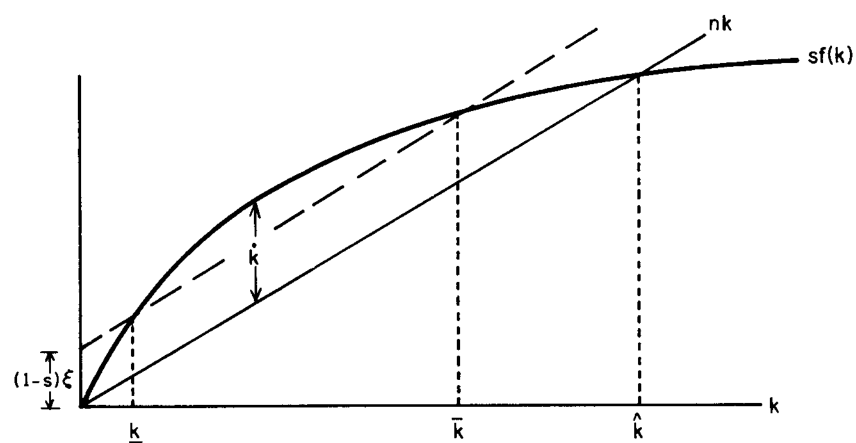

Figure 6.

$$\dot{k} = sf(k) - [(1 - s)\xi + nk].$$

In Figure 6, steady-state values of k are determined by the intersection of the curve $sf(k)$ and the dashed line parallel to the nk ray with vertical intercept $(1 - s)\xi$. In the case of a deficit, $\xi > 0$, two steady states, $\underline{k}$ and $\bar{k}$, are possible. In the neighborhood of $\underline{k}$, sign $\dot{k} \equiv$ sign $(\underline{k} - k)$, so the steady state $\underline{k}$ is unstable if ξ is held constant. In the neighborhood of $\bar{k}$, sign $\dot{k}$ = sign $(k - \bar{k})$, so $\bar{k}$ is locally stable for ξ constant.

Does this mean that the steady state $\underline{k}$ is uninteresting because it is unlikely to be observed in practice? This would be true only if we restricted our attention to governments which insisted on holding the deficit per head constant through time. The reader should convince himself that there exists a dynamic fiscal policy which makes the capital-labor ratio $\underline{k}$ globally stable. He should choose a function $\xi(k)$ such that $[nk + (1 - s)\xi(k)]$ lies below $sf(k)$ for $k < \underline{k}$, and above $sf(k)$ for $k > \underline{k}$. False policy implications can follow when stability analysis is based on an artificially restricted class of policy instruments. Our analysis shows that we have counterexamples to "burden of debt" theorems because it is possible that, across steady states, dk/dx is positive.

Intertemporal Choice and the Rate of Interest

Given its other commitments, a society faces the choice between consumption now (time t) and consumption at some later date (time $t + \Delta t$). Its production opportunities are described by the cross-hatched pie wedge of the Irving Fisher diagram of Figure 7.[8] $C(t)$ is consumption at time t and $C(t + \Delta t)$ is consumption at time $t + \Delta t$. Assume that a particular production plan $[C^0(t), C^0(t + \Delta t)]$, for example, is chosen from the *PPF*. We know that the production plan is supported by efficiency prices $[p(t), p(t + \Delta t)]$ where $p(t)$ is the price of consumption in period t, $p(t + \Delta t)$ is the price of consumption in period $t + \Delta t$.

Society's wealth is then $W^0 = p(t)C^0(t) + p(t + \Delta t)C^0(t + \Delta t)$. The price ratio $p(t)/p(t + \Delta t)$ is equal to the negative of the slope of the PPF at $[C^0(t), C^0(t + \Delta t)]$. If the economy in question were relatively small and faced fixed international prices $[p(t), p(t + \Delta t)]$—i.e., if it faced perfect borrowing and lending markets—then society's opportunity set would be the triangle of Figure 7 which includes the entire production possibility set.

[8]The basic text on intertemporal choice and the rate of interest is Irving Fisher's *Theory of Interest* (New York: Kelley, 1930).

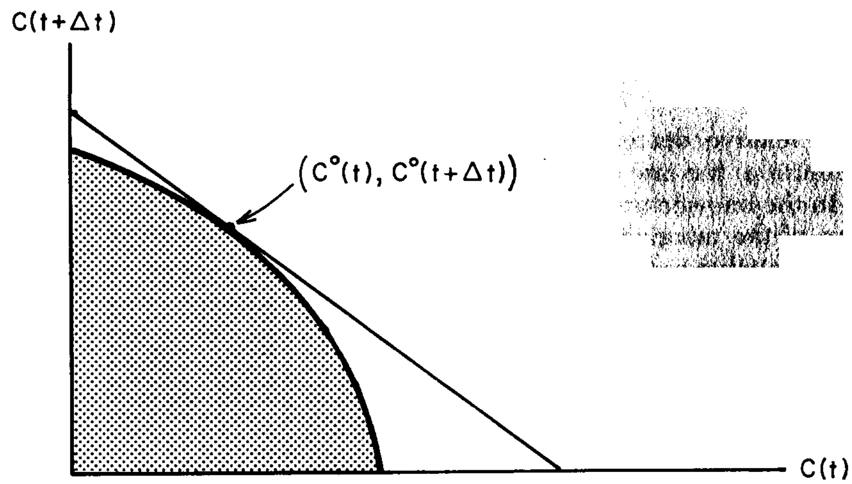

Figure 7.

The frontier of this opportunity set would be described by the line segment, $p(t)C(t) + p(t + \Delta t)C(t + \Delta t) = W^0$, $C(t) \geq 0$, $C(t + \Delta t) \geq 0$. Differentiating yields

$$\frac{dC(t + \Delta t)}{dC(t)} = -\frac{p(t)}{p(t + \Delta t)}.$$

The Interest Rate

Intertemporal price ratios for consumption can be restated (or redefined) in terms of interest charges and interest rates. In particular, we define R to be the (consumption) interest premium between times t and $t + \Delta t$ by $R = (p(t)/p(t + \Delta t)) - 1$. If consumption to be delivered later costs just as much (now) as consumption delivered now, then we say that the interest premium, R, is zero. If, however, we must now pay more for consumption delivered now than for consumption delivered later, the interest premium R is positive.

Assume that the interval between t and $t + \Delta t$ is sufficiently small, so that we can think of interest being paid continuously at an (approximately) constant rate, so that $R = \rho \cdot \Delta t$, where ρ is the interest rate. Then

$$\frac{p(t)}{p(t + \Delta t)} - 1 = \rho \cdot \Delta t.$$

Dividing both sides by Δt and letting $\Delta t \to 0$ yields

$$\rho(t) = -\frac{\dot{p}(t)}{p(t)}.$$

The relative decrease in the present price of consumption is (by definition) the rate of interest.

In the one-sector model, because the slope of the *PPF* is the negative of the marginal product of capital we have $\rho = MP_K$ or, in the differentiable case, $\rho(t) = f'(k(t))$. The basic concept from the theory of intertemporal choice is the rate of interest, which turns out to be equated to the marginal product of capital in the simple one-capital-good technology.

If there is more than one capital good, of course, no such simple relationship exists, yet the rate of interest still plays its important role in allocating resources between the present and the future. If many consumption goods exist, there is no single pure interest rate concept, but just as we are free to choose a numeraire commodity, we can base the interest rate on any commodity we choose, including money.

More Complicated Technologies

It is important that growth theory not rest solely on simple examples. This approach could lead to—and has led to—"methodological" disputes as to whether or not "fundamental" results from the simplest models generalize to more complicated models. Of course, a cost in mathematical difficulty must be borne when the analysis is generalized. The technology to be considered next is about as general as any in current growth theory. It is based on the work of David Cass and the present author, published in the *Journal of Economic Theory* during 1976.[9]

Let $C(t)$ be the output of the consumption good, $Z(t) = (Z_1(t), \ldots, Z_m(t))$, the vector of investment goods output, $K(t) = (K_1(t), \ldots, K_m(t))$ the vector of capital goods stocks; $L(t)$ the stock of a single primary factor (labor, for example), all at time t. The feasible technology is described by

$$(C, Z, K, L) \text{ belongs to } T \quad \text{and } \dot{K} = Z,$$

where the technology T is the set of (nonnegative) feasible output-

[9]The Hamiltonian approach to dynamic economics is the subject of the February 1976 issue (vol. 12) of the *Journal of Economic Theory*. An early analysis of multisectoral growth is Michio Morishima's *Equilibrium, Stability, and Growth* (Oxford: Oxford University Press, 1964).

input combinations. We assume that T is a closed, convex cone, i.e., that technology exhibits constant returns to scale but diminishing rates of substitution.

Let $p(t)$ be the present price of consumption at time t. Choose initial consumption as the numeraire, $p(0) = 1$. Let $q(t) = (q_1(t), \ldots, q_m(t))$ be the vector of present prices of investment goods, at time t.

If the production plan (C, Z, K, L) is efficient, then at each moment there exist output prices, (p, q), such that the net national product (NNP) is maximized at those prices given the existing input stocks, (K, L). Thus maximized NNP can be written as a function of output prices and input stocks,

$$NNP = H(p, q, K, L) = \max_{(C', Z', K, L) \in T} (pC' + qZ').$$

Static technological opportunities are completely described by the properties of the function H. Because of the properties of prices, H is convex and homogeneous of degree one in (p, q). Assuming T to be a convex cone is equivalent to assuming that H is concave and homogeneous of degree one in (K, L).

The partial derivative (or generalized gradient) of H with respect to p is equal to C, $\partial H/\partial p = C$. Similarly, $\partial H/\partial q_i = Z_i$ for $i = 1, \ldots, m$. The generalized gradient of H with respect to K_i is the present value of the rental rate on the i^{th} capital good, $\partial H/\partial K_i = r_i$, $i = 1, \ldots, m$, and $\partial H/\partial L = w$, the present value of the wage rate. r is then the vector of present capital goods rental rates. $p(t)$, $q(t)$, $r(t)$, and $w(t)$ are present prices: they are all expressed in terms of initial (or present) consumption, $p(0) = 1$.

As before, the interest rate $\rho(t) = -\dot{p}(t)/p(t)$. Under competition, the profit *in terms of initial consumption* from holding any asset is zero if expectations about price changes are realized. The profit from holding one unit of capital good i for a short period of time $[t, (t + \Delta t)]$ is approximately equal to $q_i(t + \Delta t) - q_i(t) + r_i(t)\,\Delta t$, where $r_i(t)$ is the "average" rental rate over this short period. Dividing by Δt, taking the limit as $\Delta t \to 0$, and adding the zero-profit condition yields

$$\dot{q}_i + r_i(t) = 0 \text{ for } i = 1, \ldots, n.$$

Capital gains plus rentals in terms of initial consumption are zero for each capital good. If asset-holders possess short-run perfect foresight these are the asset-market clearing equations. These differential equations are also efficiency conditions: an efficient intertemporal plan must be supported by prices $(p(t), q(t), r(t), w(t))$ having the above properties.

Assume that the consumption (or "utility") rate of interest is zero. Then, because $\dot{p}/p = 0$ and $p(0) = 1$, $p(t) = 1$ for all t. Also assume that there is no labor force growth; then we can consider *NNP* per head, $H(p(t), k(t)) \equiv H(1, q(t), K(t)/L(t), 1)$. The competitive (and efficient) growth paths must satisfy

$$\dot{k}(t) = \frac{\partial H(q(t), k(t))}{\partial q}$$

and

$$\dot{q}(t) = -\frac{\partial H(q(t), k(t))}{\partial k}.$$

The first equation implies that $\dot{k} = z$; the second equation implies that $\dot{q} + r = 0$. This system of differential equations is known to mathematicians and physicists as a Hamiltonian system. The Hamiltonian function H completely describes static technology.

In the case $\rho = 0$, these are the conditions for Ramsey's problem of optimal growth within the context of a general technology. A steady-state solution to these differential equations yields the golden rule vector of capital-labor ratios. Whether or not the GR vector, k^*, is stable depends on the technology. In particular, if H is strictly convex in q and strictly concave in k, the system is globally stable when the boundary conditions are satisfied.

Notice that the model is closed by the demand condition that determines the interest rate. To ensure stability in a more general model, the curvature (or, more generally, the convexity-concavity) requirement for the Hamiltonian function, must be strengthened as the interest rate, ρ, increases.

While neoclassical theory now has at its disposal a tool for the study of global stability in the most general setting, it should be stressed once again that while stability is an interesting property for an economic model, it is not a crucial property.

Heterogeneous Capital and Multi-Asset Accumulation

Of the many ways in which heterogeneous-capital models differ from homogeneous-capital models, two will be discussed. One, the subject of "reswitching of techniques," has been widely studied but, to my mind, it is not especially important to the general subject of intertemporal economics. The other has to do with asset valuation in an enterprise economy. Competitive asset valuation has received relatively less attention but, in my opinion, it is crucial to much of intertemporal economics.

Reswitching

We restrict our attention to constant-interest-rate regimes, $-\dot{p}(t)/p(t) = \rho \geq 0$. Let $Q(t)$ be the vector of *current* investment goods prices, $Q(t) = q(t)/p(t)$. The efficiency conditions are then

$$\dot{k} = \frac{\partial H(Q, k)}{\partial Q}$$

and

$$\dot{Q} = -\frac{\partial H(Q, k)}{\partial k} + \rho Q.$$

(Q^ρ, k^ρ) defines a constant-interest-rate steady state if $\partial H(Q^\rho, k^\rho)/\partial Q = 0 = \partial H(Q^\rho, k^\rho)/\partial k + \rho Q^\rho$. Consider then the steady-state capital-intensity vector as a function of the interest rate ρ, $k^\rho(\rho)$. We showed that, in the one-capital-good model, the steady-state interest rate determines steady-state capital intensity and k^ρ is nonincreasing in ρ. Thus, as the interest rate falls (across steady states), capital intensity increases—or at least does not decrease—in the homogeneous capital model.

Is there a general analogue to this proposition for heterogeneous capital technologies? The answer is no. Employing the (fixed-coefficient) activity analysis model, Champernowne, Pasinetti, and others have constructed examples with the following property. Let ρ_1 be an interest rate for which production technique A is employed. Suppose that at some interest rate $\rho_2 < \rho_1$, production technique B is employed. Nonetheless, in these examples at an even lower interest rate, ρ_3 $(\rho_1 > \rho_2 > \rho_3)$, technique A is again employed. Then A can be said to be neither more nor less "capital-intensive" than B.[10]

These examples teach us that there is no short cut in the form of "generalized capital intensity," but the interest rate remains as the measure of social time discount and as the measure of the social rate of intertemporal technological transformation.

[10]The dust had settled on the reswitching controversy by the time of the Symposium on "Paradoxes in Capital Theory," *Quarterly Journal of Economics* 80 (November 1966); see papers by Pasinetti, Levhari, Samuelson, Morishima, Bruno, Burmeister, Sheshinksi, and Garegnani.

The Asset-Market Clearing Equation

The Keynesian distinction between saving and investment really has to do with the *composition* of savings. In the simplest Solow model, there is no alternative store of value to the single capital good, so saving must translate into investment and, of course, there is no problem with the allocation of investment. Once the question of asset choice is allowed, truly interesting macrodynamics become possible.

The story is best told in terms of paper assets (money, bonds, shares in equity, etc.)—and it has been—but here we must content ourselves with a simple heterogeneous capital model. Output per head y will be a function of the per capita stocks of machinery of types 1 and 2, k_1 and k_2, respectively. If the production function is $f(\cdot)$, then

$$y(t) = f(k_1(t), k_2(t)).$$

This is a one-sector, two-capital good model, in which $y = c + z_1 + z_2$, with z_i $(i = 1, 2)$ denoting the per capita investment in capital good i. The *PPF* is then a plane in (c, z_1, z_2) – space and $\dot{k}_i = z_i - nk_i$ for $i = 1, 2$. $p(t)$ is the present price of consumption at time t, $p(0) = 1$, and $q_i(t)$ is the present price of investment good i at time t. Let $Q_i(t) = q_i(t)/p(t)$ be the current price of investment at time t. Assuming the simplest Keynesian consumption function, $c = (1 - s)f(k_1, k_2)$.

If consumption goods and investment goods are to be produced, then max $(q_1, q_2) = p$, or

$$\max\,(Q_1, Q_2) = 1.$$

Furthermore, investment will be specialized to the capital good with the higher price, e.g., if $1 = Q_1 > Q_2$, then $z_2 = 0$. Let $\dot{Q}_i^e$ be the expected rate of change in the price of capital good i, and r_i be the competitive rental rate for capital good i. Then, for asset-market equilibrium,

$$\frac{\dot{Q}_1^e}{Q_1} + \frac{r_1}{Q_1} = \frac{\dot{Q}_2^e}{Q_2} + \frac{r_2}{Q_2}.$$

That is, expected capital gains plus rentals must be equalized. Under competition, rentals are equated to marginal products, so $r_i = \partial f/\partial k_i \equiv f_i$ and

$$\frac{\dot{Q}_1^e}{Q_1} + \frac{f_1}{Q_1} = \frac{\dot{Q}_2^e}{Q_2} + \frac{f_2}{Q_2} = -\frac{\dot{p}}{p}.$$

If we assume static price expectations, $\dot{Q}_1^e = 0 = \dot{Q}_2^e$, then $Q_1/Q_2 = f_1/f_2$ and the allocation of investment is specialized to the capital good with the higher marginal product. So in this special model, if capitalists do not go to business school (i.e., if they are bad at forecasting in this primitive way), then the choice of machinery is optimal.

Consider, however, the other polar case where capitalists (business school grads all) have expectations that are always realized, $\dot{Q}_i^e = \dot{Q}_i$, or

$$\frac{\dot{Q}_1}{Q_1} + \frac{f_1}{Q_1} = \frac{\dot{Q}_2}{Q_2} + \frac{f_2}{Q_2}.$$

Assume that $1 = Q_1 > Q_2$ but that $f_2 > f_1$. Our economy begins by specializing to investment in the capital good with the lower marginal product. Then

$$\dot{Q}_2 = Q_2 f_1 - f_2,$$

with k_2 falling relative to k_1, thereby forcing Q_2 to fall at an even greater rate. This bubble must eventually burst—but how soon? These volatile, self-justifying and destabilizing capital gains are the basis for the Keynesian distinction between social returns to a capital good and private returns to that capital.[11]

Important advances in fundamental macrodynamics can be expected to occur within the next decade. Expectations formation and asset-market clearance will have to play a central role. A deeper understanding of the mechanisms determining saving, investment, and asset pricing will be our long-term reward for mastering the complexities of growth with many capital goods and several paper assets.[12]

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[11]The material on the role of the asset-market clearing equation is drawn from Karl Shell and Joseph E. Stiglitz, "The Allocation of Investment in a Dynamic Economy," *Quarterly Journal of Economics* 81 (November 1967): 592–609; this article was inspired by Frank H. Hahn's "Equilibrium Dynamics with Heterogeneous Capital Goods," *Quarterly Journal of Economics* 80 (November 1966): 633–646.

[12]It is a great shame that there is insufficient space for us to go into the analysis of decentralized growth with overlapping generations. The seminal piece here is Paul Samuelson's "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," *Journal of Political Economy* 66 (December 1958): 467–482. The otherwise very stimulating Peter Diamond paper, "National Debt in a Neoclassical Model," *American Economic Review* 55 (December 1965): 1126–1150, uses incorrect stability analysis in an attempt to show that there is a burden to the debt. Further studies are David Cass and Menahem E. Yaari, "Individual Saving, Aggregate Capital Accumulation, and Efficient Growth," in Shell, *Essays on Theory of Optimal Economic Growth*, and my "Notes on the Economics of Infinity," *Journal of Political Economy* 79 (September/October 1971): 1002–1011.

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