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CHAPTER 4

On the nonequivalence of the Arrow–securities game and the contingent-commodities game

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Abstract: We analyze two imperfectly competitive economies that face intrinsic and/or extrinsic uncertainty. The economies differ only in the structure of the markets for insurance. One economy is the Arrow-securities market game, and the other is the contingent-commodities market game. We show that for each game there is a Nash equilibrium in which all markets are open. The games are not equivalent, even though the corresponding competitive economies are.

In particular, we show that *no* Nash equilibrium allocation from the Arrow-securities game in which some income is transferred across states of nature is a Nash-equilibrium allocation for the contingent-commodities game. There is an immediate corollary for the special case in which uncertainty is purely extrinsic: A sunspot Nash-equilibrium allocation for the Arrow-securities game that is not a mere lottery over Nash-equilibrium allocations for the corresponding certainty game (i.e., the game without insurance markets) is never a Nash-equilibrium allocation for the contingent-commodities game.

These two games differ because the effects of the market power of individual traders depend on the way markets are organized. We conclude that the notion of "complete markets" makes sense only when applied to competitive economies. Imperfectly competitive economies are sensitive to details about market structure, even though these details are inessential for competitive economies.

1 Introduction and summary

Consider two competitive economies that differ only in the structure of their insurance markets. The first economy has a full set of markets for

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contingent commodities. A contingent commodity is a contract, bought or sold before nature moves, to deliver one unit of commodity i after state s has occurred.[1] The second economy has a full set of markets for trading spot commodities after the state of nature is revealed. Before the state is revealed, individuals trade on a full set of markets for Arrow securities. An Arrow security is a contract to deliver one unit of account[2] if state s occurs, otherwise nothing.[3] Under perfect competition, these two economies are equivalent in the sense that a competitive-equilibrium allocation for one economy is also a competitive-equilibrium allocation for the other.

We show that this equivalence does not extend to imperfectly competitive economies. To do so, we construct two strategic market games, the contingent-commodities game and the Arrow-securities game. The games, which differ only in the structure of their insurance markets, are analogous to the corresponding competitive economies. We build on the market-game price-formation model of Shapley and Shubik.[4] For specificity, we adopt the sell-all variants of these two games: spot-commodity offers and contingent-commodity offers are assumed to be equal to endowments; markets are "thick." Offers of Arrow securities are not prespecified because they are purely financial instruments.[5] In this chapter, all Arrow securities are inside securities; that is, endowments of Arrow securities are assumed to be zero.

Our result is the following: Let x' be an interior[6] (i.e., open-market) Nash-equilibrium allocation for the Arrow-securities game in which some

Footnote *(cont.)*

Mas-Colell was the discussant. His comments were extraordinarily useful and stimulating. Errors and other shortcomings must be blamed on the authors.

1 See Debreu (1959, chap. 7, pp. 98–102); but the idea of "contingent commodity" or "commodity claim" goes back to Arrow (1953, sect. II).

2 This implies that in each state the purchasing power of the corresponding Arrow security is positive or that, without loss of generality, its price in terms of "state s unit of account" is unity.

3 See Arrow (1953, sect. III).

4 See, for example, Shubik (1973), Shapley (1976), Shapley and Shubik (1977), Postlewaite and Schmeidler (1978), and Peck and Shell (1985, 1988). Another variant of the contingent-commodities game is analyzed in Dubey and Shubik (1977).

5 Our nonequivalence result still holds when securities offers are prespecified, no matter the level at which they are specified. Nash equilibria in which offers of state s securities are zero (and hence bids for state s securities are zero) entail the closing of the market for state s securities. Then there is no way to transfer income into or out of state s. Do not think about these equilibria until later because they are not consistent with Arrow-securities with payoffs measured in *state s units of account*.

6 An interior Nash equilibrium is one in which some bids are positive on each market. A market on which some bids are positive is called an open market. Hence, at an interior Nash equilibrium, the aggregate of the bids on each market is positive.

security market is active (i.e., some income is transferred across states of nature), and let x'' be a Nash-equilibrium allocation for the contingent-commodities game. Then *necessarily* x' differs from x''.

Applying this nonequivalence proposition to the case in which uncertainty is purely extrinsic (i.e., the case in which all the economic fundamentals are certain) yields the following corollary: An interior sunspot equilibrium allocation for the Arrow-securities game that is not a mere lottery[7] over Nash-equilibrium allocations for the certainty market game (the corresponding market game without insurance markets) cannot be a Nash-equilibrium allocation to the contingent-commodities game.

This very strong nonequivalence result applies only to finite economies. As the number of consumers becomes large through replication, the Nash equilibria of the contingent-commodities game converge to the competitive equilibria of the contingent-commodities economy, and the Nash equilibria of the Arrow-securities game converge to the competitive equilibria of the Arrow-securities economy. Hence, as one would expect, in the limit the sets of open-market equilibria for the two games are identical.[8] *The games are basically different from one another except in the limit, where competition is perfect.*

Comparative statics analysis for imperfect competition is more complicated than for perfect competition. In each case, the set of equilibrium allocations does depend on market structure, but the competitive model is special because in this case the comparative statics become trivial after "market completeness" is achieved. For example, a full set of contingent-commodities markets represents complete markets for the competitive model in the sense that the opening of additional markets does not affect the set of competitive-equilibrium allocations. This ultimate insensitivity of competitive allocations to the opening of additional markets is an important theorem in comparative statics, one that is used (and abused) regularly in economic analysis. There is no such powerful notion of market completeness that can be adopted for noncompetitive (or imperfectly competitive) environments.

[7] We could replace "mere lottery over Nash-equilibrium allocations" with "correlated equilibrium." See Peck and Shell (1985) for an analysis of correlated equilibrium in market games without intrinsic uncertainty. In this chapter, however, only perfectly correlated (symmetric-information) signals are considered. It remains to be seen whether or not our nonequivalence proposition can be extended to the asymmetric-information environment of Peck and Shell (1985).

[8] This would be true if the Arrow securities pay in state-specific units of account. If the Arrow securities are purely financial, then the state s security could have purchasing of zero in terms of state s commodities. Hence, the set of perfectly competitive equilibria in the contingent-claims economy would be a subset of the competitive equilibria in the Arrow-securities economy.

Many other aspects of market structure do not matter in the model of perfect competition but can matter very much in, for example, market games:

Wash sales. Whether or not wash sales of commodities are permitted affects the Nash equilibria of a market game. In the competitive economy, only net trades matter; restrictions on wash sales have no effect.

Short sales. Restrictions on short sales affect the equilibria in market games but have no effect on competitive economies if markets are complete so that they include some perfect borrowing and lending markets. If short sales are disallowed or restricted, then the Nash-equilibrium allocations are Pareto optimal only if the endowments are Pareto optimal.

We have shown (Peck and Shell, 1988) that in the market game that allows arbitrarily large short sales and enforces strong interpersonal bankruptcy rules there is a Nash-equilibrium allocation arbitrarily close to Pareto optimality, even if there are only few players. Restrictions on short sales can matter very much when competition is not perfect.

Bundling. In the competitive model with complete markets, the equilibrium allocation is not affected by adding markets for new goods if new goods are merely bundles of old goods. The addition of new bundled goods can affect the Nash-equilibrium allocations of a market game.

Trading rounds. Retrading (or recontracting) has no effect on the final allocation in perfectly competitive economies with complete markets. In the market game, however, retrading expands the Nash-equilibrium set. In the one-stage market game without short sales, Nash-equilibrium allocations are (nearly) efficient only if the endowment is (nearly) efficient. We have shown that, in the market game with retrading, there are dynamically consistent Nash-equilibrium paths that yield Pareto-optimal allocations in the limit as the number of rounds becomes large. (See Peck and Shell, 1987.) Retrading can matter very much when competition is not perfect. (There are many other equilibrium paths in the market game with retrading. Not all of these equilibria are efficient in the limit.)

Other dynamic issues. Because of births and deaths, participation on markets is naturally restricted. (See Cass and Shell, 1983, sect. III, pp. 198–202.) This restriction on market participation affects both competitive economies and market games.

In dynamic games, strategies can be history dependent. History dependence and problems of dynamic consistency raise issues that arise in games but do not arise in competitive economies with complete markets.

Real versus financial securities. Even in the competitive model there is a difference between a full set of *financial* Arrow securities and a full set of *real* Arrow securities. Financial securities can be worthless (in terms of commodities) in a prespecified spectrum of states, thus endogenously closing an arbitrary spectrum of markets. This does not happen with real securities. Hence, the set of competitive equilibria for an economy with (say) a full range of real Arrow securities (i.e., those that pay in commodities or unit of account) is a subset of the set of competitive equilibria for an economy with a full range of purely financial Arrow securities. The difference between the two sets is the set of equilibria in which some of the markets are (endogenously) closed. This phenomenon carries over to the market-game model, but the matter is further complicated. In the market-game model, because there is no such thing as complete markets, mixing financial securities with real securities will affect the Nash-equilibrium allocations.

In Section 2, we present the economic fundamentals on which the market games and the corresponding competitive economies are based. In Section 3, the contingent-commodities game is introduced. Existence of interior (i.e., open-market) Nash equilibrium and the limiting properties of the replicated economy follow directly from results in the literature. In Section 4, the Arrow-securities game is introduced. The existence of interior Nash equilibrium is demonstrated. The details of the proof are assembled in the appendix. Section 4 also provides the limiting properties for the replicated economy. Section 5 contains the statement and proof of our nonequivalence proposition, as well as some intuition about why it is true. Some concluding remarks are presented in Section 6.

2 The economy

We consider an economy with n traders (or consumers), each of whom is a von Neumann–Morgenstern expected-utility maximizer. There are r states of nature, indexed by the letter s, with α and β referring to specific states. Before the state is revealed, traders know the probability of each state s, which is denoted $\pi(s)$. When the state is revealed, all traders know which state has occurred, that is, information is symmetric, and expectations are rational.

There are ℓ commodities, and ℓ is a (finite) integer. The letters i and j denote specific commodities or index the commodities, depending on the context. Traders receive an endowment of commodities, which can depend on the state of nature, with $\omega_h^i(s)$ denoting the endowment of trader h of commodity i in state s. We assume that we have

$$\omega_h(s) = [\omega_h^1(s), \dots, \omega_h^i(s), \dots, \omega_h^\ell(s)] \in \mathbb{R}_{++}^\ell \quad \text{for all } h \text{ and } s.$$

By $x_h^i(s)$, we mean the consumption of trader h of commodity i in state s. We assume that we have

$$x_h(s) = [x_h^1(s), \dots, x_h^i(s), \dots, x_h^\ell(s)] \in \mathbb{R}_{+}^\ell \quad \text{for all } h \text{ and } s.$$

Consumer h's utility v_h can be expressed as

$$v_h(x_h) = \sum_s \pi(s) u_h[x_h(s)].$$

The consumption set of consumer h is the nonnegative orthant

$$\{x_h \mid x_h \in \mathbb{R}_{+}^{r\ell}\}.$$

Utility is assumed to be strictly monotonic, strictly concave, and differentiable on the strictly positive orthant $\mathbb{R}_{++}^{r\ell}$. Also, we assume that the closure of all indifference surfaces going through the strictly positive orthant is contained in the strictly positive orthant. The boundary of the consumption set, the points with at least one zero component, is the indifference surface of least utility.[9]

In Sections 3 and 4, we define, respectively, the contingent-commodities game and the Arrow-securities game. In the former game, traders buy and sell contingent commodities before the state is revealed, with delivery taking place afterward. This is simply a Shapley–Shubik model with $r\ell$ trading posts.[10] In the latter game, traders buy and sell Arrow-securities before the state is revealed. Afterward, securities are delivered and consumers participate on a spot market with complete knowledge of the state and the endowment. The prices of each of the r securities and the ℓ spot-

[9] The closure assumption on indifference surfaces along with the von Neumann–Morgenstern utility assumption is somewhat restrictive, guaranteeing that no consumer will risk zero consumption in any state. However, this assumption is used only in the existence argument and has no bearing on the nonequivalence result. It has the benefit of permitting a less severe bankruptcy rule for the securities game. Punishment would occur only for observed violations. Planned violations in nonobserved states could then go unpunished because no one would risk zero consumption in any state of nature that occurs with positive probability.

[10] All bids are made in terms of unit of account and all contingent commodities must pass through the market.

market prices (which may depend on the state) are determined by the clearing of bids and offers à la Shapley and Shubik. (See Peck and Shell, 1985, for a more complete description.)

3 The contingent-commodities game

We assume that there is a trading post for each contingent commodity. The strategy set for consumer h is

$$\{[\hat{b}_h^1(1),\ldots,\hat{b}_h^\ell(1);\ldots;\hat{b}_h^1(s),\ldots,\hat{b}_h^\ell(s);\ldots;$$
$$\hat{b}_h^1(r),\ldots,\hat{b}_h^\ell(r)] \mid \hat{b}_h^i(s)\geq 0 \text{ for all } i,s\}\subset \mathbb{R}_+^{\ell r}.$$

Here, the scalar $\hat{b}_h^i(s)$ denotes the bid of consumer h on the commodity i-state s trading post. All state-contingent endowments pass through the trading posts so consumers sell their entire endowments to finance their bids.

The final allocation is determined as follows.[11]

$$\left.\begin{aligned} x_h^i(s) &= \frac{\hat{b}_h^i(s)\sum_k \omega_k^i(s)}{\sum_k \hat{b}_k^i(s)} && \text{if the solvency constraint (2) is satisfied,}\\ &= 0 && \text{otherwise.}\end{aligned}\right\} \qquad (1)$$

The solvency constraint is that bids for contingent commodities must be financed by sales of contingent commodities. We must have

$$\sum_s \sum_j \left[\hat{b}_h^j(s) - \frac{\omega_h^j(s)\sum_k \hat{b}_k^j(s)}{\sum_k \omega_k^j(s)}\right]\leq 0. \qquad (2)$$

The unit of account, in which bids are made, can be normalized freely.

Our regularity conditions are sufficient to guarantee the existence of an interior (i.e., open-market) Nash equilibrium. For the proof, see Peck and Shell (1985). We claim that when we replicate, the limiting economy is competitive.

Claim 1. *Let $\eta^1, \eta^2, \ldots, \eta^\nu, \ldots$ be a sequence of symmetric, uniformly interior Nash equilibria for the contingent-commodities game, where there are ν traders of each of the n "types." Let η be a limit point of the sequence (η^ν). Then η results in a competitive allocation.*

Proof: The claim and its proof are standard in the literature. See, for example, Shapley (1976), Shapley and Shubik (1977), Mas-Colell (1982), or Peck and Shell (1985). ■

[11] To complete the model, we (like others before us) specify that in the unlikely event that all bids on a given trading post are zero, the endowments of this good are confiscated and the consumption of this commodity is zero.

4 The Arrow-securities game

The Arrow-securities game has a securities market that meets before the uncertainty is resolved, followed by a spot market that meets after the state has been observed. On the securities market there is a trading post for each of the r Arrow securities. An Arrow security is a contract to have one dollar[12] delivered if a given state occurs. Purchases of Arrow securities for one state must be financed by sales of securities for other states. A strategy must specify the offers at each Arrow-security post, the bids at each Arrow-security post, and the bids at each spot-market post for each state of nature.

More formally, the strategy set for consumer h is

$$\{[\ldots, \tilde{b}_h^1(s), \ldots, \tilde{b}_h^\ell(s), \ldots; \ldots, \tilde{b}_h^m(s), \ldots; \ldots, \tilde{q}_h^m(s), \ldots] \text{ satisfying } \tilde{b}_h^i(s) \geq 0 \text{ for all } i \text{ and } s, \text{ and } \tilde{b}_h^m(s) \geq 0 \text{ for all } s, \text{ and } \tilde{q}_h^m(s) \geq 0 \text{ for all } s\} \subset \mathbb{R}_+^{r(\ell+2)}.$$

The scalar $\tilde{b}_h^i(s)$ is the bid of consumer h on the commodity i-state s spot market, and $\tilde{b}_h^m(s)$ and $\tilde{q}_h^m(s)$ refer to the bids and offers, respectively, on the state s security market. The units of $\tilde{b}_h^m(s)$ can be freely normalized, but it may be convenient to normalize them as dollars. The units of $\tilde{q}_h^m(s)$ are (state s) dollars. The units of $\tilde{b}_h^i(s)$ are state s units of account, but it will be convenient to set the state s unit of account equal to one state s dollar.

The final allocation is determined as follows:

$$\left.\begin{aligned} x_h^i(s) &= \frac{\tilde{b}_h^i(s) \sum_k \omega_k^i(s)}{\sum_k \tilde{b}_k^i(s)} && \text{if the budget constraint (4) holds and the securities constraint (5) holds,}^{13} \\ &= 0 && \text{otherwise.} \end{aligned}\right\} \tag{3}$$

The solvency constraint is that, in each state, bids must be financed by the sum of sales income and net securities income. For all h and s, we must have

$$\sum_j \tilde{b}_h^j(s) \leq \sum_j \left[\omega_h^j(s) \frac{\sum_k \tilde{b}_k^j(s)}{\sum_k \omega_k^j(s)}\right] + \left[\frac{\tilde{b}_h^m(s) \sum_k \tilde{q}_k^m(s)}{\sum_k \tilde{b}_k^m(s)}\right] - \tilde{q}_h^m(s). \tag{4}$$

[12] We change from the introduction. Arrow-securities are now purely financial. The value of state s Arrow money in terms of state s commodities could possibly be zero. Any prespecified spectrum of markets could be closed with the remainder open. This is a fundamental property of money and other purely financial assets: They can always be worthless; in fact, they must be worthless if people imagine them to be.

[13] If we have $\sum_k \tilde{b}_k^i(s) = 0$ (so, of course, $\tilde{b}_h^i(s) = 0$), then by assumption we interpret the ratio $\tilde{b}_h^i(s)/\sum_k \tilde{b}_k^i(s) = 0/0$ to be equal to zero.

The securities constraint, that purchases of securities are financed by sales of securities, requires that we have, for each h,

$$\sum_s \left[\tilde{b}_h^m(s) - \frac{\tilde{q}_h^m(s) \sum_k \tilde{b}_k^m(s)}{\sum_k \tilde{q}_k^m(s)} \right] \leq 0. \tag{5}$$

Some comments are in order about constraints (4) and (5). The expression

$$\frac{\tilde{b}_h^m(s) \sum_k \tilde{q}_k^m(s)}{\sum_k \tilde{b}_k^m(s)} - \tilde{q}_h^m(s)$$

represents the net number of state s securities purchased by trader h, where one security pays off one dollar in state s. The expression

$$\frac{\sum_k \tilde{b}_k^m(s)}{\sum_k \tilde{q}_k^m(s)}$$

can be interpreted as the price of the state s security, so condition (5) amounts to

$$\sum_s (\text{net purchases of security } s)(\text{price security } s) \leq 0.$$

Also, we handle the possibility of a security post with zero bids or offers by the convention that $0/0 = 0$: When the bids on a post are zero, the offers are confiscated, and *vice versa*.

Consumer h seeks to maximize $v_h = \sum_s \pi(s) u_h[x_h(s)]$, where $u_h[x_h(s)]$ is the utility derived from the certain consumption of $x_h(s)$. A Nash equilibrium is defined in the usual way as a vector of best-response strategies. The Arrow-securities game always exhibits Nash equilibria of the following type: Bids and offers on the securities markets are zero, so no income can be transferred between states. Each of the isolated spot markets plays out a Nash equilibrium for the subgame, which is known to exist (see Peck and Shell, 1985). No trader would want to make a positive bid or offer on the security markets because it would be confiscated; no trader would want to change his spot-market behavior because each spot-market subgame exhibits a Nash equilibrium.

In the Nash equilibria described earlier, the security markets are closed. We shall show that there are always interior Nash equilibria in which there are positive bids on each security market and spot market, but first we prove the following Lemma.

Lemma 1. *All Nash equilibria are individually rational in the sense that consumers receive at least as high a utility as that of their endowments.*

Proof: Consumer h assures his endowment with the following strategy, given the strategies of the other players,

$$\left.\begin{aligned} \tilde{b}_h^i(s) &= \frac{\omega_h^i(s) \sum_{k \neq h} \tilde{b}_k^i(s)}{\sum_{k \neq h} \omega_k^i(s)} \\ \tilde{b}_h^m(s) &= \frac{\tilde{q}_h^m(s) \sum_{k \neq h} \tilde{b}_k^m(s)}{\sum_{k \neq h} \tilde{q}_k^m(s)} \end{aligned}\right\} \quad s = 1, 2, \ldots, r, \quad i = 1, 2, \ldots, \ell.$$

Thus, $\tilde{q}_h^m(s)$ can be chosen arbitrarily, and the endowment is feasible. For the above strategy not to dominate the Nash-equilibrium strategy for consumer h, his expected utility must be no less than he could achieve by consuming his endowment in each state. ∎

Remark: Lemma 1 is a well-known result in the market-game literature. We elevate it to lemma status merely for emphasis.

Definition 1. *For $b \in \Delta^{nr} \times \Delta^{n\ell} \times \cdots \times \Delta^{n\ell}$, define the following augmented bids:*

$$d_h^i(s) = \max[b_h^i(s), \epsilon],$$

$$d(s) = [d_h(s), d_{-h}(s)],$$

$$d_h^m(s) = \max[b_h^m(s), \epsilon].$$

Proposition 1. *Given our maintained assumptions on utility functions and endowments, the Arrow-securities game has an interior Nash equilibrium.*

Proof: The standard way to prove the existence of a Nash equilibrium is to define an adjustment function based on utility maximizing responses to bids of the other players, and then to find a fixed point of that mapping. The problem here is that when everyone except consumer h is bidding zero on some market, the maximizing response of consumer h is not well-defined. In this proof, we augment the bids to some level ϵ, so that the adjustment function is well-defined and continuous. Then we show that, for small enough ϵ, each component of the fixed point is greater than ϵ. The proof appears in the appendix. It is similar in structure to the proof in Peck and Shell (1985, proposition 2.23, pp. 17–29). ∎

The Arrow-securities game, just like the contingent-commodities game, becomes competitive in the limit as we replicate the economy.

Claim 2. *Let $\eta^1, \eta^2, \ldots, \eta^\nu, \ldots$ be a sequence of symmetric, uniformly interior Nash equilibria for the Arrow-securities game, where there are ν traders of each of the n "types." Let η be a limit point of the sequence (η^ν). Then η results in a competitive allocation.*

Proof: The claim and its proof are standard in the market-games literature. See, for example, Shapley (1976), Shapley and Shubik (1977), Mas-Colell (1982), and Peck and Shell (1985). ■

5 Nonequivalence of the contingent-commodities game and the Arrow-securities game

A complete set of Arrow-securities markets followed by a spot market has the same set of interior competitive equilibria as a complete set of contingent-commodities markets. We have claimed that when we replicate the consumers, Nash equilibria of the limiting contingent-commodities game are competitive and *interior* Nash equilibria of the limiting Arrow-securities game are competitive, so the two games yield the same set of *interior* Nash equilibria in the limit.

With a finite number of consumers, both the contingent-commodities game and the Arrow-securities game have inefficient interior Nash equilibria. We shall show that these two market structures are not equivalent. We go further. We show that an interior Nash-equilibrium allocation of the Arrow-securities game in which some income is transferred across states cannot be a Nash-equilibrium allocation of the corresponding contingent-commodities game.

To build some intuition, consider the noninterior competitive equilibria of the two economies. The contingent-commodities economy has only interior Nash equilibria; in this economy, markets cannot be closed endogenously. The Arrow-securities economy typically has many noninterior competitive equilibria because any pattern of open and closed securities markets is consistent with competitive equilibrium. Closed securities markets correspond to those states in which money is worthless. If (even) the state α security is worthless in state α, then the economy in state α is isolated from the economy in the other states of nature. Within state α, trades can still take place in terms of the state α unit of account, but no income can be transferred into or out of state α. Thus, when we include noninterior competitive equilibria, the contingent-commodities economy and the Arrow-securities economy are clearly nonequivalent.

In the contingent-commodities economy, trading commodity 1α for commodity 2β is very much like trading commodity 1α for commodity 2α. In the competitive Arrow-securities economy, these trades are similar only when it is possible to transfer income between state α and state β via the securities market. If, for example, the state β securities market is closed, then the first (interstate) trade is impossible even though the second (intrastate) trade is possible.

When consumers have market power, their actions affect the equilibrium price. It is far from obvious that trading on contingent-commodities markets induces the same price distortions that arise from trading on securities markets and spot markets. In fact, the distortions are different.

The nonequivalence of the Arrow-securities game and the contingent-commodities game can be motivated by the following observation. Consider a consumer who wants to trade commodity 1 in state α for commodity 2 in state β. Under the market structure of the contingent-commodities game, only the prices of these two contingent commodities will be affected by the trade. This effect on prices is of the same sort as the effect on prices of a trade of commodity 1α for commodity 2α. In the contingent-commodities game, when the consumer transfers income from state α to state β, he does so without directly distorting the terms of other interstate trades. Because there is just one budget constraint, income can be transferred between states with complete flexibility.

Under the market structure of the Arrow-securities game, the consumer must sell state-α securities and buy state-β securities and then use the securities income to buy commodity 2 in state β. This action affects the price of the securities, so the terms of trade between each commodity in state α and each commodity in state β are directly affected. Hence, in the Arrow-securities game, there is an asymmetry between trading commodity 1α for commodity 2β (an interstate trade) and trading commodity 1α for commodity 2α (an intrastate trade). There is no such asymmetry in the contingent-commodities game.

Next we state our formal nonequivalence result.

Proposition 2. *Let $\hat{b}=(\hat{b}_1,\ldots,\hat{b}_h,\ldots,\hat{b}_n)\in\mathbb{R}_+^{r\ell}$ be an interior Nash equilibrium of the contingent-commodities game, and let $\tilde{b}=(\tilde{b}_1,\ldots,\tilde{b}_h,\ldots,\tilde{b}_n)$ be an interior Nash equilibrium of the Arrow-securities game, where*

$$\tilde{b}_h=\{\tilde{b}_h^1(s),\ldots,\tilde{b}_h^\ell(s),\tilde{b}_h^m(s),\tilde{q}_h^m(s)\}_{s=1}^{s=r}\in\mathbb{R}_+^{r(\ell+2)}.$$

Assume that all of the securities markets are open and that some consumer is a net purchaser of securities in some state. Then $\hat{b}$ and $\tilde{b}$ result in different allocations.

Proof: Because $\hat{b}$ and $\tilde{b}$ are interior solutions to the utility-maximization problem, we have the following first-order conditions:

$$\hat{\lambda}_h=\pi(s)\left\{\frac{\partial u_h[x_h(s)]}{\partial x_h^i(s)}\right\}\left[\frac{\sum_{k\neq h}\hat{b}_k^i(s)}{\sum_{k\neq h}\omega_k^i(s)}\right]\left[\frac{\sum_k\omega_k^i(s)}{\sum_k\hat{b}_k^i(s)}\right]^2, \tag{6}$$

which must hold for all h, i, and s, along with conditions (1) and (2). The scalar $\hat{\lambda}_h$ is the Lagrangean multiplier on the budget constraint, condition (2).

$$\tilde{\lambda}_h(s)=\pi(s)\left\{\frac{\partial u_h[x_h(s)]}{\partial x_h^i(s)}\right\}\left[\frac{\sum_{k\neq h}\tilde{b}_k^i(s)}{\sum_{k\neq h}\omega_k^i(s)}\right]\left[\frac{\sum_k\omega_k^i(s)}{\sum_k\tilde{b}_k^i(s)}\right]^2, \tag{7}$$

and

$$\frac{\tilde{\lambda}_h(s)}{\tilde{\lambda}_h} = \frac{\sum_{k \neq h} \tilde{q}_k^m(s)}{\sum_{k \neq h} \tilde{b}_k^m(s)} \left[\frac{\sum_k \tilde{b}_k^m(s)}{\sum_k \tilde{q}_k^m(s)} \right]^2, \tag{8}$$

which must hold along with conditions (3), (4), and (5) for all h, i, and s. The scalar $\tilde{\lambda}_h(s)$ is the Lagrangean multiplier on the state s budget constraint, condition (4), and $\tilde{\lambda}_h$ is the multiplier on the securities constraint, condition (5).

Conditions (1), (2), and (6) are necessary and sufficient for $\hat{b}$ to be an optimal solution, and conditions (3), (4), (5), (7), and (8) are necessary and sufficient for $\tilde{b}$ to be an optimal solution. Even though the utility functions are not jointly concave functions of bids and offers, one can set up corresponding concave programming problems in *allocation space*. Any strategies satisfying (1), (2), and (6) or (3), (4), (5), (7), and (8) satisfy the first-order conditions of these concave programming problems, which guarantees that our conditions are necessary and *sufficient* for an optimum.

Suppose that the conclusion of the proposition is false. Then we have

$$\hat{b}_h^i(s) \frac{\sum_k \omega_k^i(s)}{\sum_k \hat{b}_k^i(s)} = \tilde{b}_h^i(s) \frac{\sum_k \omega_k^i(s)}{\sum_k \tilde{b}_k^i(s)} \quad \text{for all } h, i, \text{ and } s. \tag{9}$$

Using equations (6) and (7) for commodities i and j and simplifying, we have

$$\left[\frac{\sum_k \hat{b}_k^j(s)}{\sum_k \hat{b}_k^i(s)} \right]^2 \left[\frac{\sum_{k \neq h} \hat{b}_k^i(s)}{\sum_{k \neq h} \hat{b}_k^j(s)} \right] = \left[\frac{\sum_k \tilde{b}_k^j(s)}{\sum_k \tilde{b}_k^i(s)} \right]^2 \left[\frac{\sum_{k \neq h} \tilde{b}_k^i(s)}{\sum_{k \neq h} \tilde{b}_k^j(s)} \right]. \tag{10}$$

Equations (9) and (10) imply

$$\frac{\hat{b}_h^i(s)}{\tilde{b}_h^i(s)} = \frac{\hat{b}_h^j(s)}{\tilde{b}_h^j(s)}.$$

Thus, there is a function of s, $c(s)$, for which we have for all h and i,

$$\hat{b}_h^i(s) = c(s) \tilde{b}_h^i(s). \tag{11}$$

[If there is only one commodity, equation (11) follows immediately from equation (9)].

Going back to equations (6) and (7), we have for all h and s, $c(s)\hat{\lambda}_h = \tilde{\lambda}_h(s)$. Therefore, we also have $\tilde{\lambda}_h(\alpha)/\tilde{\lambda}_h(\beta) = c(\alpha)/c(\beta)$ for any choice of α and β. That is, the ratio of Lagrangean multipliers for any two states is the same for all consumers. From equation (8), we have for all consumers

$$\left[\frac{\sum_{k \neq h} \tilde{q}_k^m(\alpha)}{\sum_{k \neq h} \tilde{b}_k^m(\alpha)} \right] \left[\frac{\sum_k \tilde{b}_k^m(\alpha)}{\sum_k \tilde{q}_k^m(\alpha)} \right]^2 \left[\frac{\sum_{k \neq h} \tilde{b}_k^m(\beta)}{\sum_{k \neq h} \tilde{q}_k^m(\beta)} \right] \left[\frac{\sum_k \tilde{q}_k^m(\beta)}{\sum_k \tilde{b}_k^m(\beta)} \right]^2 = \frac{c(\alpha)}{c(\beta)}. \tag{12}$$

Define $p^m(s)$, the price of an Arrow security (or "Arrow money") in state s by

$$p^m(s) = \frac{\sum_k \tilde{b}_k^m(s)}{\sum_k \tilde{q}_k^m(s)}.$$

Let α be a state for which some consumer is a net purchaser of securities (there is one by hypothesis), let H_{buy} be the set of consumers who are net purchasers of state α securities, and let H_0 be the set of consumers who have zero net trades of state α securities. Let H_{sell} be the set of net sellers of state α securities. The sets H_{buy} and H_{sell} are both nonempty. For $h \in H_{\text{buy}}$, $h' \in H_{\text{sell}}$, and for $\tilde{q}_h^m(\alpha) \neq 0$, we have

$$\frac{\tilde{b}_{h'}^m(\alpha)}{\tilde{q}_{h'}^m(\alpha)} < p^m(\alpha) < \frac{\tilde{b}_h^m(\alpha)}{\tilde{q}_h^m(\alpha)}. \tag{13}$$

From inequality (13), we have

$$\frac{\sum_{k\neq h} \tilde{b}_k^m(\alpha)}{\sum_{k\neq h} \tilde{q}_k^m(\alpha)} = \frac{p^m(\alpha) - [\tilde{b}_h^m(\alpha)/\sum_k \tilde{q}_k^m(\alpha)]}{1-[\tilde{q}_h^m(\alpha)/\sum_k \tilde{q}_k^m(\alpha)]}$$
$$< \frac{p^m(\alpha) - [\tilde{q}_h^m(\alpha)p^m(\alpha)/\sum_k \tilde{q}_k^m(\alpha)]}{1-[\tilde{q}_h^m(\alpha)/\sum_k \tilde{q}_k^m(\alpha)]}.$$

The last expression equals $p^m(\alpha)$, which implies

$$\frac{\sum_{k\neq h} \tilde{q}_k^m(\alpha)}{\sum_{k\neq h} \tilde{b}_k^m(\alpha)} > \frac{1}{p^m(\alpha)}. \tag{14}$$

Furthermore, inequality (14) also holds for the case of $\tilde{q}_h^m(\alpha) = 0$. For h', a similar argument to that used to establish (14), but with the inequalities reversed, yields

$$\frac{\sum_{k\neq h'} \tilde{q}_k^m(\alpha)}{\sum_{k\neq h'} \tilde{b}_k^m(\alpha)} < \frac{1}{p^m(\alpha)}. \tag{15}$$

Combining condition (14) with equation (12), yields

$$\frac{\sum_{k\neq h} \tilde{b}_k^m(\beta)}{\sum_{k\neq h} \tilde{q}_k^m(\beta)} < \frac{[p^m(\beta)]^2 c(\alpha)}{p^m(\alpha)c(\beta)}$$

for all β. Equation (12) is satisfied for consumer h', so for all β we have

$$\frac{\sum_{k\neq h'} \tilde{b}_k(\beta)}{\sum_{k\neq h'} \tilde{q}_k^m(\beta)} > \frac{[p^m(\beta)]^2 c(\alpha)}{p^m(\alpha)c(\beta)}. \tag{16}$$

For each consumer $h_0 \in H_0$, we have

$$\frac{\sum_{k\neq h_0} \tilde{b}_k^m(\beta)}{\sum_{k\neq h_0} \tilde{q}_k^m(\beta)} = \frac{[p^m(\beta)]^2 c(\alpha)}{p^m(\alpha)c(\beta)}.$$

For $h \in H_{\text{buy}}$ and $h' \in H_{\text{sell}}$, we have that

$$\frac{\sum_{k \neq h} \tilde{b}_k^m(s)}{\sum_{k \neq h} \tilde{q}_k^m(s)} < \frac{\sum_{k \neq h'} \tilde{b}_k^m(s)}{\sum_{k \neq h'} \tilde{q}_k^m(s)} \tag{17}$$

holds for all states s.

A typical state β falls into one of two categories.

Case 1: For all $h \in H_{\text{buy}}$, we have $[\sum_{k \neq h} \tilde{b}_k^m(\beta)]/[\sum_{k \neq h} \tilde{q}_k^m(\beta)] < p^m(\beta)$. Here all consumers in H_{buy} are net buyers of state-β securities, so we have

$$\sum_{k \in H_{\text{buy}}} [\tilde{b}_k^m(\beta) - \tilde{q}_k^m(\beta) p^m(\beta)] > 0. \tag{18}$$

Case 2: For some $h \in H_{\text{buy}}$, we have

$$\frac{\sum_{k \neq h} \tilde{b}_k^m(\beta)}{\sum_{k \neq h} \tilde{q}_k^m(\beta)} \geq p^m(\beta). \tag{19}$$

From condition (17), we have for all $h' \in H_{\text{sell}}$

$$\frac{\sum_{k \neq h'} \tilde{b}_k^m(\beta)}{\sum_{k \neq h'} \tilde{q}_k^m(\beta)} > p^m(\beta),$$

so all consumers in H_{sell} are net sellers of state-β securities. Therefore,

$$\sum_{k \in H_{\text{sell}}} [\tilde{b}_k^m(\beta) - \tilde{q}_k^m(\beta) p^m(\beta)] < 0$$

holds. For consumers in H_0, we have

$$\sum_{k \in H_0} [\tilde{b}_k^m(\beta) - \tilde{q}_k^m(\beta) p^m(\beta)] \leq 0,$$

which implies

$$\sum_{k \in H_{\text{buy}}} [\tilde{b}_k^m(\beta) - \tilde{q}_k^m(\beta) p^m(\beta)] > 0.$$

Since inequality (18) holds for each of the two categories into which state β may fall, we have

$$\sum_{k \in H_{\text{buy}}} \sum_s [\tilde{b}_k^m(s) - \tilde{q}_k^m(s) p^m(s)] > 0,$$

which means for some $h \in H_{\text{buy}}$,

$$\sum_s [\tilde{b}_h^m(s) - \tilde{q}_h^m(s) p^m(s)] > 0$$

holds, contradicting the securities constraint, inequality (5). Our supposition that $\tilde{b}$ and $\hat{b}$ result in the same allocation is false, and the proposition is proved. ■

6 Concluding remarks

1. Our nonequivalence result (Proposition 2) is very strong. The set of Nash equilibria for the Arrow-securities game in which income is transferred across states is *disjoint* from the set of Nash equilibria for the contingent-commodities game. The sets are not merely different, they have no element in common. Careful reading of the proof of Proposition 2 shows that this result is not restricted to the sell-all variants but applies to any of these games in which offers are harmonized (i.e., the same in each of the two games). What would follow if offers were unrestricted (except that they not exceed endowments of commodities) in both games? One would then have to compare the set of all interior Nash equilibria for the Arrow-securities game with unrestricted offers to the set of all interior Nash equilibria for the contingent-commodities game with unrestricted offers. We conjecture that the two corresponding (big) sets of equilibrium allocations are different, but this remains to be shown.

2. In this chapter, we assume that information is symmetric. We see no reason why our nonequivalence result should not be extendable to include asymmetric information in rational-expectations games, but this also remains to be seen. For purely extrinsic uncertainty, the question would be: Is there a Nash-equilibrium allocation to the securities game that is not a correlated-equilibrium allocation to the certainty game but that is a Nash equilibrium to the contingent-commodities game?

3. The comparative statics for the competitive model is different from that for the imperfectly competitive model. Under competition, there is sense to the notion of complete markets. When markets are complete, adding further markets leaves the equilibrium allocation unaltered. In the imperfectly competitive environment, markets are *inherently incomplete.* Adding markets typically alters the set of Nash-equilibrium allocations.

Appendix

Proof of Proposition 1: Let

$$z_h = [z_h^1(1), \ldots, z_h^i(s), \ldots, z_h^\ell(r); z_h^m(1), \ldots, z_h^m(r)]$$

be the value of y_h that maximizes

$$v_h\{y_h^m(1), \ldots, y_h^m(r), [y_h^i(s)]_{i,s}; [d_{-h}^m(s)]_s, [d_{-h}(s)]_s\},$$

subject to the budget and securities constraints (4) and (5), given that $\tilde{q}_h^m(s) = Q$ holds for all h and s. The bids of others, possibly augmented by ϵ, are taken as fixed for consumer h's maximization problem; see Definition (1). The values of Q and ϵ will be specified later.

The optimal bid for consumer h, z_h, is then a well-defined function of everyone's bids b. Define the adjustment functions $b_h^i(s) \to [b_h^i(s)]'$ and $b_h^m(s) \to [b_h^m(s)]'$ by

$$[b_h^i(s)]' = \frac{z_h^i(s)}{\Sigma_k \Sigma_j z_k^j(s)},$$

and

$$[b_h^m(s)]' = \frac{z_h^m(s)}{\Sigma_k \Sigma_s z_k^m(s)}.$$

Claim 3. *The set of consumption vectors that give consumer h at least as much utility as $v_h(\omega_h)$ without violating the resource constraints of the economy K_h is* (i) *a compact set in $\mathbb{R}^{r\ell}_{++}$ and* (ii) *bounded away from the axes by the positive scalars $\zeta_h^i(s)$ for $i = 1, 2, \dots, \ell$ and $s = 1, 2, \dots, r$.*

Proof of Claim 3: (i) K_h is the intersection of two closed sets and is therefore closed. Since resources are bounded, K_h is bounded, which implies K_h is compact.

(ii) The closure assumption on preferences guarantees that each point in the upper contour set to the indifference surface through ω_h is strictly positive. The set K_h is a subset of this upper contour set. On K_h, the pointwise distance to the axes is bounded above zero, because it is a continuous function on a compact set and therefore attains its minimum. ■

Claim 4. *For all s, and for $b \in \Delta^{nr} \times \Delta^{n\ell} \times \cdots \times \Delta^{n\ell}$, there are positive scalars $\underline{\theta}$ and $\bar{\theta}$, independent of b and ϵ, such that*

$$\underline{\theta} \le \sum_k \sum_j z_k^j(s) \le \bar{\theta}.$$

Proof of Claim 4: From (4), the utility-maximizing bids $z_h^j(s)$ satisfy

$$\sum_j z_h^j(s) \le \max_{h,j}\left[\frac{\omega_h^j(s)}{\Sigma_k \omega_k^j(s)}\right]\left[\sum_{k\ne h}\sum_j d_k^j(s) + \sum_j z_h^j(s)\right] + nQ.$$

Let ϵ be chosen less than $1/n\ell$. Since we have $b \in \Delta^{nr} \times \Delta^{\ell n} \times \cdots \times \Delta^{\ell n}$, the expression $\Sigma_{k\ne h} \Sigma_j d_k^j(s)$ must be less than 2. Therefore, we have

$$\sum_j z_h^j(s) \le \frac{1}{1 - \max_{h,j}[\omega_h^j(s)/\Sigma_k \omega_k^j(s)]}\left[2 \max_{h,j}\left\{\frac{\omega_h^j(s)}{\Sigma_k \omega_k^j(s)}\right\} + nQ\right].$$

Thus, $\Sigma_k \Sigma_j z_k^j(s) \le \bar{\theta}$, where $\bar{\theta}$ is defined to be equal to

$$\bar{\theta} = \frac{n\{2 \max_{h,j}[\omega_h^j(s)/\Sigma_k \omega_k^j(s)] + nQ\}}{1 - \max_{h,j}[\omega_h^j(s)/\Sigma_k \omega_k^j(s)]}.$$

From Lemma 1 and Claim 3, it is clear that $z_h^j(s)$ must put consumer h's allocation in the set K_h. Define the positive scalar ζ by $\zeta=\min_{h,j,s}[\zeta_h^j(s)]$. There must be some h and some i for which we have $\Sigma_{k\neq h} d_k^i(s)\geq 1/n\ell$, because of the normalization. For x_h to be in K_h, the inequality

$$\zeta \leq x_h^i(s)=\frac{z_h^i(s)\,\Sigma_k\,\omega_k^i(s)}{z_h^i(s)+\Sigma_{k\neq h}\,d_k^i(s)}$$

must hold. Thus, we have

$$z_h^i(s)\leq \frac{\zeta/n\ell}{\Sigma_k\,\omega_k^i(s)}. \tag{20}$$

We define $\underline{\theta}$ to be equal to the right-hand side of inequality (20). It follows that

$$\sum_k \sum_j z_k^j(s) \geq \underline{\theta}. \qquad \blacksquare$$

Claim 5. *For $b\in\Delta^{nr}\times\Delta^{n\ell}\times\cdots\times\Delta^{n\ell}$, there are positive scalars $\underline{\phi}$ and $\bar{\phi}$ (independent of b and ϵ) for which we have*

$$\underline{\phi}\leq \sum_k \sum_s z_k^m(s)\leq \bar{\phi}.$$

Proof of Claim 5: From the securities constraint (5), we have

$$\sum_s z_h^m(s)\leq \frac{1}{n}\sum_s\left[\sum_{k\neq h} d_k^m(s)+z_h^m(s)\right].$$

Because we have $[b_1^m(1),\ldots,b_n^m(r)]\in\Delta^{nr}$, we have $\Sigma_s\, z_h^m(s)\leq 1/(n-1)$. Therefore, we also have

$$\sum_k\sum_s z_k^m(s)\leq\frac{n}{n-1}=\bar{\phi}.$$

From (4), it follows that we have

$$0\leq \sum_j z_h^j(s)\left\{1-\max_{h,j}\left[\frac{\omega_h^j(s)}{\Sigma_k\,\omega_k^j(s)}\right]\right\}$$

$$\leq \max_{h,j}\left[\frac{\omega_h^j(s)}{\Sigma_k\,\omega_k^j(s)}\right]\left[\sum_{k\neq h}\sum_j d_k^j(s)\right]+Q\left[\frac{nz_h^m(s)}{\Sigma_{k\neq h}\,d_k^m(s)+z_h^m(s)}-1\right].$$

Using the normalization of $b(s)$ and the fact that $\Sigma_{k\neq h}\,\Sigma_j\, d_k^j(s)<2$, we can manipulate the last expression to yield

$$\frac{nz_h^m(s)}{\Sigma_{k\neq h}\,d_k^m(s)}\geq 1-\max_{h,j}\left[\frac{\omega_h^j(s)}{\Sigma_k\,\omega_k^j(s)}\right]\frac{2}{Q}. \tag{21}$$

Condition (21) must hold for all h and s. By the normalization of securities bids, there must be some consumer h and some state s for which we have $\Sigma_{k\neq h} d_k^m(s) \geq (1/nr)$. For that h and that s, we have

$$z_h^m(s) \leq \frac{1}{n2r}\left\{1-\max_{h,j}\left[\frac{\omega_h^j(s)}{\Sigma_k\,\omega_k^j(s)}\right]\right\}\frac{2}{Q}.$$

As long as we choose Q to satisfy

$$Q \geq 4\max_{h,j,s}\left[\frac{\omega_h^j(s)}{\Sigma_k\,\omega_k^j(s)}\right], \tag{22}$$

we have $z_h^m(s) \geq 1/2n^2r$. Therefore, we have

$$\sum_k \sum_s z_k^m(s) \geq \frac{1}{2n^2r},$$

so $\underline{\phi}$ is defined to be equal to $1/2n^2r$. ■

Claim 6. *The mapping $b \to b'$ has a fixed point. That is, there exists $\dot{b} = [\dot{b}_h^i(s)]_{h,i,s}$ with the property*

$$\dot{b}_h^i(s) = \frac{\dot{z}_h^i(s)}{\Sigma_k\,\Sigma_j\,\dot{z}_k^j(s)} \tag{23}$$

and

$$\dot{b}_h^m(s) = \frac{\dot{z}_h^m(s)}{\Sigma_k\,\Sigma_s\,\dot{z}_k^m(s)}, \tag{24}$$

where $\dot{d}_h^i(s) = \max(\dot{b}_h^i(s), \epsilon)$ and $\dot{z}_h$ is consumer h's utility-maximizing bid given $\dot{d} = \{\dot{d}_h^i(s)\}_{h,i,s}$.

Proof of Claim 6: The function that takes b into b' is the composition of continuous functions, since the denominators are bounded above zero, and thus this function is continuous. Also, this function maps a compact, convex subset of $\mathbb{R}^{nr(\ell+2)}$ into itself. The proof of Claim 6 then follows directly from Brouwer's fixed-point theorem. ■

Claim 7. *For some choice of ϵ, $0<\epsilon<1/n\ell$, we have at the fixed point $\dot{b}_h^i(s) > \epsilon$ for all h, i, and s.*

To prove Claim 7, we suppose it were not true, and we consider two cases. Intuitively, Case 1 represents too low a price for some commodity i in state s, which is never possible. Case 2 represents a price for commodity i that is not too low, so each consumer h must bid at least some amount to put himself in K_h. By choosing ϵ below that amount, Claim 7 is proved.

Proof of Claim 7: Suppose Claim 7 is false. For some h, i, and s, $\dot{b}_h^i(s) \le \epsilon$ occurs for all positive ϵ. Because K_h is compact, each consumer's marginal rate of substitution within K_h is bounded. There are $\gamma_h^{i,j}(s,s')$ and $\delta_h^{i,j}(s,s')$ such that

$$\gamma_h^{i,j}(s,s') \le \mathrm{MRS}_{i,j,s,s'} \le \delta_h^{i,j}(s,s').$$

Let the positive scalar γ be defined by $\gamma = \min_{i,j,k,s,s'}[\gamma_k^{i,j}(s,s')]$. Also, let M_1 be given by

$$M_1 = \max_{h,i,j,s,s'} \left\{ \frac{\sum_{k\neq h} \omega_k^i(s)[\sum_k \omega_k^j(s')]^2}{[\sum_k \omega_k^i(s)]^2[\sum_{k\neq h} \omega_k^j(s')]} \right\},$$

which is well-defined because endowments are positive and finite. Then, define M by

$$M = \frac{[1+\bar{\theta}]^2 M_1}{\min(\underline{\theta},1)\gamma}.$$

We know that for some consumer h' and commodity j, we have $\dot{b}_{h'}^j(s) \ge 1/n\ell$.

Case 1: $\sum_k \dot{d}_k^i(s) \le 1/n\ell M$

There must be some consumer (call her h'') for whom $\dot{b}_{h''}^i(s)$ is less than the sum of the other consumers' possibly augmented bids on that market. Since h'' will choose a point in $K_{h''}$, and since the first-order conditions for utility maximization are necessary and sufficient for an optimum, we have

$$\gamma \le \frac{\dfrac{\partial u_{h''}}{\partial x_{h''}^i(s)}}{\dfrac{\partial u_{h''}}{\partial x_{h''}^j(s)}} = \frac{\left[\dfrac{\dot{z}_{h''}^i(s) + \sum_{k\neq h''} \dot{d}_k^i(s)}{\sum_k \omega_k^i(s)}\right]^2 \left[\dfrac{\sum_{k\neq h''} \omega_k^i(s)}{\sum_{k\neq h''} \dot{d}_k^i(s)}\right]}{\left[\dfrac{\dot{z}_{h''}^j(s) + \sum_{k\neq h''} \dot{d}_k^j(s)}{\sum_k \omega_k^j(s)}\right]^2 \left[\dfrac{\sum_{k\neq h''} \omega_k^j(s)}{\sum_{k\neq h''} \dot{d}_k^j(s)}\right]}.$$

Therefore, we have

$$\gamma \le \left[\frac{M_1 \sum_{k\neq h''} \dot{d}_k^j(s)}{\sum_{k\neq h''} \dot{d}_k^i(s)}\right]\left[\frac{\dot{z}_{h''}^i(s) + \sum_{k\neq h''} \dot{d}_k^i(s)}{\dot{z}_{h''}^j(s) + \sum_{k\neq h''} \dot{d}_k^j(s)}\right]^2. \tag{25}$$

We have the following facts:

$$\dot{z}_{h''}^i(s) + \sum_{k\neq h''} \dot{d}_k^i(s) < \bar{\theta}\dot{b}_{h''}^i(s) + \frac{1}{n\ell M} < \frac{1}{n\ell M}(\bar{\theta}+1)$$

and

$$\dot{z}_{h''}^j(s) + \sum_{k\neq h''} \dot{d}_k^j(s) > \underline{\theta}\dot{b}_{h''}^j(s) + \sum_{k\neq h''} \dot{d}_k^j(s) > \frac{1}{n\ell}[\min(\underline{\theta},1)].$$

The last inequality holds whether or not h' and h'' are equal. Combining the above inequalities with inequality (25) implies that

$$\gamma<\frac{M_1(\bar{\theta}+1)[\dot{z}^i_{h''}(s)+\sum_{k\neq h''}\dot{d}^i_k(s)]}{M[\min(\underline{\theta},1)]\sum_{k\neq h''}\dot{d}^i_k(s)}$$

holds. From (23) and Claim 4, we have

$$\gamma<\left\{\frac{M_1(\bar{\theta}+1)}{M[\min(\underline{\theta},1)]}\right\}\left[1+\frac{\bar{\theta}\dot{b}_{h''}(s)}{\sum_{k\neq h''}\dot{d}^i_k(s)}\right].$$

Consumer h'' was chosen because her bid is less than $\sum_{k\neq h''}\dot{d}^i_k(s)$, so we have

$$\gamma<\frac{M_1(1+\bar{\theta})^2}{M[\min(\underline{\theta},1)]},$$

which contradicts the definition of M.

Case 2: $\sum_k \dot{d}^i_k(s)>1/n\ell M$

For consumer h to be in K_h, we must have

$$\zeta<\frac{\dot{z}^i_h(s)\sum_k\omega^i_k(s)}{\dot{z}^i_h(s)+\sum_{k\neq h}\dot{d}^i_k(s)}<\frac{\dot{z}^i_h(s)\sum_k\omega^i_k(s)}{\sum_{k\neq h}\dot{d}^i_k(s)}.$$

Because $\dot{d}^i_h(s)=\epsilon$ holds, we have $\sum_{k\neq h}\dot{d}^i_k(s)+\epsilon>1/n\ell M$. Thus, there is a positive value of ϵ, say ϵ_1, below which $\sum_{k\neq h}\dot{d}^i_k(s)\geq 1/n\ell M$ holds. As long as we choose ϵ below ϵ_1, we have

$$\zeta<\frac{\bar{\theta}\epsilon\sum_k\omega^i_k(s)}{1/n\ell M}.$$

Choose ϵ to satisfy

$$0<\epsilon<\min\left[\frac{1}{n\ell},\epsilon_1,\frac{\zeta}{\bar{\theta}\sum_k\omega^i_k(s)n\ell M}\right], \tag{26}$$

so that we have a contradiction. ■

Claim 8. *There is a choice of* ϵ, *satisfying* (26), *below which we have at the fixed point* $\dot{b}^m_h(s)>\epsilon$ *for all h and s.*

To prove Claim 8, we suppose it were not true, and we consider two cases. Case 1 represents too low a price for state-α securities, which is impossible. Case 2 represents a price that is not too low, so each consumer must bid at least some amount to achieve positive income in state α. By choosing ϵ below that amount, Claim 8 is proved.

Proof of Claim 8: Suppose Claim 8 is false. For some h and some α, $\dot{b}_h^m(\alpha) \le \epsilon$ holds for all positive ϵ. We know that for some consumer h' and some β, we have $\dot{b}_{h'}^m(\beta) \ge 1/nr$. Define M_2 by the equation

$$M_2 = \frac{(1+\bar{\phi})^2(\bar{\theta}+1)(1+M\bar{\theta})M_1}{\min(\underline{\phi},1)\min(\underline{\theta},1)}.$$

Case 1: $\sum_k \dot{d}_k^m(\alpha) \le 1/nrM_2$

There must be some consumer h'' for whom we have

$$\dot{b}_{h''}^m(\alpha) \le \sum_{k \ne h''} \dot{d}_k^m(\alpha).$$

Because consumer h'' will choose a point in $K_{h''}$ and because the first-order conditions for utility maximization are necessary and sufficient for an optimum, we have, for i and j,

$$\gamma \le \left\{ \frac{\pi(\alpha)\dfrac{\partial u_{h''}[x_{h''}(\alpha)]}{\partial x_{h''}^i}}{\pi(\beta)\dfrac{\partial u_{h''}[x_{h''}(\beta)]}{\partial x_{h''}^j}} \right\} = \left[\frac{\dot{z}_{h''}^m(\alpha)+\sum_{k\ne h''}\dot{d}_k^m(\alpha)}{\dot{z}_{h''}^m(\beta)+\sum_{k\ne h''}\dot{d}_k^m(\beta)}\right]^2 \left[\frac{\sum_{k\ne h''}\dot{d}_k^m(\beta)}{\sum_{k\ne h''}\dot{d}_k^m(\alpha)}\right]$$
$$\times \left[\frac{\sum_k \omega_k^i(\alpha)}{\sum_k \omega_k^j(\beta)}\right]^2 \left[\frac{\sum_{k\ne h''}\omega_k^j(\beta)}{\sum_{k\ne h''}\omega_k^i(\alpha)}\right]\left[\frac{\sum_{k\ne h''}\dot{d}_k^j(\beta)}{\sum_{k\ne h''}\dot{d}_k^i(\alpha)}\right]$$
$$\times \left[\frac{\dot{z}_{h''}^i(\alpha)+\sum_{k\ne h''}\dot{d}_k^i(\alpha)}{\dot{z}_{h''}^j(\beta)+\sum_{k\ne h''}\dot{d}_k^j(\beta)}\right]^2. \tag{27}$$

We can choose commodity j to satisfy $\sum_k \dot{d}_k^j(\beta) \ge 1/\ell$; and by Case 1 of Claim 7, we can choose i to satisfy $1/n\ell M \le \sum_k \dot{d}_k^i(\alpha) \le 1/\ell$.

It follows from inequality (27) that we have

$$\gamma < \left[\frac{\dot{z}_{h''}^m(\alpha)+\sum_{k\ne h''}\dot{d}_k^m(\alpha)}{\dot{z}_{h''}^m(\beta)+\sum_{k\ne h''}\dot{d}_k^m(\beta)}\right]^2 \left[\frac{\sum_{k\ne h''}\dot{d}_k^m(\beta)}{\sum_{k\ne h''}\dot{d}_k^m(\alpha)}\right]\left[\frac{M_1(\bar{\theta}+1)(1+M\bar{\theta})}{\min(\underline{\theta},1)}\right].$$

We also know that

$$\dot{z}_{h''}^m(\alpha) + \sum_{k\ne h''}\dot{d}_k^m(\alpha) < \bar{\phi}\dot{b}_{h''}^m(\alpha) + \frac{1}{nrM_2} < \frac{1}{nrM_2}(1+\bar{\phi})$$

and

$$\dot{z}_{h''}^m(\beta) + \sum_{k\ne h''}\dot{d}_k^m(\beta) > \underline{\phi}\dot{b}_{h''}^m(\beta) + \sum_{k\ne h''}\dot{d}_k^m(\beta) > \min(\underline{\phi},1)\left(\frac{1}{nr}\right)$$

hold. The last inequality holds whether or not h' and h'' are equal. Thus, we have

$$\gamma < \frac{(1+\bar{\phi})}{\min(\underline{\phi},1)M_2}\left[1+\frac{\dot{z}_{h''}^m(\alpha)}{\sum_{k\ne h''}\dot{d}_k^m(\alpha)}\right]\frac{M_1(\bar{\theta}+1)(1+M\bar{\theta})}{\min(\underline{\theta},1)}. \tag{28}$$

Consumer h'' bids less on the α security market than the sum of the augmented bids of the other consumers. This fact, conditions (24) and (28), and the bounds on the securities bids imply that

$$\gamma < \frac{(1+\bar{\phi})^2 M_1(\bar{\theta}+1)(1+M\bar{\theta})}{\min(\underline{\phi},1)M_2 \min(\underline{\theta},1)},$$

which contradicts the definition of M_2.

Case 2: $\sum_k \mathring{d}_k^m(\alpha) > 1nrM_2$

From condition (4) and the fact that $\mathring{b}_h^m(\alpha) \le \epsilon$, we have that

$$0 < 2\max_{j,h}\left[\frac{\omega_h^j(\alpha)}{\sum_k \omega_k^j(\alpha)}\right] + \frac{\epsilon nQ}{(1/nrM_2)} - Q$$

must hold. Thus, we have

$$\frac{2\max_{j,h,s}[\omega_h^j(s)/\sum_k \omega_k^j(s)]}{Qn^2rM_2} < \frac{Q-2\max_{j,h}[\omega_h^j(\alpha)/\sum_k \omega_k^j(\alpha)]}{Qn^2rM_2} < \epsilon.$$

By choosing ϵ less than

$$\frac{2\max_{j,h}[\omega_h^j(\alpha)/\sum_k \omega_k^j(\alpha)]}{Qn^2rM_2},$$

but positive, we have a contradiction. ■

Claim 9. *Choose ϵ greater than 0 and satisfying*

$$\epsilon < \min\left[\frac{1}{n\ell}, \epsilon_1, \left\{\frac{\zeta}{\bar{\theta}\sum_k \omega_k^j(s)n\ell M}\right\}_{j,s}, \frac{2\max_{j,h,s}[\omega_h^j(s)/\sum_k \omega_k^j(s)]}{Qn^2rM_2}\right]$$

and choose Q so that

$$Q > 4\max_{j,h,s}\left[\frac{\omega_h^j(s)}{\sum_k \omega_k^j(s)}\right]$$

holds. Then the fixed point $\mathring{b}$ (combined with the security market bids of Q for each consumer and state) constitutes an interior Nash equilibrium.

Proof of Claim 9: We have shown that $\mathring{b} = \mathring{d}$. Conditions (23) and (24) hold at the fixed point, where $\mathring{z}_h$ is now consumer h's utility-maximizing bid given $\mathring{b}$.

When $\sum_k \sum_i \mathring{z}_k^i(s) > 1$ for some s, we have $\mathring{b}_h^i(s) < \mathring{z}_h^i(s)$ for all h and i. Thus, consumer h wants to increase all his bids, so at $\mathring{b}$ there is slack in his state s budget constraint. That is, we have

$$\sum_j \dot{b}_h^j(s) < \sum_j \left[\frac{\omega_h^j(s) \sum_k \dot{b}_k^j(s)}{\sum_k \omega_k^j(s)} \right] + \frac{\dot{b}_h^m(s) nQ}{\sum_k \dot{b}_k^m(s)} - Q.$$

Summing over h, we reach a contradiction.

When $\sum_k \sum_i \dot{z}_k^i(s) < 1$ for some s, we have $\dot{b}_h^i(s) > \dot{z}_h^i(s)$ for all h and i. Consumer h wants to decrease all his bids, so at $\dot{b}$ his bids must not have been feasible in state s. That is, we have

$$\sum_j \dot{b}_h^j(s) > \sum_j \left[\frac{\omega_h^j(s) \sum_k \dot{b}_k^j(s)}{\sum_k \omega_k^j(s)} \right] + \frac{\dot{b}_h^m(s) nQ}{\sum_k \dot{b}_k^m(s)} - Q.$$

Summing over h, we reach a contradiction, which implies $\sum_k \sum_i \dot{z}_k^i(s) = 1$.

When $\sum_k \sum_s \dot{z}_k^m(s) > 1$, we have $\dot{b}_h^m(s) < \dot{z}_h^m(s)$ for all h and s. Consumer h, given the other securities bids, wants to increase all his bids, which means there is slack in his securities constraint. Thus, we have

$$\sum_s \dot{b}_h^m(s) < \frac{1}{n} \sum_s \sum_k \dot{b}_k^m(s).$$

Summing over h, we reach a contradiction.

When $\sum_k \sum_s \dot{z}_k^m(s) < 1$, we have $\dot{b}_h^m(s) > \dot{z}_h^m(s)$. Consumer h wants to reduce all his securities bids, which means his securities constraint is violated at $\dot{b}$. Thus, we have

$$\sum_s \dot{b}_h^m(s) > \frac{1}{n} \sum_s \sum_k \dot{b}_k^m(s).$$

Summing over h, we reach a contradiction, which implies $\sum_k \sum_s \dot{z}_k^m(s) = 1$.

Conditions (23) and (24) reduce to the following:

$$\dot{b}_h^i(s) = \dot{z}_h^i(s)$$

and

$$\dot{b}_h^m(s) = \dot{z}_h^m(s).$$

Given $\dot{b}$ and Q, each consumer optimizes by not deviating from $\dot{b}$. Because $\dot{b}_h^m(s)$ is positive for all h and s, no consumer gains from changing $\tilde{q}_h^m(s)$ away from Q.[14] Thus, we have found an interior Nash equilibrium for the Arrow-securities game, completing the proof of Proposition 1. ■

[14] See Peck and Shell (1985) for the proof. Because optimizing with respect to $q_h^m(s)$ and $b_h^m(s)$ yields the same interior first-order conditions, fixing $q_h^m(s)$ and solving for $b_h^m(s)$ is a solution to the problem in which $q_h^m(s)$ is not fixed. See Peck and Shell (1985, sect. 2.1, pp. 11–13) for further elucidation of this point.

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