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ON TAXATION AND COMPETITIVE EQUILIBRIA *

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I. INTRODUCTION

A central question in the theory of public finance is how the government's fiscal policies affect the allocation of resources ; see, e.g., Guesnerie's (1980) monograph, Chapters III and IV. The principal goal of the present paper is to analyze the effects of (costless) lump-sum taxes on competitive equilibria. Costless lump-sum taxation is, of course, an idealization, representing the unrealistic case where fiscal policy is completely potent. This study should be taken, therefore, as only a prolegomenon to future developments, which will reflect the fact that real-world taxes are limited in scope (cf., e.g., Diamond and Mirrlees (1971) or, more generally, that administration of tax schemes is costly (cf. Heller and Shell (1974)).

The particular issues we investigate in this paper are very similar to those which have been dealt with recently in the theory of general equilibrium (without taxation) by Debreu (1970) and others. Here, taxes are among the parameters which define the economy. We investigate the number of equilibria with lump-sum taxes, the continuity and smoothness properties of the mapping from lump-sum taxes to equilibria, and the properties of the set of lump-sum taxes which are consistent with equilibrium.

Another motivation for this paper comes from the theory of monetary economics. The lump-sum taxation model can be slightly reformulated to become a finite monetary economy. The results we obtain for lump-sum taxes apply *mutatis mutandis* to monetary policies. The results can be contrasted with the basically different

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results which have been derived for the (more realistic) infinite monetary economy.

The analysis is cast in terms of an exchange economy. Consumers are assumed to satisfy regularity properties which allow us to employ the techniques of elementary differential topology. Definitions, assumptions and notation are introduced in Section 2. In Section 3, we recall the well-known relationship between lump-sum tax policies and Pareto-optimal allocations. We then go on to investigate *bonafide fiscal policies*, those tax systems consistent with the existence of competitive equilibrium. For given tastes and endowments, the set of bonafide fiscal policies is shown to be bounded, arc-connected, and to contain the zero vector in its interior.

Section 4, the mathematical heart of this paper, is devoted to the structure of the equilibrium set and to comparative statics. The graph of the correspondence from bonafide taxes to equilibrium prices is shown to be a smooth manifold diffeomorphic to $\mathbb{R}^{n-1}$, where n is the number of consumers. We establish the following *generic* results for the regular exchange economy with lump-sum taxes : (i) the number of equilibria is finite (although not necessarily odd) ; (ii) locally, there are smooth mappings from bonafide tax vectors to equilibrium price vectors ; (iii) the number of equilibria is locally constant. We also show in Section 4 that the set of bonafide fiscal policies is the union of an open set and a negligible set. In Section 5, our results are applied to the finite-dimensional monetary economy.

II. DEFINITIONS, ASSUMPTIONS, AND NOTATION

There are ℓ commodities and n consumers. Both ℓ and n are (finite) positive integers. Consumers have preference preorderings defined on the strictly positive orthant $\mathbb{R}^{\ell}_{++}$. Each preference preordering is assumed to be monotonic and strictly convex. Furthermore, each indifference set is a smooth surface with everywhere nonzero Gaussian curvature. The closure of each indifference surface in $\mathbb{R}^{\ell}$ is contained in $\mathbb{R}^{\ell}_{++}$.

Let p_j denote the price of commodity j ($j = 1, \ldots, \ell$) and $p = (p_1, \ldots, p_j, \ldots, p_\ell)$ be the price vector. Choose commodity ℓ as the *numéraire*, $p_\ell = 1$, and define the set of strictly positive normalized prices $\mathcal{S} = \{p \in \mathbb{R}^{\ell}_{++} \mid p_\ell = 1\}$. We denote by $x_i \in \mathbb{R}^{\ell}_{++}$ a consumption plan for consumer i ($i = 1, \ldots, n$) while $w_i \in \mathbb{R}_{++}$ denotes his income. Consumer i's demand for commodities is obtained by maximizing his preordering $\precsim_i$ on the budget set $\{x_i \in \mathbb{R}^{\ell}_{++} \mid p.x_i \leqq w_i\}$ for given $p \in \mathcal{S}$. This constrained maximization problem has a unique solution denoted by $f_i(p, w_i) \in \mathbb{R}^{\ell}_{++}$. The individual demand mapping thus defined, $f_i : \mathcal{S} \times \mathbb{R}_{++} \to \mathbb{R}^{\ell}_{++}$ is a diffeomorphism. Let $\omega_i \in \mathbb{R}^{\ell}_{++}$ denote consumer i's endowments and define $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^{\ell}_{++} \times \ldots \times \mathbb{R}^{\ell}_{++}$.

Next define a *fiscal policy* (or, a *system of lump-sum taxes*) by the vector $\tau = (\tau_1, \ldots, \tau_n)$. The *space of feasible fiscal policies* (or, *space of feasible lump-sum taxes*) is denoted by F. We place no restrictions on this space, so that $F = \mathbb{R}^n$. That is, τ_i ($i = 1, \ldots, n$) can be any real number. If τ_i is positive, consumer i is being taxed a positive amount, while if τ_i is negative he is receiving a positive transfer (i.e., he is "paying" a negative tax).

2.1. Definition : Given endowments $\omega \in (\mathbb{R}^{\ell}_{++})^n$ and the fiscal policy $\tau \in F$, $p \in \mathcal{S}$ is said to be a *competitive equilibrium price vector* associated with (ω, τ) if p satisfies the equation

$$\sum_{i=1}^{n} f_i(p, p.\omega_i - \tau_i) = \sum_{i=1}^{n} \omega_i = r,$$

where $r \in \mathbb{R}^{\ell}_{++}$ is the vector of aggregate endowments, or resources. The *set of equilibrium price vectors* associated with (ω, τ) is denoted by $\mathcal{S}(\omega, \tau) \subset \mathcal{S} \subset (\mathbb{R}^{\ell}_{++})^n$.

Our aim is to study the equilibrium set $\mathcal{S}(\omega, \tau)$, the dependence of the equilibrium set on taxes $\tau \in F$, and also the set of taxes for which the equilibrium set is nonempty, i.e., such that $\mathcal{S}(\omega, \tau) \neq \emptyset$. This leads us to the next definition.

2.2. Definition : Fix the endowment vector $\omega \in (\mathbb{R}^{\ell}_{++})^n$. Then $\tau \in F$ is said to be an *ω-bonafide fiscal policy* if $\mathcal{S}(\omega, \tau) \neq \emptyset$. The *set of*

ω-bonafide fiscal policies is denoted by $\mathcal{F}_B(\omega)$.

If τ is not ω-bonafide, then the government could not have a "*good-faith*" expectation that τ would be consistent with equilibrium when the endowments of the economy are given by ω. If $\tau \epsilon$ F is ω-bonafide in the finite economy, then obviously the sum of the taxes must equal the sum of the transfers. This is formalized in the next lemma.

2.3. Lemma : Let $\tau = (\tau_1, \ldots, \tau_i, \ldots, \tau_n) \in F$ be a fiscal policy and $\omega \in (\mathbb{R}^{\ell}_{++})^n$ be an endowment vector. The set of ω-bonafide taxes $\mathcal{F}_B(\omega)$ is a subset of $\mathcal{F} = \left\{ \tau \epsilon \text{ F} \mid \sum_{i=1}^{n} \tau_i = 0 \right\}$.

Proof : Since $w_i = p.\omega_i - \tau_i = p.f_i(p, w_i)$, we have from Definition (2.1).

$$p.\sum_{i=1}^{n} f_i(p, w_i) = p.\sum_{i=1}^{n} \omega_i - \sum_{i=1}^{n} \tau_i = p.\sum_{i=1}^{n} \omega_i$$

and thus $\quad \sum_{i=1}^{n} \tau_i = 0.$ □

Thus, for the *finite* economy we can restrict attention to τ lying in $\mathcal{F}$, an $(n-1)$-dimensional subspace of F. Contrast this with the overlapping-generations model (cf. Balasko and Shell (1981)), where there can be bonafide fiscal policies which do not satisfy $\sum \tau_i = 0$.

In our analysis, we shall find it convenient to employ the price-income equilibrium approach of Balasko (1979). This concept is formalized in the next definition.

2.4. Définition : For a fixed resource vector $r \in \mathbb{R}^{\ell}_{++}$, the vector

$(p, w) = (p_1, \ldots, p_\ell, w_1, \ldots, w_n) \in \mathcal{T} \times \mathbb{R}^n_{++}$ is said to be a *price-income equilibrium* if it satisfies the equation

$$\sum_{i=1}^{n} f_i(p, w_i) = r.$$

We denote by $\mathcal{B}(r)$ *the set of price-income equilibria consistent with* r. This set is diffeomorphic to $\mathbb{R}^{n-1}$ (same proof as in Balasko (1979), the commodity space being there $\mathbb{R}^\ell$ instead of $\mathbb{R}^\ell_{++}$; see also the proof of (4.1)).

III. BONAFIDE TAXES AND PARETO-OPTIMAL ALLOCATIONS

The results of this section are not new. They are consequences of the Second Theorem of Welfare Economics (cf. e.g., Arrow and Hahn (1971), Chapter 4). We include the following two propositions for completeness. The proofs are short and introduce the analysis of the next section, all of which is new.

We begin with a proposition which states that with lump-sum taxes as its only instrument the government's fiscal policy is fully potent, a well-known and important result in the public finance literature. We then turn to the analysis of the structure of the set of bonafide fiscal policies.

3.1. Proposition : Fix resources $r \in (\mathbb{R}^\ell_{++})$. Any Pareto-optimal allocation is a competitive equilibrium allocation associated with the fixed endowment vector $\omega \in (\mathbb{R}^\ell_{++})^n$ satisfying $\sum_{i=1}^{n} \omega_i = r$ and some lump-sum fiscal policy $\tau \in \mathcal{F}$. The vector τ is then unique.

Proof : If $p \in \mathcal{T}(\omega, \tau)$, then the competitive equilibrium allocation is the vector $(f_1(p, p.\omega_1 - \tau_1), f_2(p, p.\omega_2 - \tau_2), \ldots, f_n(p, p.\omega_n - \tau_n))$.

Let P denote the set of Pareto-optimal allocations in economies defined by preferences $\precsim_i$ $(i = 1, \ldots, n)$ and resources $r \in \mathbb{R}^\ell_{++}$.

The structure of the set P is well-known. Cf. Arrow-Hahn (1971), Chapter 4 or Balasko (1979). For the Balasko (1979) reference, the commodity space is $\mathbb{R}^\ell$ rather than $\mathbb{R}^\ell_{++}$. There is, however, no essential difference in the results (or the proofs) between the two cases.

In particular, P is a smooth submanifold of $(\mathbb{R}^\ell)^n$ diffeomorphic to $\mathbb{R}^{n-1}$. Let $\bar{P}$ be the closure of P in $(\mathbb{R}^\ell)^n$. $\bar{P}$ is compact because it is closed and bounded ($x_i \geqq 0$ and $\Sigma\ x_i = r$). We leave it as an exercise to show that $\bar{P} \backslash P$ consists of a finite union of smooth manifolds of dimension $\leqq n-2$. In other words, $\bar{P}$ has a stratified structure, the dimension $(n-1)$ stratum being P. (Hint : Start as in Schecter (1978), Theorem (3.1), but note that the assumption that the closure of every indifference set in $\mathbb{R}^\ell$ is contained in $\mathbb{R}^\ell_{++}$ implies the transversality of $\bar{P}$ with the boundary of the set $\{x = (x_1, \ldots, x_n) \mid \sum_{i=1}^{n} x_i = r$ and $x_i \geqq 0$ for $i = 1, \ldots, n\}$.) Now let $g : P \to \mathcal{S}$ be a mapping which associates with the Pareto-optimal allocation $x = (x_1, \ldots, x_n) \in P$ the unique price vector $p \in \mathcal{S}$ which supports x. The mapping g is smooth. Define $\phi(x) = (g(x).(x_1-\omega_1), \ldots, g(x).(x_n-\omega_n))$. Clearly, the image $\phi(x)$ belongs to $\mathcal{F}$. The mapping $\phi : P \to \mathcal{F}$ thus defined is smooth. Next, note that if x is a competitive allocation associated with (ω, τ), then $x = (f_1(p, p.\omega_1 - \tau_1), \ldots, f_n(p, p.\omega_n - \tau_n))$ is Pareto-optimal and, therefore, we have $\phi(x) = (-\tau_1, \ldots, -\tau_n)$.

The converse is obvious. □

3.2. Proposition : The set $\mathcal{F}_B(\omega)$ of bonafide lump-sum taxes is bounded, arc-connected, and contains 0 in its interior.

Proof : It follows from the proof of Proposition (3.1) that $\mathcal{F}_B(\omega)$ is the image $\phi(P)$. Therefore, $\mathcal{F}_B(\omega)$ is arcconnected as the image by the smooth (hence continuous) mapping ϕ of Pareto-optimal allocations P, which is arcconnected.

We next show that 0 belongs to the interior of $\mathcal{F}_B(\omega)$. Let $p \in \mathcal{S}(\omega, \tau)$. It results from the numéraire assumption, $p_\ell = 1$, that p belongs to $\mathcal{S}(\omega', 0)$, where $\omega' = (\omega'_1, \ldots, \omega'_i, \ldots, \omega'_n)$ is given by

$$\omega_i'^{k} = \omega_i^{k} \qquad \text{for } k = 1, \ldots, \ell - 1 \text{ and } i = 1, \ldots, n$$

and

$$\omega_i'^{\ell} = \omega_i^{\ell} - \tau_i \quad \text{for } i = 1, \ldots, n,$$

provided that $\omega_i^{\ell} - \tau_i > 0$ for $i = 1, 2, \ldots, n$. The set of the vectors τ satisfying the above contains 0 and is obviously open. Since competitive equilibria exist for these constructed no-taxation economies, the τ's thus constructed belong to $\mathcal{F}_B(\omega)$.

Finally, we show that $\phi(P)$ is bounded. The mapping g, and hence the mapping ϕ, have obvious continuous extensions to $\bar{P}$. Therefore, $\phi(\bar{P})$ is compact, hence bounded. Since we have $\phi(P) \subset \phi(\bar{P})$, $\phi(P)$ is bounded. □

Compare our Proposition (3.2) with De Montbrial (1971), Chapter VIII. Especially note that if the lump-sum tax collected from the ith consumer is sufficiently large, then $(\omega_i^{\ell} - \tau_i)$ would be negative. This observation justifies our specific analysis of equilibria with *lump-sum* taxation rather than merely studying equilibria after only one commodity has been reassigned.

IV. THE STRUCTURE OF THE SET OF EQUILIBRIA AND COMPARATIVE STATICS

We investigate here some basic properties of the set of equilibrium prices $\mathfrak{T}(\omega, \tau)$, including how the set $\mathfrak{T}(\omega, \tau)$ depends on the fiscal policy $\tau \in \mathcal{F}_B(\omega) \subset \mathcal{F}$. We also go beyond Proposition (3.2), providing a more complete description of the set of bona-fide fiscal policies $\mathcal{F}_B(\omega)$.

Let E be the graph of the correspondence from fiscal policies to equilibrium prices $\tau \mapsto \mathfrak{T}(\omega, \tau)$, for fixed endowments ω. The next proposition provides the structure for the graph E, which will be essential to the sequel.

4.1. Proposition : The set E is a smooth submanifold of $\mathcal{F} \times \mathcal{S}$ diffeomorphic to $\mathbb{R}^{n-1}$.

Proposition (4.1) tells us that the set of pairs (τ, p) satisfying $p \in \mathcal{S}(\omega, \tau)$ can be described locally by smooth functions of $(n-1)$ parameters. An obvious corollary to Proposition (4.1) is that the set E is arcconnected. This has the interpretation that the government can "continuously" affect the equilibrium $\mathcal{S}(\omega, \tau)$ through its control variable τ.

Proof : Let $\mathcal{B}(r) \subset \mathcal{S} \times \mathbb{R}^n_{++}$ be the set of price-income equilibria (p, w) for fixed resources r (Definition (2.4)). We define the mapping $\psi : \mathcal{B}(r) \to \mathcal{F} \times \mathcal{S}$ by the formula

$$(p, w_1, \ldots, w_n) \overset{\psi}{\mapsto} (\tau_1 = p.\omega_1 - w_1, \ldots, \tau_n = p.\omega_n - w_n).$$

The set $\psi(\mathcal{B}(r))$ is just E and ψ has a smooth inverse defined on E. Therefore, E and $\mathcal{B}(r)$ are diffeomorphic by ψ, and E is a smooth submanifold of $\mathcal{F} \times \mathcal{S}$.

Since the mapping $x \in P \mapsto (g(x), g(x).x_1, \ldots, g(x).x_n) \in \mathcal{B}(r)$ is a diffeomorphism (see Balasko (1979)), we can conclude by recalling that the set of Pareto-optimal allocations P is diffeormorphic to $\mathbb{R}^{n-1}$. □

The next proposition provides generic properties on the number of equilibria in $\mathcal{S}(\omega, \tau)$ and how these equilibria are affected by changes in the government's fiscal policy $\tau \in \mathcal{F}$.

4.2. Proposition : There is an open dense subset $\mathcal{R}(\omega)$ of $\mathcal{F}$ consisting of fiscal policies τ with the following properties :

(4.2.1.). The set of equilibria $\mathcal{S}(\omega, \tau)$ is finite if τ belongs to $\mathcal{R}(\omega)$.

(4.2.2.). For any $\tau \in \mathcal{R}(\omega)$ there is an open neighborhood $\mathcal{S} \subset \mathcal{R}$ of τ, and k smooth mappings $s_j : \mathcal{S} \to \mathcal{S}$ $(j = 1, \ldots, k)$ such that

$$\mathcal{T}(\omega, .) = \bigcup_{j=1}^{k} s_j(.) \text{ on } \mathcal{S}.$$

Proposition (4.2) should be compared with Debreu (1970), Remark, p. 394. We call $\mathcal{R}(\omega)$ the *set of regular fiscal policies* associated with the endowments ω. Notice that (4.2.2) implies that the number of equilibria k is locally constant, hence constant on every connected component of $\mathcal{R}(\omega)$. The number of equilibria k is not necessarily odd ; in fact, k = 0 when τ does not belong to $\mathcal{F}_B(\omega)$. This differs from the case of pure exchange without taxation.

Proof : For fixed r, we study solutions to the equation system

$$\left.\begin{aligned} &\sum_{i=1}^{n} f_i(p, w_i) = r \\ &w_i = p.\omega_i - \tau_i, i = 1, ..., n. \end{aligned}\right\}$$

This system of equations determines the intersection of the manifold $\mathcal{B}(r)$ of price-income equilibria (p, w) with the linear space

$$\beta(\tau) = \left\{(p, w_1, \ldots, w_n) \in \mathcal{T} \times \mathbb{R}^n_{++} \mid w_i = p.\omega_i - \tau_i \quad \text{for } i = 1, \ldots, n\right\}.$$

Let H be the "hyperplane" in $\mathcal{T} \times \mathbb{R}^n$ defined by the equation

$$p.r = w_1 + \ldots + w_n.$$

Clearly, $\beta(\tau)$ belongs to H if and only if τ belongs to $\mathcal{F}_B(\omega)$. But, $\mathcal{B}(r)$ is a subset of H by Walras' law. Therefore, H is the ambient space in which we shall develop an intersection theory as in Balasko (1979).

Define the mapping $\rho : H \rightarrow \mathcal{F}$ by the formula

$$\rho(p, w_1, \ldots, w_n) = (\tau_1 = p.\omega_1 - w_1, \ldots, \tau_n = p.\omega_n - w_n).$$

Now let $\overline{\mathcal{B}}(r)$ be the closure of $\mathcal{B}(r)$ in H. One readily sees that $\overline{\mathcal{B}}(r)$ is a stratified set diffeomorphic to the stratified set $\bar{P}$ (see the proof of Proposition (3.1)). Thus $\overline{\mathcal{B}}(r)$ is compact, $\mathcal{B}(r)$ is an $(n-1)$ - dimensional stratum, and $\overline{\mathcal{B}}(r)\setminus\mathcal{B}(r)$ consists of a finite number of strata having dimension $\leqslant n - 2$.

Denote by $\bar{\rho}$ the restriction of the mapping ρ to $\overline{\mathcal{B}}(r)$. Clearly, $\bar{\rho}^{-1}(\tau)$ consists of the intersection points of $\overline{\mathcal{B}}(r)$ with $\beta(r)$. Note also $\bar{\rho}\,(\mathcal{B}(r)) = \mathcal{F}_B(\omega) \subset \bar{\rho}\,(\overline{\mathcal{B}}(r))$.

Denote by $\Sigma(\omega)$ the set of singular values of $\bar{\rho}$ restricted to $\mathcal{B}(r)$. Let us consider the set $\Sigma' = \Sigma(\omega) \cup \bar{\rho}\,(\overline{\mathcal{B}}(r)\setminus\mathcal{B}(r))$.

(i) It follows from Sard's Theorem (cf. Milnor (1965), page 10) that $\Sigma(\omega)$ has Legesgue measure zero in $\mathcal{F}$. Recall that $\mathcal{F}$ can be identified with $\mathbb{R}^{n-1}$. Since $\overline{\mathcal{B}}(r)\setminus\mathcal{B}(r)$ is stratified by a finite number of manifolds having dimension $\leqslant n-2$, $\bar{\rho}\,(\overline{\mathcal{B}}(r)\setminus\mathcal{B}(r))$ also has measure zero in $\mathcal{F}$, hence Σ' has measure zero. (ii) Notice that Σ' is the image by $\bar{\rho}$ of a closed subset of $\overline{\mathcal{B}}(r)$. Since $\overline{\mathcal{B}}(r)$ is compact, this closed subset is also compact, and Σ' is therefore compact. (iii) Define $\mathcal{R}(\omega) = \mathcal{F}\setminus\Sigma'$. We have established that $\mathcal{R}(\omega)$ is dense in $\mathcal{F}$.

We are ready to prove (4.2.1). Clearly, $\mathcal{R}(\omega)$ is contained in the set of regular values of $\bar{\rho}$ restricted to $\mathcal{B}(r)$. Take $\tau \in \mathcal{R}(\omega)$. Since τ is a regular value of the mapping $\bar{\rho} : \mathcal{B}(r) \to \mathcal{F}$, then $\bar{\rho}^{-1}(\tau) \cap \mathcal{B}(r)$ is discrete. Here, the mapping $\bar{\rho}$ is obviously defined on $\mathcal{B}(r)$. Then, either $\bar{\rho}^{-1}(\tau) \cap \mathcal{B}(r)$ is finite or $\rho^{-1}(\tau) \cap \mathcal{B}(r)$ is infinite. In the second case, $\bar{\rho}^{-1}(\tau) \cap \mathcal{B}(r)$ contains an infinite sequence $(b_\nu) \in \mathcal{B}(r)$. Since $\bar{\rho}^{-1}(\tau)$ is closed in $\overline{\mathcal{B}}(r)$, hence compact, the sequence (b_ν) has a subsequence converging to some $b \in \overline{\mathcal{B}}(r)$. We next show that this implies a contradiction. First, b cannot belong to $\mathcal{B}(r)$, otherwise the discreteness of $\bar{\rho}^{-1}(\tau) \cap \mathcal{B}(r)$ would be contradicted. Assume, therefore, that $b \in \overline{\mathcal{B}}(r) \setminus \mathcal{B}(r)$. Then, by the continuity of the mapping $\bar{\rho}$, we have $\tau = \bar{\rho}(b)$, which belongs to $\bar{\rho}\,(\overline{\mathcal{B}}(r)\setminus\mathcal{B}(r))$, hence to Σ', which is impossible by the definition of $\mathcal{R}(\omega)$. This finite number, however, may be equal to zero : it is sufficient to take $\tau \notin \bar{\rho}\,(\mathcal{B}(r)) = \mathcal{F}_B(\omega) \neq \mathcal{F}$.

Next, we establish (4.2.2), which is in fact an improvement of (4.2.1). The line of proof is the same as in Debreu (1970) or in

Balasko (1975) and is inspired by Milnor (1965), page 8. The only difference is that $\bar{\rho} : \mathcal{B}(r) \to \mathcal{F}$ is not necessarily proper. It is sufficient, however, for those proofs to be applied to our problem to show that if K is any compact subset of $\mathcal{R}(\omega)$, then $\bar{\rho}^{\,-1}(K) \cap \mathcal{B}(r)$ is also compact. Clearly, $\bar{\rho}^{\,-1}(K)$ is compact. Now, $\bar{\rho}^{\,-1}(K) \cap (\bar{\mathcal{B}}(r) \setminus \mathcal{B}(r))$ is empty, otherwise K would intersect $\bar{\rho}\,(\bar{\mathcal{B}}(r) \setminus \mathcal{B}(r))$, which is impossible by the definition of $\mathcal{R}(\omega)$. Combined with Milnor's argument, this remark implies (4.2.2). □

Let V be a connected component of $\mathcal{R}(\omega)$. Let k(V) be the number of equilibria associated with any $\tau \in$ V. Define $\mathcal{R}_{\geqslant 1}(\omega)$ as the union of those V for which $k(V) \geqq 1$. We obviously have that $\mathcal{R}_{\geqslant 1}(\omega) \subset \bar{\rho}\,(\mathcal{B}(r)) = \mathcal{F}_B(\omega)$. The next proposition provides a more complete description of the set of bonafide fiscal policies.

4.3. Proposition : The set $\mathcal{F}_B(\omega)$ of bonafide fiscal policies is the union of

$\mathcal{R}_{\geqslant 1}(\omega)$, an open subset of $\mathcal{F}$, and

$\bar{\rho}\,(\mathcal{B}(r)) \setminus \mathcal{R}_{\geqq 1}(\omega)$, a subset of $\mathcal{F}$, the closure of which has measure zero.

Proof : It results from Proposition (4.2) that $\bar{\rho}\,(\mathcal{B}(r)) \setminus \mathcal{R}_{\geqslant 1}(\omega)$ is a subset of $\Sigma' = \bar{\rho}\,(\bar{\mathcal{B}}(r) \setminus \mathcal{B}(r)) \cup \Sigma(\omega)$, closed with Lebesgue measure zero. □

V. *APPLICATIONS TO THE MONETARY ECONOMY*

The government might not have the ability to levy taxes denominated in units of account, but might be able instead to transfer money to some consumers and tax other consumers, also in terms of the monetary unit. In such an economy, the price of money (relative to commodities) is determined in the market.

Except for the "technology of taxes and transfers" the model is the same as that introduced in Section 2. There are ℓ commodities and n consumers. We continue to assume that ℓ and n are finite, even though this assumption may seem somewhat strange in the monetary context (cf. Cass and Shell (1980)). Let $\mu = (\mu_1, \ldots, \mu_i, \ldots, \mu_n)$ be the vector of *money transfers* (or, simply a *monetary policy*) in the space of feasible monetary policies $\mathbb{R}^n$. We choose the same normalization, $p_\ell = 1$, or the vector of commodity prices p is constrained to lie in $\mathcal{S} = \{p \in \mathbb{R}^\ell_{++} \mid p_\ell = 1\}$. The price of money in terms of the ℓth commodity is $p^m \geqq 0$. Therefore, we have

$$w_i = p.\omega_i + p^m \mu_i \quad \text{for } i = 1, \ldots, n. \tag{5.1}$$

Define $q = (p, p^m)$ and $\mathcal{Q} = \{(p, p^m) \mid p \in \mathcal{S} \text{ and } p^m \in \mathbb{R}_+\}$.

5.2. *Definition* : The vector $q = (p, p^m) \in \mathcal{Q}$ is said to be a *monetary competitive equilibrium* associated with $\omega \in (\mathbb{R}^\ell_{++})^n$ and $\mu \in \mathbb{R}^n$ if it satisfies the equations

$$\left.\begin{array}{l} \sum_{i=1}^{n} f_i(p, w_i) = \sum_{i=1}^{n} \omega_i = r \\[2em] w_i = p.\omega_i + p^m \mu_i \quad \text{for } i = 1, \ldots, n. \end{array}\right\}$$

A monetary competitive equilibrium $q = (p, p^m)$ is said to be *proper* if $p^m \neq 0$. The *set of monetary competitive equilibria* associated with (ω, μ) is denoted by $\mathcal{Q}(\omega, \mu) \subset \mathcal{Q} = \mathcal{S} \times \mathbb{R}_+$.

5.3. *Proposition* : The set $\mathcal{Q}(\omega, \mu)$ is nonempty.

Proof : Set $p^m = 0$. The economy then reduces to a standard pure-exchange economy for which equilibrium is assured (cf.. e.g., Arrow-Hahn (1971), Chapter 5). That is, there exists $p \in \mathcal{T}$ such that $(p, 0) \in \mathcal{Q}(\omega, \mu)$. □

5.4. Proposition (The "Neutrality" of Money) : For each positive scalar λ,

$$\mathcal{Q}(\omega, \lambda\mu) = \{(p, \; p^m/\lambda) \mid (p, p^m) \in \; \mathcal{Q}(\omega, \mu)\}.$$

Proof : From Definition (5.2), $w_i = p.\omega_i + p^m\mu_i$ for $i = 1, \ldots, n$. Since the only effect of μ or p^m on f_i is through w_i, the result follows immediately. □

By Proposition (5.3), a monetary competitive equilibrium always exists ; however, for given (ω, μ) all such equilibria may be trivial in the sense that $p^m = 0$, essentially the nonmonetary case since then $p^m\mu = 0$. If for given ω, the government chooses a monetary policy μ such that $\mathcal{Q}(\omega, \mu)$ contains no proper monetary competitive equilibria, then the government could not have a "*good-faith*" expectation that its monetary policy would affect the allocation of resources. This idea is formalized in the next definition.

5.5. Definition : The monetary policy μ is said to be *ω-bonafide* if there is a proper monetary equilibrium $q = (p, p^m) \in \mathcal{T} \times \mathbb{R}_{++}$ associated with (ω, μ). The ω-bonafide monetary policy μ is said to be *normalized* if $q = (p, 1)$ is a (proper) monetary competitive equilibrium associated with (ω, μ). Let $\mathcal{M}_B(\omega) \subset \mathbb{R}^n$ denote the *set of normalized ω-bonafide monetary policies.*

5.6. Remark : From Proposition (5.4), we know that the set of ω-bonafide monetary policies is the positive cone generated by $\mathcal{M}_B(\omega)$. The set $\mathcal{M}_B(\omega)$ is thus appropriately normalized.

5.7. Proposition : Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ be a monetary policy with the property that $\sum_{i=1}^{n} \mu_i \neq 0$. Then, there is no $\omega \in (\mathbb{R}^{\ell}_{++})^n$ such that μ is ω-bonafide.

Proof : The proposition is an immediate consequence of Lemma (2.3), where for τ_i we substitute $-p^m\mu_i$. □

5.8. Remark : Proper monetary equilibrium is thus consistent in the finite economy only with very special monetary policies, those in which taxes exactly equal transfers. In order to consider more interesting monetary policies, one must go beyond the finite model (cf. Cass and Shell (1980)) to infinite economies such as the overlapping-generations economy studied in Balasko-Shell (1981). That the balance of taxes and transfers permits proper monetary equilibrium in the finite model is neatly articulated in Starr (1974). Starr credits Lerner (1947) with the idea that the state can through its taxing power create fiat money which can bear a positive equilibrium price.

5.9. Proposition : The set $\mathcal{M}_B(\omega)$ of normalized ω-bonafide monetary policies is bounded, arcconnected, and contains 0 in its interior.

Proof : Let $\tau = (\tau_1, \ldots, \tau_n) = -\mu = (-\mu_1, \ldots, -\mu_n)$. The result follows immediately from Lemma (3.2) since for normalized ω-bonafide monetary policies, we set $p^m = 1$. □

Define the set of equilibrium money prices

$$\mathcal{T}^m(\omega, \mu) = \{p^m \mid (p, p^m) \in \mathcal{Q}(\omega, \mu)\} \subset \mathbb{R}_+ .$$

5.10. Proposition : Let $\mu \neq 0$ be a nontrivial ω-bonafide (not necessarily normalized) monetary policy. Then the set $\mathcal{T}^m(\omega, \mu)$ of equilibrium money prices is bounded. Furthermore, $0 \in \mathcal{T}^m(\omega, \mu)$

and 0 is not isolated in $\mathcal{T}^m(\omega, \mu)$.

Proof : Proposition (5.4) allows us to construct a natural bijection between $\mathcal{T}^m(\omega, \mu)$ and $\mathcal{M}_B(\omega) \cap \{\lambda\mu \mid \lambda \in \mathbb{R}_+\}$. Namely, map p^m from $\mathcal{T}^m(\omega, \mu)$ to $p^m\mu$ in $\mathcal{M}_B(\omega)$. Proposition (5.10) then follows from Proposition (5.9). □

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