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MONNAIE ET ALLOCATION INTERTEMPORELLE

par

Karl SHELL *

* Université de Pennsylvanie (Philadelphie) et CEPREMAP (Paris)

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MONEY AND INTERTEMPORAL RESOURCE ALLOCATION *

By

Karl Shell

University of Pennsylvania and CEPREMAP

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Résumé en Français

Lorsqu'on étudie la théorie monétaire et la macroéconomie,il est naturel d'utiliser des modèles où les générations d'agents se succèdent. Ce sont peut être même les seuls modèles où on puisse obtenir un prix positif pour la monnaie de façon naturelle. On peut distinguer la formulation "Samuelsonienne" où la contrainte du budget porte sur toute la durée de la vie, et la formulation "Keynésienne", où elle doit être satisfaite pour chaque période. Dans l'un et l'autre cas, l'équilibre concurrentiel avec prévision parfaite peut ne pas être Pareto-optimal.

Dans un modèle "Samuelsonien" stationnaire où la durée de vie est de deux périodes et où il n'y a qu'un seul "type" de consommateur, on est assuré de l'existence d'un équilibre et il est possible d'obtenir une allocation Pareto-optimale avec une politique monétaire passive. On donne des exemples qui contredisent la version habituelle de la théorie quantitative de la monnaie. Dans un modèle avec anticipations rationnelles, la politique monétaire optimale doit être active, et les politiques monétaires à variance nulle ne sont pas optimales.

On donne enfin un exemple d'économie "Samuelsonienne" non stationnaire. Même si les agents sont reliés par la production ("productively related"), il n'existe pas d'équilibre concurrentiel où la monnaie ait un prix non négatif.

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* Les idées ci-après ont été développées en collaboration avec mon collègue David CASS, qui ne peut cependant être tenu pour responsable de la présente note du fait qu'un océan nous sépare actuellement. Je tiens à remercier le CEPREMAP (et particulièrement Jean-Michel GRANDMONT) pour l'hospitalité qui m'a été offerte. Je remercie également la Fondation Guggenheim et le CNRS pour l'aide qu'ils m'ont apportée à l'occasion de cette recherche.

The infinite - horizon overlapping - generations model of Samuelson (1958) deserves closer study.Only in the context of this model can we explain in a natural way the role of fiat money as a potential store of value, i.e., allowing for fiat money with a positive price. The "store-of-value" role may not be the most important function of money, but if money does not serve this function, it is difficult to imagine how it can serve any other function - e.g., economizing on transactions costs.

One of the most influential of the recent contributions to the theory of monetary policy (Lucas (1972)) is indeed based on the overlapping-generations model. From the perspective of the "consumption-loan" model, however, the policy implications drawn by Lucas seem to be counterintuitive. A short-run goal of the present research is to re-evaluate Lucas's conclusions in the light of more general analysis. The long-run goal is a more positive one, viz., the development of a carefully articulated model of overlapping generations, which can serve as a basis for monetary and macroeconomic analysis.

This project has engaged my time since arriving in Paris in September. We (Cass and I) do not as yet have general theorems, but we do have examples (and counterexamples) which I find interesting. Even so, the analysis reported here is based upon the special cas of perfect foresight (or, more generally, "rational expectations"). Models with only life-time budget constraints (the "Samuelsonian" models) are treated separately from models with period-by-period budget constraints (the "Keynesian" models). The "Keynesian" models are probably more appropriate for macroeconomic analysis. From that long-run perspective, therefore, this report may place undue emphasis on the "Samuelsonian" formulation.

A "Samuelsonian" model with two-period lives. There are two commodities : one, a completely perishable physical good (subscripted by the period of its availability) ; the other, an imperishable fiat money. In each period, $t = 1, 2, \ldots$ an individual (alternatively n like individuals) is born. (There is no population growth). We give him for a name his birth date, t . He receives endowments of physical goods and fiat money in each of his two periods of life, t and $t + 1$. In the first period, individual 0 is also alive. He lives only during that period,

$t = 1$. $u^i(\cdot)$ is the utility function of individual i, $i = 0, 1, \ldots$. x_t^i is consumption of the physical good by individual i in period t, $i = 0, 1, \ldots$; $t = 1, 2, \ldots$.

Tastes :

$$u^0(x_1^0) \qquad x_1^0 \in R_+$$

strictly increasing function

$$t = 1,2,\ldots \qquad \begin{cases} u^t(x_t^t, x_{t+1}^t) & (x_t^t, x_{t+1}^t) \in R_+^2 \\ u^t(\cdot) & \text{nondecreasing, semi-strictly quasi-concave function.} \end{cases}$$

Endowments : ω_t^i is individual i's endowment of physical commodity in period t . ω_{mt}^i is individual i's (not necessarily nonnegative) endowment of fiat money in period t .

$$\omega^0 = \omega_1^0 \qquad \omega^0 \in R_+$$

$$\omega_m^0 = \omega_{m1}^0 \qquad \omega_m^0 \in R$$

$$t = 1,2,\ldots \qquad \begin{cases} \omega^t = (\omega_t^t, \omega_{t+1}^t) \geq 0 & \omega^t \in R_+^2 \\ \omega_m^t = (\omega_{mt}^t, \omega_{mt+1}^t & \omega_m^t \in R^2 \end{cases}$$

Aggregates (t = 1, 2, ...) :

$$x_t \equiv x_t^t + x_t^{t-1} > 0$$

$$\omega_t \equiv \omega_t^t + \omega_t^{t-1} > 0$$

$$\omega_{mt} \equiv \omega_{mt}^t + \omega_{mt}^{t-1}$$

$$\equiv m_t^t + m_t^{t-1} = \sum_{s=1}^{s=t} \omega_{ms} \geq 0$$

where m_t^i is fiat money held by individual i in period t. The requirement (above) that aggregate money supply be nonnegative implies that in no period is the government a net creditor. Thus, even if there is government control over fiat money endowments, ω^0, $\omega^1, \ldots$, the model does not allow for all lump-sum takes and transfers. Even though some individual monetary endowments may be negative (i.e., monetary taxes as well as transfers may be levied), taxes are restricted by the natural constraint, $m_t \geq 0$.

The above formulation also makes money a nondisposable good. This constraint will be binding in an interpretation of one of our models, where the equilibrium price of money must be negative.

Let p_t, $t = 1,2, \ldots$, be the price of physical commodity in period t In the examples studied here it will be possible to interpret this price as a <u>present price</u> since we can legitimately set $p_1 \equiv 1$. Let p_{mt} be the present price (in terms of period 1 physical good) of money in period t. In the "Samuelson model" households face life-time <u>constraints</u> ($t = 1, 2, \ldots$) :

$$p_1x_1^0 + p_{m1}x_{m1}^0 \leqq p_1\omega_1^0 + p_{m1}\ {}_{m1}^{0}$$

$$m_1^1 = x_{m1}^0 \geqq 0$$

$$+ p_{t+1}x_{t+1}^t + p_{mt}x_{mt}^t + p_{mt+1}x_{mt+1}^t \leqq p_t\omega_t^t + p_{t+1}\omega_{t+1}^t + p_{mt}\omega_{mt}^t + p_{mt+1}\omega_{mt+1}^t$$

$$m_{t+1}^t = m_t^t + x_{mt+1}^t = x_{mt}^t + x_{mt+1}^t \geqq 0$$

x_{mt}^i is the net addition to money balances made by individual i in period t ($i = 0,1, \ldots$; $t = 1,2,\ldots$). <u>Market clearance (materials balance) for the physical commodity ($t = 1,2,\ldots$)</u> :

$$x_t \leqq \omega_t$$

Because of the form of the ("Samuelson") budget constraint, monetary transfers and taxes can be restricted to the last period of the individual's life without loss of generality.

$$(\omega^t_{mt}\ ,\ \omega^t_{mt+1}) = (0,\ \Delta m_{t+1}) \qquad t = 1,2,...$$

$$\omega^0_m = m_1 + \Delta m_1$$

so that

$$m_{t+1} = m_t + \Delta m_t = m^t_{t+1} + m^{t+1}_{t+1} \qquad t = 1,2,...$$

reflecting balance in the market for money. Combining the budget constraint with this special form of monetary policy yields

$$\begin{cases} p_t(x^t_t - \omega^t_t) + p_{t+1}(x^t_{t+1} - \omega^t_{t+1})\ p_{mt}x^t_{mt} + p_{mt+1}(x^t_{mt+1} - \Delta m_{t+1}) \leqq 0\ , \\ x^t_{mt} + x^t_{mt+1} \geqq 0 \qquad \text{for } t = 1,2,... \end{cases}$$

and

$$\begin{cases} p_1(x^0_1 - \omega^0_1) + p_{m1}(x^0_{m1} - \Delta m_i - m_1) \leqq 0 \\ x^0_{m1} \geqq 0 \end{cases}$$

Remember we have assumed that in each period there is at least one individual with positive wealth who is not satiated in the physical commodity for that period. We then have the following :

<u>Proposition 1</u> :

(a) $\quad p_t \in (0,\infty)\ ,\ t = 1,2,...$

(b) $\quad p_{mt} = p_m$ (a constant scalar), $t = 1,2,...$

(c) $$p_m = \frac{x^0_1 - \omega^0_1}{m_1 + \Delta m_1}$$

Justification :

$p_t = 0$ => excess demand for the physical good in period t .

$p_{mt} > p_{mt+1}$ => excess supply of money in period t

$p_{mt} < p_{mt+1}$ => excess demand for money in period t .

(c) (i) For $p_m > 0$, and $m_1 + \Delta m_1 > 0$,

utility maximization by individual zero implies that his terminal money balance is zero , $x^0_{m1} = 0$. (ii) For $p_m = 0$ or $m_1 + \Delta m_1 = 0$, $x^0_1 = \omega^0_1$. (iii) For $p_m < 0$ and $m_1 + \Delta m_1 > 0$, we impose the nondisposability constraint, implying that $x^0_{m1} = 0$.

Therefore, we can write

$$(x^0_1 - \omega^0_1) = p_m(m_1 + \Delta m_1)$$

and

$$p_t(x^t_t - \omega^t_t) + p_{t+1}(x^t_{t+1} - \omega^t_{t+1}) = p_m \Delta m_{t+1} , \quad t = 1,2, \ldots..$$

In order to be able to calculate optimal monetary rules very simply we specialize the model even further. The simplification is not essential to the general arguments that follow, although, the precise form of the results do depend on the assumptions which one makes.

Special case : linear, stationary model (Shell, 1971) :

$$u^0(x^0_1) = x^0_1$$

$$\omega^0 = \omega^0_1 = 1$$

$$\left.\begin{aligned} u^t(x^t_t , x^t_{t+1}) &= x^t_t + x^t_{t+1} \\ \omega^t = (\omega^t_t , \omega^t_{t+1}) &= (1,1) \end{aligned}\right\} \quad t = 1,2,$$

Definition :

Monetary policy is said to be $\left\{\begin{matrix}\text{expansive}\\ \text{passive}\\ \text{contractionary}\end{matrix}\right\}$ in period t

if $\left\{\begin{matrix}\Delta m_t > 0\\ \Delta m_t = 0\\ \Delta m_t < 0\end{matrix}\right\}$

Proposition 2 : In the linear, stationary "Samuelson" model, if monetary policy is either expansive or passive in period $t+1$, then $p_{t+1} = p_t = 1$, $t = 1,2,\ldots.$

Justification

$$p_t x_t^t + p_{t+1} x_{t+1}^t = p_t \cdot 1 + p_{t+1} \cdot 1 + p_m \Delta m_{t+1}$$

If, for example, $p_{t+1} > p_t$, then there is excess demand for the physical commodity in period t , because from the linearity of indifference curves we have that

$$x_t^t > 2 + p_m \Delta m_{t+1} > 2 = \omega_t \ (\geqq x_t \geqq x_t^t).$$

Proposition 3 : In the linear, stationary "Samuelson" model : (a) Autarky is an (inefficient) competitive equilibrium with $p_t = 1$, $t = 1,2,\ldots$, and either $p_m = 0$ or $m_t = 0$, $t = 1,2, \ldots.$. (b) The price system $p_t = 1$, $t = 1,2,\ldots$ and $p_m \geqq 0$ are consistent with competitive equilibrium iff $0 \leqq p_m m_t \leqq 1$, $t = 1,2,\ldots.$. For noncontractionary monetary policy ($\Delta m_t \geqq 0$), such prices characterize the set of competitive allocations. (c) Such competitive equilibria are Pareto-optimal iff $\lim_{t \to \infty} p_m m_t = 1$. (d) In particular, $p_m = 1$, $m_t = 1$, $t = 1,2, \ldots$, characterizes the unique efficient "stationary" equilibrium, i.e., the unique Pareto-optimal allocation with passive monetary policy.

Justification : Relative to autarky, an imposed allocation can increase total consumption of the physical good (summed over all individuals and all time periods of at most one unit.

Applications to monetary policy :

(a) Monetary policy is nonneutral. Even with perfect foresight, real allocations are affected by monetary policy.

(b) Some monetary equilibria with passive monetary policy ($\Delta m_t = 0$, $t = 1,2,\ldots$) are Pareto-optimal but others are not.

(c) Some monetary equilibria with active monetary policy ($\Delta m_t \neq 0$ for some $t = 1,2, \ldots$) are Pareto-optimal. Others are not.

(d) The quantity theory of money (as usually applied) does not necessarily obtain. Efficient and inefficient equilibria can be found in which for some t (=1,2,...) and some s (=1,2,...)

$$\frac{p_m m_t}{p_t} \neq \frac{p_m m_s}{p_s}$$

A further application of the linear, stationary model : Rational Expectations. Agents base their expectations about the general price level (p_t/p_{mt}) on their expectations as to the presence of sunspots. There is no instrinsic uncertainty in the model. Are agents irrational ? No. Although uncertainty is not intrinsic, a single agent rejecting the "sunspot theory" would be injured.

Assume, given the data available in period t , that all agents expect $p_{t+1} = 1$ with certainty. Assume further that expectations about the price of money, p_{mt} , are determined by the following stationary stochastic process

Prob { sunspots in period t+1 = 1/2

and

Prob { no sunspots in period t+1 = 1/2

Let $\sigma_t = \alpha_t \min (p_{mt}, 1 - p_{mt})$ $\alpha_t \in [0,1]$

Because of risk-neutrality the following is a "rational expectations process."

$$= p_{mt} + \sigma_t \qquad \text{if sunspots in period } t+1$$

$$= p_{mt} - \sigma_t \qquad \text{if no sunspots in period } t+1$$

Monetary policy yields (<u>ex post</u>) efficient allocation iff $\lim_{t \to \infty} p_{mt} m_t$

<u>Policy implications</u> (1) Passive (zero-variance) monetary policy ($\Delta m_t = 0$, $t=1,2$ is in general inefficient. Efficient monetary policy is likely to be active ($\Delta m_t \neq 0$ for some $t = 1,2,\ldots$) . Var $\{\Delta m_t\}$ is likely to be positive for efficient monetary policy. (2) In monetary models, the <u>future creates its own uncertainty</u>. Even without "underlying" uncertainty, rational expectations are consistent with random price behavior.

It should be remarked that the above results are derived far more simply for the Keynesian model.

<u>Passive monetary policy</u>. The existing literature (see, e.g., SAMUELSON (1958), SHELL (1971), GALE (1973), and OKUNO-ZILCHA (1977)) focuses on the case where $\Delta m_t = 0$, $t = 1,2\ldots$. Without loss of generality set $m_t = m_1 = 1$, $t = 1,2,\ldots$. We have the following result for the linear, stationary "Samuelsonian" model which extends to the nonlinear cases if stationarity and generation structure are preserved.

<u>Proposition 4</u> : Consider the stationary "Samuelsonian" model (above) in which individuals are (directly or indirectly) productively related. Restrict monetary policy to be passive. Competitive equilibria exist for $p_m = 0$ and possibly some $p_m > 0$. The competitive allocation is Pareto-optimal iff p_m takes on its maximum value over all such equilibria.

Justification (for the linear case) : The set of equilibria is parametrized by $p_m \in [0,1]$. For each equilibrium allocation, $u^t = x^t_{t+1} = 2$ $(t = 1,2,\ldots)$, but $u^0 = x^0_1 = 1 + p_m \in [1,2]$

Can Proposition 4 be extended to more general cases, e.g., to nonstationary cases ? Hosts of "counterexamples" are being put together suggesting that restricting of our attention to passive monetary policy would be a mistake.

(1) I present an example (below) in which individuals are productively related but in which no passive-monetary-policy equilibrium exists with $p_m \geqq 0$.
(2) There are now unpublished examples (due to Cass and Okuno-Zilcha) which yield passive-monetary-policy equilibria with $p_m \geqq 0$, none of which are Pareto-optimal.
(3) In another example with passive monetary policy, there exists an equilibrium with $p_m = 0$ and another with $p_m > 0$, both of which are Pareto-optimal.

Because of the limit on time, I shall present only the first "counter-example.

"Counterexample" 1

Endowments and life-spans

Individual \ Period	1	2	3	4	5	6	7
0	1	0					
1	0	1	1/2				
2		0	1/2	0			
3			0	1	1/2		
4				0	1/2	0	
5					0	1	1/2
6						0	1/2
7							0

$$\omega^0 = (\omega^0_1, \omega^0_2) = (1,0)$$

$$\omega^t = (\omega^t_t, \omega^t_{t+1}, \omega^t_{t+2}) = (0,1,1/2) \qquad t = 1,3,5,.$$

$$\omega^t = (\omega^t_t, \omega^t_{t+1}, \omega^t_{t+2}) = (0,1/2,0) \qquad t = 2,4,6,.$$

Tastes ("The grass is always greener....") :

$$u^0 = x_2^0$$

$$u^t = x_t^t \qquad t = 1,3,5,..$$

$$u^t = x_t^t + x_{t+2}^t \qquad t = 2,4,6,...$$

Monetary policy is passive, $m_t = 1$, $t = 1,2,...$. The following matrix displays "candidate allocations", allocations satisfying materials balance for the physical commodity and reflecting the basic structure of preferences.

Candidate Allocations

Individual \ Period	1	2	3	4	5	6	7
0	–	θ_2^0					
1	1	–	–				
2		$1-\theta_2^0$	–	θ_4^2			
3			1	–	–		
4				$1-\theta_4^2$	–	θ_6^4	
5					1	–	–
6						$1-\theta_6^4$	–
7							1

where $\theta_t^{t-2} \in [0,1] \qquad t = 2,4,6,...$

$$p_t \in (0,\infty)$$

and, therefore, we can set $p_1 \equiv 1$.

Budget constraints for $i = 0, 1$ yield

$$p_2\theta_2^0 = 1 \cdot 1 + p_m \cdot 1 \qquad \text{or} \qquad p_2\theta_2^0 = 1 + p_m$$

and

$$1 \cdot 1 = p_2 \cdot 1 + p_3 \cdot 1/2 \qquad \text{or} \qquad 1 = p_2 + 1/2\, p_3$$

Since $1 > p_2 > 0$ and $0 \leqq \Theta_2^0 \leqq 1$, we conclude that necessarily

$$p_m < 0 .$$

From this we conclude that there is no non-monetary equilibrium and no equilibrium with nonnegatively priced money. If negative prices are allowed for money (money is nondisposable), then equilibrium with passive monetary policy is possible. One efficient equilibrium is given by

$$\begin{cases} p_t = 1 & t = 1,3,5, \ldots. \\ p_t = 1/2 & t = 2,4,6, \ldots. \\ \Theta_t^{t-2} = 1 & t = 2,4,6, \ldots. \\ p_m = -1/2 & \end{cases}$$

One inefficient equilibrium is given by

$$\begin{cases} p_t = 1 & t = 1,3,5, \\ p_t = 1/2 & t = 2,4,6, \\ \Theta_t^{t-2} = 1/2 & t = 2,4,6, \\ p_m = -3/4 & \end{cases}$$

<u>The "Keynesian" model</u>. From the viewpoint of macroeconomics, this is the more important model. We return to the two-period , linear stationary model in which monetary transfers are made solely during the last period of life. Let P_{mt} be the current price of money (the inverse of the general price level) in period t ,

$$P_{mt} = p_{mt}/p_t \qquad t = 1,2,$$

Individuals face period-by-period liquidity constraints :

$$x_1^0 \leqq 1 + P_{m1}\, m_1 + P_{m1}\, \Delta m_1$$

$$x_t^t + P_{mt}\, x_{mt} \leqq 1$$

$$\begin{aligned} x_{t+1}^t &\leqq 1 + P_{mt+1}\, x_{mt} + P_{mt+1}\, \Delta m_{t+1} \\ &= 1 + P_{mt+1}\, m_{t+1} + P_{mt+1}\, \Delta m_{t+1} \\ &= 1 + P_{mt+1}(m_{t+1} + \Delta m_{t+1}) \end{aligned}$$

<u>Proposition 5</u> : In the linear, stationary "Keynesian" model (a) $m_t > 0$ and $P_{mt} > 0 \Rightarrow P_{mt} \geqq P_{mt+1}$ and (b) $\Delta m_{t+1} \geqq 0$ (noncontractionary monetary policy) $\Rightarrow P_{mt+1} = P_{mt}$

<u>Justification</u> : (a) Otherwise the demand for money would be zero in period t, yielding excess supply at a positive price. (b) Otherwise $x_{t+1}^t > 2 = \omega_{t+1}$

How does the "Keynesian" model differ from the "Samuelsonian" model ? This question requires work, but some preliminary observations can be made. Existence of equilibrium seems to be easier to establish in the "Keynesian" model. Optimality of equilibrium is, however, more subtle in the "Keynesian" case because money then serves two roles : (1) relieving liquidity constraints, as well as (2) bringing physical goods forward from the infinite future.

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