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Market Uncertainty: Correlated and Sunspot Equilibria in Imperfectly Competitive Economies

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An imperfectly competitive economy is very prone to *market uncertainty*, including uncertainty about the liquidity (or "thickness") of markets. We show, in particular, that there exist stochastic equilibrium outcomes in nonstochastic market games if (and only if) the endowments are not Pareto optimal. We also provide a link between extrinsic uncertainty arising in games (e.g. *correlated equilibria*) and extrinsic uncertainty in market economies (e.g. *sunspot equilibria*). A correlated equilibria to the market game is either a sunspot equilibrium or a non-sunspot equilibrium to the related securities games, but the converse is not true in general.

1. INTRODUCTION AND SUMMARY

The usual general equilibrium model can be used to explain how randomness in endowments, preferences, and technology is *transmitted* through the economy to make economic outcomes random. This is not the only source of economic volatility. Some economic uncertainty is generated by the market process itself. We refer to this other type of economic uncertainty as *market uncertainty*. The "sunspot-equilibrium" notion introduced by Cass and Shell (see Shell (1977) and Cass and Shell (1983)) provides a formal theory of market uncertainty. From the sunspots literature, we know that in rational-expectations models the competitive-equilibrium allocation of resources can be random, even if the economic fundamentals are immune from random disturbances.

In the present paper, we extend the formal analysis of market uncertainty to economies in which competition is not perfect. The scope of the effects of market uncertainty is richer and of greater policy interest in the non-competitive environment. In this environment, economic actors are uncertain as to whether markets will be thick or thin (i.e. liquid or illiquid), or even whether these markets will be open or closed. An economic actor must forecast the behaviour of the other economic actors in order to forecast market liquidity. A wide range of beliefs about market liquidity is rational. If, for example, demand in a particular market (or overall) is weak, then in response, supply is weak.

We build on the familiar market game of Shapley and Shubik (e.g. Shubik (1973), Shapley and Shubik (1977), Shapley (1976), Mas-Colell (1982), Peck and Shell (1985), and Peck, Shell, and Spear (1989)). The *market-game* model provides the common stage[1] for comparing two leading equilibrium concepts in which extrinsic uncertainty plays an important role: (a) sunspot equilibrium (which was first developed in the context of competitive *market* economies), and (*b*) correlated equilibrium (which was first developed in the context of matrix *games*).[2]

In our formal analysis, we follow the "sunspots" tradition in assuming that uncertainty is purely *extrinsic*,[3] but we invite the reader to imagine at times that this represents the polar case of *intrinsic* uncertainty with relatively small effects on the fundamentals and relatively large effects on economic outcomes.[4] We formally follow the "sunspots" tradition in assuming that market uncertainty is *exogenous* to the economy, i.e. that it is adopted from outside the economy and serves to coordinate the plans of the individual market participants. In some cases, however, it may be possible to dispense with this assumption and then the market uncertainty could be considered *endogenous* to the economy.

In Section 2, we present the (certainty) market game Γ. We also present some results about pure-strategy Nash equilibrium in Γ, which results are used in the succeeding sections. In Section 3, we apply the notion of correlated equilibrium to the game Γ. Strategies are based on an extrinsic, exogenous randomizing device which is outside the rules of Γ. Asymmetric information is permitted.

In Section 3, we also discuss the importance of Aumann's (1987) equivalence of correlated equilibrium and Bayes-rational equilibrium in the context of market games. In the Bayes-rational-equilibrium model, the only uncertainty that a consumer faces is uncertainty about what moves the other consumers will actually make. This uncertainty is obviously extrinsic. It might also be considered to be endogenous in the sense that an exogenous device generating the correlated signals need not be identified. A consumer might know how he gets his signal from the economic, physical, and cultural environment, and he might know how his signal is correlated with the actions of others, even though he does not know the source of the signals of the others.

If uncertainty is exogenous, it can be observed and hence be incorporated into the rules of the game. As part of the rules of the game, it is natural to include markets which permit hedging of risks across the states of nature (as is done in competitive equilibrium models: see Arrow (1964) and Cass and Shell (1983)). The existence of these markets can alter the set of equilibrium allocations. In Section 4, we define the securities game $\hat{\Gamma}(P)$, an extension of the game Γ to allow for transfer of incomes across the extrinsic states of nature over which the randomizing device P is defined. Every correlated equilibrium allocation to the market game Γ based on the randomizing device P is also a pure-strategy Nash equilibrium allocation to the securities game $\hat{\Gamma}(P)$. We also show in Section 4 that proper sunspot equilibria in the securities game are not flukes: if and only if endowments are not Pareto optimal in Γ, there is a P so that the securities game $\hat{\Gamma}(P)$ has a pure strategy Nash equilibrium in which the allocation of resources is affected

1. See Peck and Shell (1985) for the earliest such work. We had to choose a concrete example for comparison of sunspot equilibrium and correlated equilibrium. The market game is probably the best stage for our analysis, but our basic results should also apply to other *general-equilibrium* models of imperfect competition.

2. See (a) Shell (1977) and Cass and Shell (1983), and (b) Aumann (1974, 1987).

3. For the definitions of intrinsic uncertainty and extrinsic uncertainty, see Cass and Shell (1983), Section II, p. 196.

4. See Manuelli and Peck (1988).

by the outcome of the randomization device P. Hence the existence of sunspot equilibrium is *generic*. (Nonsunspot equilibria, however, *always* exist: for every $\hat{\Gamma}(P)$, there is a pure-strategy Nash equilibrium in which the randomization does not affect the allocation.)

In Section 5, we provide two examples of equilibria which are not mere randomizations over pure-strategy Nash equilibria to Γ. The first example is based on the transfer of incomes across states of nature. It establishes that not every pure-strategy Nash equilibrium to $\hat{\Gamma}(P)$ can be achieved as a correlated equilibrium to Γ. That is, at least for some market games the set of correlated equilibrium allocations is a *proper* subset of the pure-strategy Nash equilibria to the corresponding securities games. The second example is driven by asymmetric information. We display a correlated equilibrium to a market game. The equilibrium allocation is neither a randomization over pure-strategy Nash equilibrium allocations nor is it a mixed-strategy Nash equilibrium allocation.

In Section 6, we conclude with remarks about replication of the market and securities games to achieve "large economy" results. We relate our limiting symmetric-information solutions to competitive equilibrium in economies with incomplete markets. We *contrast* our limiting asymmetric-information solutions to competitive-equilibrium outcomes.

2. PURE CERTAINTY: THE MARKET GAME Γ

There are l consumption goods and inside money. There are n consumers. Consumer h is endowed with a positive amount of commodity i, ω_h^i, for $i=1,\dots,l$. If we denote by ω_h the endowment vector $(\omega_h^1,\dots,\omega_h^i,\dots,\omega_h^l)$, then we have $\omega_h \in \mathscr{R}_{++}^l$ for $h=1,\dots,n$.

For each commodity, there is a single trading post on which the commodity is exchanged for money. Consumer h supplies a non-negative quantity of commodity i, q_h^i, at trading post i. He also supplies a non-negative quantity of money, b_h^i, at trading post i. We say that q_h^i is his *offer* (of commodity i) and that b_h^i is his (money) *bid* (for commodity i). Let $b_h=(b_h^1,\dots,b_h^i,\dots,b_h^l)$ and $q_h=(q_h^1,\dots,q_h^i,\dots,q_h^l)$ denote (respectively) his *bids* and his *offers*. Offers must be made in terms of physical commodities. Hence offers cannot exceed endowments, i.e. we have $q_h^i \leqq \omega_h^i$ for $i=1,\dots,l$. The *strategy set* S_h of consumer h is then given by $S_h \doteq \{(b_h, q_h)\in \mathscr{R}_+^{2l} \mid q_h \leqq \omega_h\}$.

The trading process is simple. The total amount of commodity i which is offered, $\sum_{k=1}^{k=n} q_k^i$, is allocated to consumers in proportion to their shares of the bids for commodity i. Consumer h's share of the bids at post i is $(b_h^i/\sum_{k=1}^{k=n} b_k^i)$. Thus the gross receipts of commodity i for consumer h are $[(b_h^i)/(\sum_{k=1}^{k=n} b_k^i)](\sum_{k=1}^{k=n} q_k^i)$ for $i=1,\dots,l$ and $h=1,\dots,n$. Similarly, consumer h's share of the offers at post i is $(q_h^i/\sum_{k=1}^{k=h} q_k^i)$, and his gross receipts of money from post i are $[(q_h^i)/(\sum_{k=1}^{k=n} q_k^i)](\sum_{k=1}^{k=n} b_k^i)$ for $i=1,\dots,l$ and $h=1,\dots,n$.[5] Each consumer faces a single overall budget constraint which he must meet or be punished. He is required to finance his bids (for commodities) by his offers (of commodities). The *budget constraint* for consumer h is

$$\sum_{j=1}^{j=l} \{(q_h^j/\sum_{k=1}^{k=n} q_k^j)\sum_{k=1}^{k=n} b_k^j\} \geqq \sum_{j=1}^{j=l} b_h^j, \tag{2.1}$$

for $h=1,\dots,n$.

Let x_h^i denote the consumption of commodity i by consumer h, and let $x_h=(x_h^1,\dots,x_h^i,\dots,x_h^l)$ be his consumption vector. Assume that consumer k chooses the

5. It is assumed that (1) if there are no bids on a post, offers on this post are confiscated and (2) if there are no offers on the post, bids are confiscated. This rule is simply expressed by the convention $0/0=0$ in the systems (2.1) and (2.2) below.

strategy $(b_k, q_k) \in \mathcal{R}_+^{2l}$ for $k=1, \ldots, n$; then the consumption of consumer h is given by

$$x_h^i = \omega_h^i - q_h^i + [(b_h^i)/\textstyle\sum_{k=1}^{k=n} b_k^i](\textstyle\sum_{k=1}^{k=n} q_k^i) \quad \text{if (2.1) is satisfied}$$

and (2.2)

$$x_h^i = 0 \quad \text{if (2.1) is not satisfied}$$

for $i=1, \ldots, l$ and $h=1, \ldots, n$. (Failure to meet budget constraint (2.1) leads to confiscation of all the consumer's goods.)

The consumption set of consumer h is the nonnegative orthant $\{x_h \mid x_h \in \mathcal{R}_+^l\}$. His utility function, u_h, is strictly increasing, smooth, and strictly concave on the strictly positive orthant $\mathcal{R}_{++}^l$. Also, the closure in $\mathcal{R}^l$ of each indifference surface from $\mathcal{R}_{++}^l$ is contained in $\mathcal{R}_{++}^l$. (This last assumption allows us to avoid some boundary solutions.) The boundary of the consumption set, $(\mathcal{R}_+^l \backslash \mathcal{R}_{++}^l)$, is also the indifference surface of least utility.

We have specified *strategy sets* S_h, the *outcomes* x_h (through Equation (2.2)), and the *payoffs* $u_h(x_h)$ for the market game Γ. Let $\sigma_h = (b_h, q_h)$ be a strategy in S_h. Define the set S by $S = S_1 \times \cdots S_h \times \cdots \times S_n \subset (\mathcal{R}_+^{2l})^n$. Define the strategy vector by $\sigma \in S$ by $\sigma = (\sigma_1, \ldots, \sigma_h, \ldots, \sigma_n)$. From Equation (2.2), we see that x_h^i is a function of the b's and q's so that the outcome can be written as a function of the strategies σ, namely $x_h(\sigma)$ for $h=1, \ldots, n$ and $x(\sigma) = (x_1(\sigma), \ldots, x_h(\sigma), \ldots, x_n(\sigma))$. We adopt the standard concept of (pure-strategy) *Nash equilibrium.*

The game Γ exhibits individual rationality, since the trivial strategy $\sigma_h = (b_h, q_h) = (0, 0)$ is in S_h. Furthermore, if every consumer plays the trivial strategy, this is a Nash equilibrium. The corresponding allocation is autarky. Next we provide formal definitions of a *closed market* and an *open market.*

Definition 2.3. Let $\sigma = ((b_1, q_1), \ldots, (b_h, q_h), \ldots, (b_n, q_n))$ be a Nash equilibrium vector of strategies in the market game Γ. We say that market i is *closed* (resp. *open*) if $\sum_{k=1}^{k=n} b_k^i = 0$ (resp. $\sum_{k=1}^{k=n} b_k^i > 0$). It follows immediately that market i is closed (resp. open) if and only if $\sum_{k=1}^{k=n} q_k^i = 0$ (resp. $\sum_{k=1}^{k=n} q_k^i > 0$).

Next we define an *interior* Nash equilibrium strategy vector and then we report its basic existence and welfare properties.

Definition 2.4. The strategy $\sigma = \{(b_h, q_h)\}_{h=1}^{h=n} \in S$ is said to be an *interior Nash equilibrium* to the market game Γ if σ is a Nash equilibrium for Γ and σ is strictly positive, i.e. $\sigma \in \mathcal{R}_{++}^{ln}$, so that each of the l markets is open. The corresponding allocation $x(\sigma) \in \mathcal{R}_{++}^{ln}$ is called an *interior Nash equilibrium allocation* of Γ.

Proposition 2.5. *An interior Nash equilibrium allocation of* Γ *is autarkic if and only if the endowment vector* ω *is Pareto optimal. Furthermore, if* ω *is not Pareto optimal, an interior Nash equilibrium allocation of* $\Gamma, x = (x_1, \ldots, x_h, \ldots, x_n)$, *must satisfy* $u_h(x_h) \geqq u_h(\omega_h)$, *with strict inequality for at least one h,* $h = 1, \ldots n$.

Proof. See Peck and Shell (1985, Proposition (2.20)) and Peck, Shell, and Spear (1989, Proposition (2.9)). ‖

Next we cite a central result for pure strategy Nash equilibrium in market games. This result will be used to establish that generically market games possess equilibra in which the strategies and outcomes are affected by extrinsic uncertainty.

Proposition 2.6. *There is always an interior Nash equilibrium strategy* $\sigma = \{(b_h, q_h)\}_{h=1}^{h=n} \in S$ *for the market game* Γ.

Proof. See the proof of Proposition (2.23), pp. 18–31 in Peck and Shell (1985). (Also see the proof of Theorem (4.10) in Peck, Shell, and Spear (1989)). ||

In strategy space, there are ln dimensions of indeterminacy to the interior pure-strategy Nash equilibria for the market game Γ. (See Peck, Shell, and Spear (1989, Proposition (2.12)).) It is natural then to take the vector of offers (and its components) as measuring "market thickness", "market confidence", or even "market liquidity". Large q's imply that markets are thick, allowing the potential for large net trades. In this sense, markets are liquid. Large q's also signify that market participants are confident in that sellers expect buyers to materialize for what they are offering. On the other hand, small q's mean that markets are thin with (at most) small net trades. In this sense, markets are illiquid. With small q's, we can think that participants lack confidence in the sense that it is expected that effective demand for commodities will be weak.

This section contains the analysis of (pure-strategy) Nash equilibrium to the simple market game Γ. In succeeding sections, we expand the analysis to allow for the effects of extrinsic uncertainty. First, we extend the solution concept to correlated equilibrium, but do not alter the game. Later, we extend the game by adding sunspot-contingent markets, but use the simple solution concept of (pure-strategy) Nash equilibrium. Then we relate the results from the analysis of the more complicated solution concept applied to the simple game with the results from the analysis of the simpler solution concept applied to the more complicated game.

3. EXTRINSIC UNCERTAINTY: CORRELATED EQUILIBRIUM IN THE MARKET GAME

The game Γ is unchanged, but a more general solution concept is analyzed. Purely extrinsic uncertainty is introduced. The fundamentals of the economy—here tastes and endowments—are unaffected by the random variable; think of it as the level of "sunspot activity". The set of states of nature is $\Omega = \{1, \dots, s, \dots, r\}$. In observing sunspot activity, consumers receive differing and possibly imperfect signals about the true state. The events which consumer h can observe are described by I_h, a partition of Ω. After receiving his signal, consumer h knows in which element of I_h the true state lies. Because of this restriction on his information, consumer h's actions must be "measurable" with respect to I_h. Define the information structure I to be the join of the consumers' information partitions. Assume that prior probability beliefs are commonly held by each of the consumers. Let $\pi(s)$, $s = 1, \dots, r$, be the probability of the occurrence of state s with $0 < \pi(s) < 1$ and $\sum_{s=1}^{s=r} \pi(s) = 1$. Define the vector of probabilities $\pi \in \mathcal{R}_{++}^{r}$ by $\pi = \{\pi(s)\}_{s=1}^{s=r}$. The probability space P is then defined by $P = (\Omega, \pi, I)$.

Let $\tilde{x}_h^i(s) \in \mathcal{R}_{++}$ be consumer h's consumption of commodity i if state s occurs. Define $\tilde{x}_h(s) \in \mathcal{R}_{++}^{l}$, $\tilde{x}_h \in \mathcal{R}_{++}^{lr}$, and $\tilde{x} \in \mathcal{R}_{++}^{lrn}$ by $\tilde{x}_h(s) = \{x_h^i(s)\}_{i=1}^{i=l}$, $\tilde{x}_h = \{\tilde{x}_h(s)\}_{s=1}^{s=r}$, and $\tilde{x} = \{\tilde{x}_h\}_{h=1}^{h=n}$. Consumer h is assumed to have a von Neumann-Morgenstern utility function v_h defined by $v_h(\tilde{x}_h) = \sum_{s=1}^{s=r} \pi(s) u_h(\tilde{x}_h(s))$, where $u_h : \mathcal{R}_{+}^{l} \to \mathcal{R}$ is the utility function described in Section 2.

See Aumann (1974, 1987) for the definitions of *correlated strategy* and *correlated equilibrium.* Assume that consumer h plays the strategy $\tilde{\sigma}_h(s) \in S_h$ if state s occurs and the strategy $\tilde{\sigma}_h(s') \in S_h$ if the state s' occurs. (The symbol s indexes strategies and allocations, just as the superscript i and the subscript h index these variables.) If s and s' are signals which fall in the same element of I_h, then the measurability assumption entails $\tilde{\sigma}_h(s)=\tilde{\sigma}_h(s')$: if consumer h cannot distinguish between the two states, then his actions must be the same in the two states. In what follows, we record some of the basic properties of correlated equilibrium for the market game Γ.

Remark 3.1. (1) A correlated equilibrium for the market game Γ is individually rational. (2) A (pure-strategy) Nash equilibrium to the market game Γ is also a correlated equilibrium to Γ. (3) A mixed-strategy Nash equilibrium to Γ is also a correlated equilibrium to Γ, where the I_h are chosen appropriately, i.e. to maintain "stochastic independence" of the strategies $\tilde{\sigma}_h$, $h=1,\ldots,n$. (4) A randomization over pure-strategy Nash equilibria to Γ is also a correlated equilibrium to Γ, where the I_h are chosen to be the same for each consumer.

There is another formal way to look at correlated equilibrium. We can replace the market game Γ with a related two-stage extensive form game, $\Gamma(P)$, in which the now *exogenous* probability space P is incorporated into the rules of play. Nature is assumed to choose the state $s=1,\ldots,r$ and consumers learn the elements in their partitions corresponding to the occurrence of state s. In the second stage, each consumer chooses a spot-market strategy based on his own information. We spell out the isomorphism between Γ and $\Gamma(P)$ in what follows.

Remark 3.2. (1) A pure-strategy Nash-equilibrium allocation $x\in\mathscr{R}^{lrn}_{++}$ for the new game $\Gamma(P)$ is also a correlated-equilibrium allocation for the market game Γ. (2) Conversely, a correlated-equilibrium allocation to the market game Γ is also a pure-strategy Nash-equilibrium allocation for the game $\Gamma(P)$ for some appropriately chosen probability space P.

In Remark (3.2), we describe the equivalence of correlated equilibrium allocations to the market game Γ with pure-strategy Nash-equilibrium allocations to the new game $\Gamma(P)$. But do things stop here? What about *correlated* equilibrium allocations to the *new* game $\Gamma(P)$? It turns out that one does not have to consider such complicated objects, because they contain only allocations which are pure-strategy Nash equilibrium solutions to the game $\Gamma(P')$, where P' is a sufficiently rich probability space. No complicated iteration of this sort is necessary.[6]

There is a third and very important interpretation of correlated equilibrium due to Aumann (1987). Assume, as before, that n consumers play the market game Γ. The only uncertainty is about what action the other players will adopt. Assume that the probability beliefs of the players can differ only because of their differing information; i.e. it is assumed that consumers share common prior probability beliefs. We call a consumer *Bayes-rational* if he chooses an action that maximizes his expected utility given his information. If it is common knowledge that each consumer is Bayes-rational at each state of the world, then the distribution of the strategies σ^* is also a correlated equilibrium (based on some probability space P) for the market game Γ. In this way, we can consider the probability space P as endogenously generated within the market game. The space P can be thought of as a set of self-fulfilling (or consistent) prophecies.

6. There is another potential technical issue which should be recorded. We assume for ease of analysis that the set of states Ω is finite, even though the strategy space S is infinite. We see no economic difficulties, only complications, in assuming that Ω is infinite, but that remains to be checked.

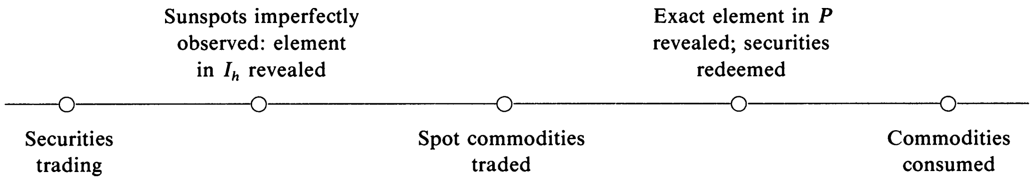
Aumann's equivalence between correlated equilibrium and Bayes-rational equilibrium plays an important role in the analysis of market uncertainty. While it might be difficult to imagine that businessmen focus on an explicit, exogenous correlating device, we readily accept the idea that businessmen, even those who know all the "fundamentals" with certainty, can still be uncertain about the actions of others. This description of participants reading the economic environment for their beliefs about what the others will do, and then maximizing, corresponds to the notion of Bayesian rationality. Since the actual mechanism that provides the participants with their "readings" is not known by anyone, we think of the market uncertainty as being generated endogenously. Aumann's (1987) result shows that the formalism of correlated equilibrium can capture this aspect of businessmen's behaviour even when no participant can describe the correlating device.

Aumann's (1987) hypothesis of common priors includes a fair amount of "correlation" in the way players form beliefs. To justify the assumption of common priors, we suppose that an actual mechanism (physical, genetic, or cultural) is providing the correlation. The actual mechanism or system can be so complicated that no one knows more than a small piece of it. The fact that no player can point to the cause of the market uncertainty does not invalidate the correlated-equilibrium (or the sunspot-equilibrium) explanation.

4. EXTRINSIC UNCERTAINTY: PURE-STRATEGY NASH EQUILIBRIUM IN THE SECURITIES GAME

We return to the simple (pure-strategy) Nash-equilibrium concept employed in Section 2, but we complicate the market game in order to allow for extrinsic uncertainty. In Section 3, we introduced the game $\Gamma(P)$ in which the probability space P is written into the rules. No insurance or hedging possibilities are allowed for in $\Gamma(P)$. Consumer h may have high utility in one state and low utility in another. Because of the concavity of his utility function, consumer h has a motive to transfer purchasing power from one state to another but the rules of $\Gamma(P)$ do not permit such transfers.

There is an economic need for state-contingent securities. In what follows, we expand the game $\Gamma(P)$ to include Arrow securities (or state-contingent money). Spot-market trading is the same as described in Section 3. Also following Section 3, the n consumers are assumed to be expected-utility maximizers. The expanded game is called the securities game and is denoted by $\hat{\Gamma}(P)$, where the probability space P (described in Section 2) is written into the rules.



This is our time line. Each of the n consumers is alive and active during the entire period. Let $\hat{x}_h(s) = (\hat{x}_h^1(s), \ldots, \hat{x}_h^i(s), \ldots, \hat{x}_h^l(s)) \in \mathcal{R}_{++}^l$ be consumer h's consumption basket if state s occurs ($s = 1, \ldots, r$ and $h = 1, \ldots, n$) and define $\hat{x}_h = \{\hat{x}_h(s)\}_{s=1}^{s=r} \in \mathcal{R}_{++}^{lr}$ and $\hat{x} = \{\hat{x}_h\}_{h=1}^{h=n} \in \mathcal{R}_{++}^{lrn}$.

The securities market is composed of r trading posts, one for each state of nature. Bids are denominated in "general monetary units", but offers are made in state-specific units of account. After consumers receive their private signals, they trade on the spot market, composed as in Section 2 of l posts, one for each commodity. Let $\hat{b}_h^i(s) \in \mathscr{R}_+$ and $\hat{q}_h^i(s) \in \mathscr{R}_+$ be, respectively, the bid and the offer of consumer h on spot market trading post i given that state of nature s has occurred. Let $\hat{b}_h^m(s) \in \mathscr{R}_+$ and $\hat{q}_h^m(s) \in \mathscr{R}_+$ be, respectively, the bid and the offer of consumer h on security market s. Define $\hat{b}_h \in \mathscr{R}_+^{r(l+1)}$ and $\hat{q}_h \in \mathscr{R}_+^{r(l+1)}$ in the obvious way. Define the strategy vectors $\hat{\sigma}_h = (\hat{b}_h, \hat{q}_h)$ and $\hat{\sigma} = (\hat{\sigma}_1, \ldots, \hat{\sigma}_h, \ldots, \hat{\sigma}_n)$. Then the strategy set $\hat{S}_h$ for consumer h in the securities game $\hat{\Gamma}(P)$ is given by

$$\hat{S}_h = \{\hat{\sigma}_h \in \mathscr{R}_+^{2r(l+1)} \mid \hat{b}_h^i(s) \text{ and } \hat{q}_h^i(s) \text{ are measurable with respect to } I_h, \text{ and } \hat{q}_h^i(s) \leqq \omega_h^i \text{ for } i = 1, \ldots, l \text{ and } s = 1, \ldots, r\}. \tag{4.1}$$

There are two markets: the securities market, which meets before consumers receive their private signals, and the spot commodities market, which meets after consumers receive their private signals but before the state s is perfectly revealed. Consumer h must satisfy two constraints, one for each market; if either one or both are not satisfied, consumer h is punished. The securities-market constraint is:

$$\sum_{s=1}^{s=r} \hat{b}_h^m(s) \leqq \sum_{s=1}^{s=r} \left[\hat{q}_h^m(s) \frac{\sum_{k=1}^{k=n} \hat{b}_k^m(s)}{\sum_{k=1}^{k=n} \hat{q}_k^m(s)} \right], \tag{4.2.i}$$

i.e. the sum of the securities-market bids in "general dollars" must be no greater than the sum of the revenue in "general dollars" from the sales of securities. Purchases of securities are financed by the sales of securities. A single unit of security s pays one unit of account in state s and zero otherwise. Security s can be thought of as state-s money, or state-s dollars, dollars accepted in state s and only in state s. In order to avoid punishment, consumer h must also meet the commodity-market budget constraint:

$$\sum_{j=1}^{j=l} \hat{b}_h^j(s) \leqq \sum_{j=1}^{j=l} \left[\hat{q}_h^j(s) \frac{\sum_{k=1}^{k=n} \hat{b}_k^j(s)}{\sum_{k=1}^{k=n} \hat{q}_k^j(s)} \right] + \hat{b}_h^m(s) \frac{\sum_{k=1}^{k=n} \hat{q}_k^m(s)}{\sum_{k=1}^{k=n} \hat{b}_k^m(s)} - \hat{q}_h^m(s) \tag{4.2.ii}$$

for $s = 1, \ldots r$. The consumption, $\hat{x}_h$, of consumer h is given by

$$\hat{x}_h^i(s) = \omega_h^i - \hat{q}_h^i(s) + \hat{b}_h^i(s) \left[\frac{\sum_{k=1}^{n} \hat{q}_k^i(s)}{\sum_{k=1}^{n} \hat{b}_k^i(s)} \right] \quad \text{if the budget constraints (4.2.i)-(4.2.ii) hold}$$

and (4.3)

$$\hat{x}_h^i(s) = 0 \text{ otherwise}$$

for $i = 1, \ldots, l$ and $s = 1, \ldots, r$.[7]

The game $\hat{\Gamma}(P)$ can be thought of as the non-competitive analogue of the Arrow (1964) securities model, but in which uncertainty is purely extrinsic. Hence, for the special case of symmetric information, our model can also be thought of as the non-competitive analogue of the particular Cass-Shell (1983) sunspot model in which there are no restrictions on market participation. The securities game $\hat{\Gamma}(P)$ is completely specified. The *strategy sets* are $\hat{S}_h$ are defined in Equation (4.1). The *outcomes* $(\hat{x}_h(1), \ldots, \hat{x}_h(r))$ are given by Equation (4.3), and the *payoffs* are the expected utilities $v_h(\hat{x}_h)$ at probabilities $\{\pi(s)\}_{s=1}^{s=r}$. The game $\hat{\Gamma}(P)$ is individually rational for each of the

7. The system of equations (4.3) is consistent with the following auditing-punishment procedure: Trade takes place on the securities market, and if constraint (4.2.i) is violated, consumer h is punished immediately no matter which state of nature occurs, i.e. $\hat{x}_h(s) = 0$ for $s = 1, \ldots, r$. Then the referee audits the consumer's spot market plans. If it is the case that in some state of nature s, consumer h violates constraint (4.2.ii), then he is punished on the spot market no matter which state of nature occurs, i.e., $\hat{x}_h(s') = 0$ for $s' = 1, \ldots, r$.

n consumers. We adopt the standard definition of *Nash equilibrium.* A Nash equilibrium to the securities game $\hat{\Gamma}(P)$ is said to be *interior* if all bids are strictly positive (including those on the markets for securities) and hence all offers are strictly positive. Hence, if the equilibrium is interior, all markets are open.

Definition 4.4. We say that *sunspots do not matter* if in the allocation of consumption goods, we have $\hat{x}_h(s) = \hat{x}_h(s')$ for $h = 1, \ldots, n$ and $s, s' = 1, \ldots, r$. Otherwise, *sunspots matter.* A pure-strategy Nash equilibrium to $\hat{\Gamma}(P)$ in which the allocation of consumption is independent (resp. dependent) on the state of nature is called a *non-sunspot Nash equilibrium* (resp. *sunspot Nash equilibrium*) to $\hat{\Gamma}(P)$.

It is easy to display an interior non-sunspot Nash equilibrium to $\hat{\Gamma}(P)$. This is done in the next proposition.

Proposition 4.5. *The securities game $\hat{\Gamma}(P)$ has an interior non-sunspot Nash equilibrium.*

Proof. Let $\sigma = \{(b_k, q_k)\}_{k=1}^{k=n} \in S$ be an interior Nash equilibrium of the certainty game Γ (analysed in Section 2). We know from Proposition (2.6) that there is such a strategy σ. We now construct $\hat{\sigma} \in \hat{S}$, measurable with respect to the I_h, to be an interior non-sunspot Nash equilibrium to $\hat{\Gamma}(P)$:

$$\hat{b}_h^i(s) = b_h^i, \qquad \hat{q}_h^i(s) = q_h^i, \qquad \hat{b}_h^m(s) = \pi(s) \quad \text{and} \quad \hat{q}_h^m(s) = 1 \tag{4.6}$$

for $i = 1, \ldots, l$; $s = 1, \ldots, r$; $h = 1, \ldots, n$.

No income is being transferred between states if $\hat{\sigma}$ defined by Equations (4.6) is the strategy vector for $\hat{\Gamma}(P)$. Hence, constraint (4.2.i) holds with equality. The constraint (4.2.ii) holds with equality since σ is an interior Nash equilibrium of Γ. Since all bids and offers are positive, the first-order conditions for utility maximization under binding constraints (4.2) are necessary and sufficient for optimality. These conditions are:

$$\frac{\hat{\lambda}_h(s)}{\hat{\lambda}_h(s')} = \frac{\pi(s)}{\pi(s')} \frac{\partial u_h(\hat{x}_h(s))/\partial \hat{x}_h^i(s)}{\partial u_h(\hat{x}_h(s'))/\partial \hat{x}_h^j(s')} \frac{\sum_{k \neq h} \hat{b}_k^i(s)}{\sum_{k \neq h} \hat{q}_k^i(s)} \left[\frac{\sum_{k=1}^n \hat{q}_k^i(s)}{\sum_{k=1}^n \hat{b}_k^i(s)}\right]^2 \times \frac{\sum_{k \neq h} \hat{q}_k^j(s')}{\sum_{k \neq h} \hat{b}_k^j(s')} \left[\frac{\sum_{k=1}^n \hat{b}_k^j(s')}{\sum_{k=1}^n \hat{q}_k^j(s')}\right]^2 \tag{4.7}$$

and

$$\frac{\hat{\lambda}_h(s)}{\hat{\lambda}_h(s')} = \frac{\sum_{k \neq h} \hat{q}_k^m(s)}{\sum_{k \neq h} \hat{b}_k^m(s)} \left[\frac{\sum_{k=1}^n \hat{b}_k^m(s)}{\sum_{k=1}^n \hat{q}_k^m(s)}\right]^2 \frac{\sum_{k \neq h} \hat{b}_k^m(s')}{\sum_{k \neq h} \hat{q}_k^m(s')} \left[\frac{\sum_{k=1}^n \hat{q}_k^m(s')}{\sum_{k=1}^n \hat{b}_k^m(s')}\right]^2 \tag{4.8}$$

for $h = 1, \ldots, n$; $i, j = 1, \ldots, l$; and $s, s' = 1, \ldots, r$; and $\hat{\lambda}_h(s)$ and $\hat{\lambda}_h(s')$ are (respectively) the Kuhn-Tucker-Lagrange multipliers associated with constraint (4.2.ii) for states s and s'. Substitute the data from Equations (4.6) into the right side of Equation (4.8), which is consistent if

$$\hat{\lambda}_h(s)/\hat{\lambda}_h(s') = \pi(s)/\pi(s'). \tag{4.9}$$

If we substitute from Equation (4.9), Equation (4.7) must hold because of the first-order condition for the market game Γ,

$$\frac{\partial u_h(x_h)/\partial x_h^i}{\partial u_h(x_h)/\partial x_h^j} = \frac{\sum_{k\neq h} q_k^i}{\sum_{k\neq h} b_k^i}\left[\frac{\sum_{k=1}^{k=n} b_k^i}{\sum_{k=1}^{k=n} q_k^i}\right]^2 \frac{\sum_{k\neq h} b_k^j}{\sum_{k\neq h} q_k^j}\left[\frac{\sum_{k=1}^{k=n} q_k^j}{\sum_{k=1}^{k=n} b_k^j}\right]^2. \tag{4.10}$$

Thus $\hat{\sigma} \in \hat{S}$ defined by Equations (4.6) is an interior Nash equilibrium for $\hat{\Gamma}(P)$. Since we have $\hat{x}_h(s) = \hat{x}_h(s')$ for $s, s' = 1, \ldots, r$ and $h = 1, \ldots, n$, $\hat{\sigma}$ is also a non-sunspot Nash equilibrium for the securities game $\hat{\Gamma}(P)$. ‖

A careful reading of the proof of Proposition (4.5) shows that the Nash equilibria to Γ reappear as the non-sunspot Nash equilibria to $\hat{\Gamma}(P)$. (Example (5.7) shows that the converse is not true, which contrasts with the competitive case with convexity and symmetric information.)

If endowments are Pareto optimal, then there are no sunspot Nash equilibria to $\hat{\Gamma}(P)$. If the probability mechanism P along with the signalling devices I_h are nondegenerate, then we have for endowments which are not Pareto-optimal that there must exist sunspot Nash equilibria to $\hat{\Gamma}(P)$. These ideas are formalized in the following proposition.

Proposition 4.11. (a) *Let the endowment vector $\omega \in \mathscr{R}^{ln}_{++}$ in the market game Γ be Pareto-optimal. Then there is no sunspot Nash equilibrium to the corresponding securities game $\hat{\Gamma}(P)$.* (b) *Let the endowment vector $\omega \in \mathscr{R}^{ln}_{++}$ in the market game Γ be not Pareto-optimal. Let there be a common coarsening of the information partitions I_h which contains (at least) two elements. Then there is a sunspot Nash equilibrium $\hat{\sigma} \in \hat{S}$ to the corresponding securities game $\hat{\Gamma}(P)$.*

Proof. (a) Assume that ω is Pareto-optimal in Γ. Clearly, then ω is also Pareto-optimal in $\hat{\Gamma}(P)$. Assume that $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_h, \ldots, \hat{x}_n)$ is a sunspot Nash equilibrium allocation in $\hat{\Gamma}(P)$. Let P_h^δ be the δ-th element of h's information partition. Because of individual rationality, we have

$$\textstyle\sum_{s\in P_h^\delta} \text{Prob}\,(s \mid P_h^\delta) u_h(x_h(s)) \geqq u_h(\omega_h).$$

Multiplying both sides by Prob (P_h^δ) and summing over δ yields

$$v_h(\hat{x}) = \textstyle\sum_{s=1}^{s=r} \pi(s) u_h(\hat{x}_h(s)) \geqq v_h(\omega_h, \ldots, \omega_h) = u_h(\omega_h)$$

for $h = 1, \ldots, n$. Because of the strict concavity of u_h and the fact that $\hat{x}$ is a sunspot Nash equilibria allocation, we can find a certainty allocation $x = (x_1, \ldots, x_h, \ldots, x_n)$ such that $u_h(x_h) \geqq u_h(\omega_h)$ for $h = 1, \ldots, n$ with at least one strict inequality. We have a contradiction. Thus there are no sunspot Nash equilibria for $\hat{\Gamma}(P)$.

(b) Assume that ω is not Pareto-optimal in Γ. Then there are at least two Nash equilibria for Γ, $\sigma' = 0 \in S$ (with the corresponding allocation $x' = \omega \in \mathscr{R}^{ln}_{++}$), and an *interior* Nash equilibrium strategy $\sigma'' = \{(\sigma_h)''\}_{h=1}^{h=n} = \{((b_h)'', (q_h)'')\}_{h=1}^{h=n}$ (with the corresponding allocation $x'' \in \mathscr{R}^{ln}_{++}$). From Proposition (2.5), we know that x' and x'' are not equal. By hypothesis, we can partition the states of nature Ω into two subsets, A and B, which are each elements of a common coarsening of the I_h. We have $A \cup B = \Omega$, $A \cap B = \emptyset$, $A \neq \emptyset$, and $B \neq \emptyset$. We construct the sunspot Nash equilibrium $\hat{\sigma}$ for the securities game $\hat{\Gamma}(P)$ from these two Nash equilibria of the market game Γ as follows: $(\hat{b}_h^i(s) = 0$, $\hat{q}_h^i(s) = 0$, $\hat{b}_h^m(s) = 0$, $\hat{q}_h^m(s) = 0)$ for $s \in A$, $h = 1, \ldots, n$ and $i = 1, \ldots, l$; and $(\hat{b}_h^i(s) = (b_h^i)''$, $\hat{q}_h^i(s) = (q_h^i)''$, $\hat{b}_h^m(s) = 0$, $\hat{q}_h^m(s) = 0$ for $s \in B$, $h = 1, \ldots, n$ and $i = 1, \ldots, l$, where $(b_h^i)''$ and $(q_h^i)''$ are bids and offers in the interior Nash equilibrium strategy σ'' for the market game Γ.

The strategy $\hat{\sigma} \in \hat{S}$ is clearly a sunspot Nash equilibrium for the securities game $\hat{\Gamma}(P)$. ||

The sunspot Nash equilibrium allocation $\hat{x}$ constructed in Proposition (4.10) is a lottery over (certainty) Nash equilibria from the underlying market game Γ.[8] As such, the allocation $\hat{x}$ is also a correlated equilibrium allocation to the market game Γ. Next we show that a correlated equilibrium to Γ is always a Nash equilibrium to $\hat{\Gamma}(P)$ for some probability space P.

Proposition 4.12. *Let $\tilde{x} \in \mathcal{R}^{lrn}_{++}$ be a correlated-equilibrium allocation for the market game Γ, where the probability-signalling mechanism is given by Ω, $\{\pi(s)\}_{s=1}^{s=r}$, and $\{I_h\}_{h=1}^{h=n}$. Then $\tilde{x} \in \mathcal{R}^{lrn}_{++}$ is also a Nash-equilibrium allocation to the securities game $\hat{\Gamma}(P)$ generated by Γ, Ω, $\{\pi(s)\}_{s=1}^{s=r}$ and $\{I_h\}_{h=1}^{h=n}$.*

Proof. Let $\tilde{\sigma} = \{\tilde{\sigma}_h\}_{h=1}^{h=n} \in \tilde{S}$ be the correlated equilibrium corresponding to $\tilde{x}$. Define $\hat{\sigma} \in \hat{S}$ by $\hat{b}_h(s) = \tilde{b}_h(s)$, $\hat{q}_h(s) = \tilde{q}_h(s)$, $\hat{b}^m_h(s) = 0$, and $\hat{q}^m_h(s) = 0$ for $h = 1, \ldots, n$ and $s = 1, \ldots, r$. Then $\hat{\sigma} \in \hat{S}$ is clearly a pure-strategy Nash equilibrium to the securities game $\hat{\Gamma}(P)$ which supports the allocation $\tilde{x}$. ||

Those sunspot equilibria in which the securities market is closed are equivalent to correlated equilibria and hence equivalent to Bayes-rational equilibria. In this case, we can think of the market uncertainty as being endogenous (even though we might not be able to identify *how* beliefs become correlated). Can sunspot equilibria with open and active securities markets also fit the Bayes-rational-equilibrium interpretation? In the securities game, consumers actively trade securities that explicitly depend on the realization of sunspot activity, so the randomizing device must be exogenous. It would be worthwhile to investigate securities markets which do not explicitly depend on sunspots. For example, we might allow for more realistic securities whose returns depend on spot market prices. If spot market prices depend on sunspots, then "option markets" allow income transfers across states of nature, without the neeed for an exogenous, identifiable randomizing device. We would then have a sunspot equilibrium with markets created for partial hedging against the effects of market uncertainty; however, no consumer would be required to understand the "ultimate source" of this uncertainty.

5. EXAMPLES

This section is devoted to numerical and other examples. We begin by computing some (pure-strategy) Nash equilibria to the market game Γ. It is easy to find other correlated equilibria to Γ (and hence (pure-strategy) Nash equilibria to $\hat{\Gamma}(P)$) which are simple randomizations over the Nash equilibria to Γ. Our more interesting examples involve either (a) *transfers of income across states of nature* which generate sunspot Nash equilibria to the securities game $\hat{\Gamma}(P)$ that are not correlated equilibria to the underlying certainty market game Γ, or (b) *asymmetric information* which generates correlated equilibria for the market game Γ which are not achievable with symmetric information.

We begin with examples of Nash equilibria in a market game Γ. These computed solutions will be used repeatedly in the sequel.

8. It is a lottery over an interior NE and a closed-market NE. We know from Peck, Shell, and Spear (1989, especially Section 5) that multiplicity of *interior* NE to Γ is generic. Hence, generically it is possible to construct sunspot Nash equilibria to the securities game which are lotteries over *interior* NE to the market game.

Example 5.1. Let there be two consumers and two commodities. The following data about consumer preferences and endowments complete the description of the market game Γ: $u_h(x_h^1, x_h^2) = \log x_h^1 + \log x_h^2$ for $h = 1, 2$ and $\omega_1 = (\omega_1^1, \omega_1^2) = (80, 20)$; $\omega_2 = (\omega_2^1, \omega_2^2) = (20, 80)$.

Solution 1 *to Example* (5.1). The example is symmetric. Hence, if each of the consumers offers 100 percent of his endowments for sale, we have the symmetric interior Nash equilibrium to Γ displayed below. This is a "thick-market" solution. Trading is substantial, but since this game is neither cooperative nor perfectly competitive, the allocation of consumption goods is still far from Pareto-optimal.

Solution 1 *to the* 2×2 *game* Γ *of Example* (5.1): *Each consumer offers* 100% *of his endowments.*

	q_1	b_1	x_1	q_2	b_2	x_2
Commodity 1	80·0000	0·3333	66·6667	20·0000	0·1667	33·3333
Commodity 2	20·0000	0·1667	33·3333	80·0000	0·3333	66·6667

Solution 2 *to Example* (5.1). The Nash equilibrium is interior, but the symmetry is broken. Consumer 1 offers 100% of his endowments, but Consumer 2 offers only 25%. Markets are thin relative to those in Solution 1. Each consumer is worse off than in Solution 1.

Solution 2 *to the* 2×2 *game* Γ *of Example* (5.1): *Consumer* 1 *offers* 100% *of his endowments*: *Consumer* 2 *offers* 25% *of his endowments.*

	q_1	b_1	x_1	q_2	b_2	x_2
Commodity 1	80·0000	0·6836	74·2236	5·0000	0·0992	25·7764
Commodity 2	20·0000	0·1618	29·7970	20·0000	0·0554	70·2029

The following is an example of a securities game $\hat{\Gamma}(P)$ which is based on the market game Γ described in Example (5.1).

Example 5.2. Let Γ be described by Example (5.1). Let there be two states $\Omega = \{\alpha, \beta\}$ and assume that the extrinsic random variable s obeys the probability law $\pi(\alpha) = \pi(\beta) = 1/2$. Let I_h $(h = 1, 2)$ be the finest partition of $\Omega = \{\alpha, \beta\}$. Hence, signals are *perfectly* correlated. Let $\hat{\Gamma}(P)$ be the corresponding securities game defined by the market game Γ and the probability space $P = (\Omega, \pi, I)$.

Next we compute two sunspot Nash equilibria for $\hat{\Gamma}$. The allocation of resources varies across states of nature as market thickness varies. These solutions establish: (a) Some sunspot Nash equilibria to $\hat{\Gamma}(P)$ are interior; others are not. (b) Some sunspot Nash equilibrium allocations to $\hat{\Gamma}(P)$ are also correlated equilibrium allocations to Γ; others are not.

Solution 1 *to Example* (5.2). The securities markets are closed. Hence, this solution can be taken as a lottery over interior Nash-equilibrium solutions (1) and (2) (to Example (5.1)) in the market game Γ. This solution is a sunspot Nash equilibrium to $\hat{\Gamma}(P)$ which is not interior. The corresponding sunspot Nash equilibrium allocation to $\hat{\Gamma}(P)$ is also a correlated-equilibrium allocation to the market game Γ.

Solution 1 *to the game* $\hat{\Gamma}(P)$ *defined in Example* (5.2): *Consumer* 2 *reduces his offers to* 25% *in state* β. *The securities markets are closed.*

	State α			State β		
	Commodity 1	Commodity 2	Security	Commodity 1	Commodity 2	Security
$\hat{b}_1$	0·3333	0·1667	0	0·6836	0·1618	0
$\hat{q}_1$	80·0000	20·0000	0	80·0000	20·0000	0
$\hat{x}_1$	66·6667	33·3333	—	74·2236	29·7970	—
$\hat{b}_2$	0·1667	0·3333	0	0·0992	0·0554	0
$\hat{q}_2$	20·0000	80·0000	0	5·0000	20·0000	0
$\hat{x}_2$	33·3333	66·6667	—	25·7764	70·2029	—

Solution 2 *to Example* (5.2). In Solution 2, the securities markets are open and net securities purchases are non-zero. The price of the α-security in terms of the β-security is

$$\frac{(116{\cdot}3231+116{\cdot}3031)/(100+100)}{(83{\cdot}6768+83{\cdot}6968)/(100+100)}=1{\cdot}39.$$

Consumer 1's purchases of the α-security (or better, α-money) are $200[(116{\cdot}3231)/(116{\cdot}3231+116{\cdot}3031)]=100{\cdot}0086$ units. His *net* purchases of the α-money are hence 0·0086 units. Consumer 1's bids for commodities in state α sum to 0·5087 state-α dollars, of which 0·0086 state-α dollars (amounting to 1·7 percent of the total) are financed by his purchases of α-money in the securities market. Consumer 1 transfers income into state α, while consumer 2 transfers income into state β. Consumer 1 seeks commodity 2 in state β, but Consumer 2 offers little of this commodity. Hence Consumer 1 parts with commodity 1 in state β in exchange for state-β money, a substantial portion of which he then exchanges for state-α money. The state-α money is used to finance his purchases of commodities in state α.

Compare Solutions (1) and (2) (Example 5.2). With open securities markets (Solution 1), Consumer 1 increases his consumption of both commodities in (the good state) α and reduces his consumption of both commodities in (the bad state) β.

Solution 2 *to the Game* $\hat{\Gamma}(P)$ *defined in Example* (5.2): *Consumer* 2 *reduces his offers to* 25% *in State* β. *The securities markets are open, and net purchases of securities are nonzero.*

	State α			State β		
	Commodity 1	Commodity 2	Security	Commodity 1	Commodity 2	Security
$\hat{b}_1$	0·3376	0·1710	116·3231	0·6741	0·1586	83·6768
$\hat{q}_1$	80·0000	20·0000	100·0000	80·0000	20·0000	100·0000
$\hat{x}_1$	67·5204	34·1981	—	73·3364	29·0214	—
$\hat{b}_2$	0·1624	0·3290	116·3031	0·1072	0·0600	83·6968
$\hat{q}_2$	20·0000	80·0000	100·0000	5·0000	20·0000	100·0000
$\hat{x}_2$	32·4796	65·8019	—	26·6636	70·9786	—

Solution 2 is especially noteworthy. It is an interior sunspot Nash equilibrium solution to $\hat{\Gamma}$ with open and *active securities markets.* Hence this solution *cannot* be considered to be a lottery over Nash-equilibrium solutions to Γ. In the next lemma, we establish that the sunspot Nash equilibrium displayed in Solution 2 is not a correlated-equilibrium allocation to the corresponding market game Γ.

In what follows, we use Solution 2 to Example (5.2) to establish that for some market game Γ and some probability law P, the set of correlated equilibrium allocations to Γ is a proper subset of the set of pure strategy Nash equilibrium allocations to $\hat{\Gamma}(P)$. The basic idea is that the securities game allows income to be transferred across states, but the self-enforcing nature of correlated equilibrium in Γ excludes income transfer across states.

Lemma 5.3. *It is not always the case that a pure-strategy Nash equilibrium allocation for the securities game $\hat{\Gamma}(P)$ is a correlated-equilibrium allocation for the corresponding market game Γ based on the probability law P.*

Proof. We will be considering the sunspot Nash-equilibrium allocation to Example (5.2) which is presented in Solution 2 and the state space $\Omega=\{\alpha, \beta\}$. We need to show that $\hat{x}=((67\cdot5204, 34\cdot1981, 73\cdot3364, 39\cdot0214), (32\cdot4796, 65\cdot8019, 26\cdot6636, 70\cdot9786))$ is not a correlated-equilibrium allocation to the market game Γ defined in Example (5.1). Assume the contrary, i.e. that $\hat{x}$ is a correlated-equilibrium allocation to Γ. If neither player could distinguish between state α and state β, the information-measurability assumption would imply that the correlated-equilibrium allocation would be independent of the state of nature. This is not the case for the allocation $\hat{x}$. If one player could not recognize the difference between states, the other player's best response would lead to an allocation independent of states. Hence we have shown that each of the two players can see sunspots: neither is blind to solar activity.

Tildes indicate correlated strategies and outcomes for Γ. Given the strategy of consumer k, we have from Conditions (2.1)–(2.2) that the frontier of consumer h's budget set in $(\tilde{x}_h^1(s), \tilde{x}_h^2(s))$-space is given by the intersection of the equation defined by

$$\frac{\tilde{b}_k^1(s)(\omega_h^1-\tilde{x}_h^1(s))}{\omega_h^1+\tilde{q}_k^1(s)-\tilde{x}_h^1(s)}+\frac{\tilde{b}_k^2(s)(\omega_h^2-\tilde{x}_h^2(s))}{\omega_h^2-\tilde{x}_h^2(s)+\tilde{q}_k^2(s)}=0 \tag{5.4}$$

and the positive orthant, where $h\neq k$. If Mr. h is optimizing, it is also the case that this budget frontier is tangent to his indifference curve at the consumption point, which yields

$$\frac{\hat{x}_h^2(s)}{\hat{x}_h^1(s)}=\frac{\tilde{q}_k^1(s)\tilde{b}_k^1(s)}{\tilde{q}_k^2(s)\tilde{b}_k^2(s)}\left[\frac{\omega_h^2-\hat{x}_h^2(s)+\tilde{q}_k^2(s)}{\omega_h^1-\hat{x}_h^1(s)+\tilde{q}_k^1(s)}\right]^2 \tag{5.5}$$

for $s=\alpha, \beta$, as a consequence of the utility-function specification in (5.1). Let $h=2$, $k=1$, and $s=\alpha$. Combining Equations (5.4) and (5.5), setting $\tilde{x}_2(\alpha)$ equal to $\hat{x}_2(\alpha)$, and substituting numerical values for $\hat{x}_2(\alpha)$ and ω_2 yields

$$\frac{14\cdot1981}{12\cdot4796}=\frac{65\cdot8019}{32\cdot4796}\left[\frac{\tilde{q}_1^2(\alpha)}{\tilde{q}_1^2(\alpha)+14\cdot1981}\right]\left[\frac{\tilde{q}_1^1(\alpha)-12\cdot4796}{\tilde{q}_1^1(\alpha)}\right]. \tag{5.6}$$

From the definition of the strategy set S_1, we have $\tilde{q}_1^2(\alpha)\leqq 20=\omega_1^2$. Substituting into Equation (5.6) yields

$$\frac{\tilde{q}_1^1-12\cdot4796}{\tilde{q}_1^1}\geqq\frac{14\cdot1981}{12\cdot4796}\times\frac{32\cdot4796}{65\cdot8019}\times\frac{34\cdot1981}{20}.$$

We have $\tilde{q}_1^1\geqq(12\cdot4796/0\cdot03977)>300$, which is a contradiction to the requirement that

$\tilde{q}_1^1 \leqq 80$. The allocation $\hat{x}$ is not a correlated-equilibrium allocation[9] for the market game Γ. $\|$

The driving force of this example is income transfer across states of nature. Information in this example (and the other computed examples) is symmetric across consumers. In the next example, we construct (but do not compute) a non-trivial correlated equilibrium driven by asymmetric information.

Example 5.7. (Non-trivial Correlated Equilibrium): We construct a correlated equilibrium to the market game Γ in which the allocation is neither a Nash equilibrium nor a simple randomization over Nash equilibria. It is a *non*-sunspot equilibrium which is not equivalent to a NE from the certainty market game.

Consider first the market game Γ^n based on n consumers, $n \geqq 2$, and l commodities, $l \geqq 2$. We know from Proposition (2.6) that there is an interior pure-strategy Nash equilibrium $\sigma = \{(b_h, q_h)\}_{h=1}^{h=n} \in \mathscr{R}_{++}^{2ln}$ with the corresponding allocation $x = \{x_h\}_{h=1}^{h=n} \in \mathscr{R}_{++}^{ln}$. Let $\sigma' = \{(b'_h, q'_h)\}_{h=1}^{h=n}$ be the strategy defined by $b'_h = Mb_h$ and $q'_h = q_h$ where M is a positive scalar. We know from Equations (2.1) and (2.2) that the budget set is homogeneous of degree zero in the bids b. Hence σ' is also a pure-strategy Nash equilibrium for Γ^n. The equilibrium allocation corresponding to σ', call it x', is also an equilibrium allocation corresponding to the pure-strategy Nash equilibrium σ.

Introduce two states of nature, $s = \alpha, \beta$. Assume that each of the n consumers can distinguish between α and β, i.e. I_h is the finest partition of $\Omega = \{\alpha, \beta\}$ for $h = 1, \dots, n$. (We can define a *trivial* correlated equilibrium for Γ^n by randomizing over σ and σ'. Bids and the general price level in state β would be M times those in state α. Consumption is unaffected by sunspots, although consumers must be alert to price level changes caused by sunspots.)

Next introduce a new consumer, Mr. $(n+1)$, and create from Γ^n the new market game Γ^{n+1}. Choose the preferences and endowments of Mr. $(n+1)$ so that his best response in Γ^{n+1} to strategies of the others given by σ would involve some non-zero net trades. Assume that Mr. $(n+1)$ cannot recognize the difference between state α and state β, so that I_{n+1} is the coarsest partition of Ω. Define $\tilde{\sigma} \in \mathscr{R}_+^{4l(n+1)}$ by $\tilde{b}_h(\alpha) = b_h$, and $\tilde{q}_h(\alpha) = q_h$, $\tilde{b}_h(\beta) = Mb_h = b'_h$, and $\tilde{q}_h(\beta) = q_h = q'_h$ for $h = 1, \dots, n$, and $\tilde{b}_{n+1}(\alpha) = 0$, $\tilde{q}_{n+1}(\alpha) = 0$, $\tilde{b}_{n+}^1(\beta) = 0$, and $\tilde{q}_{n+1}(\beta) = 0$. We claim that for suitable values for M and $\pi(\alpha)$, $\tilde{\sigma}$ is a correlated equilibrium to Γ^{n+1} with an allocation given by $\tilde{x} = (x_1, \dots, x_n, \omega_{n+1}; x_1, \dots, x_n, \omega_{n+1}) \in \mathscr{R}_{++}^{2l(n+1)}$. Hence in both states of nature, consumer h, $h = 1, \dots, n$, consumes in the correlated equilibrium to Γ^{n+1} the same amounts as he does in the pure-strategy Nash equilibrium to Γ^n. Consumer $(n+1)$ consumes his endowments in the two states. The allocation $\tilde{x}$ does not depend on sunspots, yet it is not based on any pure-strategy Nash equilibrium to Γ^{n+1}. That is, $(x_1, \dots, x_n, \omega_{n+1})$ is not a pure-strategy Nash-equilibrium allocation to Γ^{n+1} (even though $(x_1, \dots, x_n)$ is a pure-strategy Nash equilibrium to Γ^n).

Before we formally state or prove our claim, we provide a heuristic argument. If the first n players were playing the pure strategy σ (or even the pure strategy σ'), Mr.

9. We believe that the allocation of Solution 2 is not a correlated equilibrium allocation for *any* state space, information structure, and probabilities. We have not yet found a proof for this conjecture. Forges (1990) constructs a closed-market sunspot equilibrium to $\hat{\Gamma}(P)$ where, for state α, some consumer is below his individually rational utility level. This clearly cannot be a correlated equilibrium allocation for any P. Also, for an economy with indivisibilities in consumption (based on the model of Shell and Wright (1989)), one can easily construct a sunspot equilibrium that yields utilities different from the utilities achievable in any correlated equilibrium.

$(n+1)$ would respond with a strategy which would yield him non-zero net trades. Neither σ nor σ' can form the basis for a Nash-equilibrium strategy for Γ^{n+1}. Because Mr. $(n+1)$ cannot distinguish between α and β, his randomized strategy must be degenerate in that it is constant across states. Because of the extreme bankruptcy penalty, he will not risk bankruptcy in either state. His bids are bounded from above in order to avoid bankruptcy originating from state α. State β is the inflationary state. If consumer $(n+1)$ makes positive bids, they would have to be small to avoid state-α bankruptcy. This means that he would make substantial offers with meagre state-β bids (relative to the state β bids of others). This would lead to a loss in utility on his state β trades which cannot be offset by the gains from his state-α trades if the probability $\pi(\alpha)$ is sufficiently small and the inflation rate M is sufficiently large.

Claim 5.8. *For some positive scalar M and some probability $\pi(\alpha)$ with $0<\pi(\alpha)<1$, the strategy vector $\tilde{\sigma}$ defined in Example* (5.7) *is a correlated equilibrium to the market game Γ^{n+1}.*

Proof of Claim. Mr. $(n+1)$ is blind to sunspot activity, so he maximizes

$$\pi(\alpha)u_{n+1}\left[\ldots,\omega^i_{n+1}-q^i_{n+1}+\frac{b^i_{n+1}\sum_{k=1}^{n+1}q^i_k}{\sum_{k=1}^{n+1}b^i_k},\ldots\right]$$
$$+\pi(\beta)u_{n+1}\left[\ldots,\omega^i_{n+1}-q^i_{n+1}+\frac{b^i_{n+1}\sum_{k=1}^{n+1}q^i_k}{b^i_{n+1}+M\sum_{k=1}^{n}b^i_k},\ldots\right] \tag{5.8}$$

subject to

$$\sum_{j=1}^{l}b^j_{n+1}\leqq\sum_{j=1}^{l}\frac{q^j_{n+1}}{\sum_{k=1}^{n+1}q^j_k}\sum_{k=1}^{n+1}b^j_k \tag{5.9}$$

and

$$\sum_{j=1}^{l}b^j_{n+1}\leqq\sum_{j=1}^{l}\frac{q^j_{n+1}}{\sum_{k=1}^{n+1}q^j_k}\left(b^j_{n+1}+M\sum_{k=1}^{n}b^j_k\right). \tag{5.10}$$

Suppose there is some i for which $b^i_{n+1}>0$. Then the first-order condition

$$\pi(\alpha)\left[\frac{\partial u_{n+1}}{\partial x^i_{n+1}(\alpha)}\right]-\lambda_{n+1}(\alpha)\left[\frac{\sum_{k=1}^{n}q^i_k}{\sum_{k=1}^{n}b^i_k}\right]\left[\frac{\sum_{k=1}^{n+1}b^i_k}{\sum_{k=1}^{n+1}q^i_k}\right]^2+\pi(\beta)\left[\frac{\partial u_{n+1}}{\partial x^i_{n+1}(\beta)}\right]$$
$$\times\frac{M(\sum_{k=1}^{n+1}b^i_k)^2}{(b^i_{n+1}+M\sum_{k=1}^{n}b^i_k)^2}=0 \tag{5.11}$$

must be satisfied, where $\lambda_{n+1}(\alpha)$ is the Lagrangean multiplier associated with constraint (5.9). From (5.11), we have that for every scalar $\varepsilon>0$ there is a pair $(\bar{\pi}(\alpha),\bar{M})$ such that for $\pi(\alpha)<\bar{\pi}(\alpha)$ and $M>\bar{M}$, we have $\lambda_{n+1}(\alpha)<\varepsilon$.

Next consider the expression

$$\pi(\alpha)\left[\frac{\partial u_{n+1}}{\partial x^j_{n+1}(\alpha)}\right]+\pi(\beta)\left[\frac{\partial u_{n+1}}{\partial x^j_{n+1}(\beta)}\right]\left[\frac{M\sum_{k=1}^{n+1}b^j_k}{b^j_{n+1}+M\sum_{k=1}^{n}b^j_k}\right]$$
$$-\lambda_{n+1}(\alpha)\left[\frac{\sum_{k=1}^{n+1}b^j_k}{\sum_{k=1}^{n+1}q^j_k}\right]^2\left[\frac{\sum_{k=1}^{n}q^j_k}{\sum_{k=1}^{n}b^j_k}\right]. \tag{5.12}$$

Choose the positive scalar ε so that expression (5.12) is positive for $j=1,\ldots,l$, and choose the remaining parameters so that $0<\pi(\alpha)<\bar{\pi}(\alpha)$ and $M>\bar{M}$ hold. Therefore, all offers from Mr. $(n+1)$ are zero, $q^j_{n+1}=0$ for $j=1,\ldots,l$. This contradicts the assumption that there is a positive bid, since (5.12) or (5.13) implies $b^j_{n+1}=0$ for $j=1,\ldots,l$. By choosing $M > 0$ sufficiently large and $\pi(\alpha) > 0$ sufficiently small, we have ensured that Mr. $(n+1)$'s best response is zero bids, zero offers, and hence zero trade. Since Mr. $(n+1)$ bids and offers zero, the first n consumers will by construction be content with playing σ in state α and σ' in state β. We have constructed a correlated equilibrium to Γ^{n+1} driven by asymmetric information. ||

The strategy $\tilde{\sigma}$ is a non-trivial correlated equilibrium to Γ^{n+1} in which the asymmetry of information is essential. But this correlated equilibrium has a very special feature: Mr. $(n+1)$, who is blind to sunspots, stays out of the market for fear of bankruptcy in the deflationary state, α. According to Definition (4.4), this equilibrium is a non-sunspot equilibrium, even though it could not arise without extrinsic uncertainty. We are not aware of another example of an equilibrium in which the allocation is independent of sunspots, yet is not an equilibrium for the deterministic economy. Andreu Mas-Colell has suggested that a non-trivial sunspot equilibrium should be one in which, with positive probability, the ex post realization is not an equilibrium of the deterministic economy. Under this alternative definition, the equilibrium of Example (5.7) would be considered a non-trivial sunspot equilibrium.

Interior correlated equilibria are not easy to construct in this model because of the severe specification of the bankruptcy rule: if the consumer risks bankruptcy in any state he is severely punished in all states. (The proper correlated equilibrium exhibited in the Aumann-Peck-Shell (forthcoming) note is relatively easy to construct, because in that example the use of only *commodity* money completely obviates the bankruptcy problem.) A more realistic (less severe) bankruptcy rule would limit punishments to states in which violation of the budget constraint is actually observed and would make such punishments proportional to the extent of bankruptcy. We are confident that in such a setting proper correlated equilibria would be frequently encountered and robust to changes in the parameters.

When information is symmetric, the bankruptcy punishment can be weakened without changing the set of equilibria, because each player knows the equilibrium actions of the others. There is no need to forego a trade that is desired in one state but leads to bankruptcy in another state. In this sense, our examples of sunspot equilibria are quite robust. With asymmetric information, the sunspot equilibria and correlated equilibria will be sensitive to the bankruptcy rule, but the comparison between the two solution concepts is robust. Because markets can be endogenously closed, correlated-equilibrium allocations to Γ (with a different bankruptcy rule) will be sunspot Nash-equilibrium allocations to $\hat{\Gamma}(P)$ (with the new bankruptcy rule). There will also be the possibility of sunspot-equilibrium allocations, involving transfers of income across states, that are not correlated-equilibrium allocations. The only result in the present paper that is sensitive to the bankruptcy rule is Example (5.7).

We conclude with some remarks on the Cournotian foundations of incomplete-market competitive equilibrium and sunspot competitive equilibrium.[10]

6. CONCLUDING REMARKS ON REPLICATIONS, INCOMPLETE MARKETS, AND SUNSPOTS

As the economy becomes large through replication, the *interior* NE to Γ approach competitive equilibrium. Any spectrum of open and closed markets is self-justifying. In

10. See Mas-Colell (1982). See also Maskin and Tirole (1987), Peck and Shell (1985, Section 4), and Aumann, Peck, and Shell (forthcoming). Compare with Peck and Shell (1990).

this simple way, the market-game model provides an endogenous theory of market incompleteness.

For the *symmetric* information case, as the economy becomes large the set of interior sunspot equilibria vanishes, but some sunspot equilibria in which some markets are closed will remain. For example, the markets for commodity 1 in state α and commodity 2 in state β might be (endogenously) closed. Sunspots will then very likely matter, and if the securities market is open, consumers will typically be transferring income across states of nature. We can interpret the limit of our model in which some markets are closed as a competitive, incomplete-markets model. (See, e.g. Balasko and Cass (1989) and Geanakoplos and Mas-Colell (1989), where the emphasis is, however, on complete spot commodity markets and incomplete securities markets.)

When information is *asymmetric*, the picture is completely changed. Even as the economy is made large through replication, some interior proper correlated equilibria persist. For instance, the correlated-equilibrium allocation in Example (5.7) does not tend to a competitive allocation as the economy is replicated. In Aumann, Peck, and Shell (forthcoming), a two-commodity variant of Γ is used in which bankruptcy is impossible. A family of correlated equilibria is constructed with the property that as the number of consumers approaches infinity, significant uncertainty about allocations persists. The realized allocation is never a competitive-equilibrium allocation in this limit economy.

In the competitive economy, consumers take prices (of commodities traded on open markets) as given. In large market games, consumers have (little or) no influence on prices, but they do *not* take prices as given. Since the price is determined by the action of the other consumers, a consumer might make his bids and offers without knowing the bids and offers of the other consumers. That is, he might have to make his move without knowing prices. Asymmetric information is crucial to this distinction. When information is symmetric, each consumer knows the equilibrium bids and offers of the other consumers, so he knows the equilibrium prices.

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REFERENCES

ARROW, K. J. (1964), "The Role of Securities in the Optimal Allocation of Risk-bearing", *Review of Economic Studies*, **31**, 91-96.

AUMANN, R. J. (1974), "Subjectivity and Correlation in Randomized Strategies", *Journal of Mathematical Economics*, **1**, 67-96.

AUMANN, R. J. (1987), "Correlated Equilibrium as an Expression of Bayesian Rationality", *Econometrica*, **55**, 1-18.

AUMANN, R. J., PECK, J. and SHELL, K. (forthcoming), "Asymmetric Information and Sunspot Equilibria: A Family of Simple Examples", in Kuhn, H. W. (ed.), *Models of Incomplete Information and Bounded Rationality* (Heidelberg: Springer-Verlag).

BALASKO, Y. and CASS, D. (1989), "The Structure of Financial Equilibrium with Exogenous Yields: The Case of Incomplete Markets", *Econometrica*, **57**, 135-162.

CASS, D. and SHELL, K. (1983), "Do Sunspots Matter?", *Journal of Political Economy*, **91**, 193-227.

FORGES, F. (1990), "Sunspot Equilibrium as a Game-Theoretical Solution Concept" (CORE Discussion Paper No. 1029, Université Catholique de Louvain).

GEANAKOPLOS, J. and MAS-COLELL, A. (1989), "Real Indeterminacy with Financial Assets", *Journal of Economic Theory*, **47**, 22-38.

MAS-COLELL, A. (1982), "The Cournotian Foundations of Walrasian Equilibrium Theory: An Exposition of Recent Theory", in Hildenbrand, W. (ed.), *Advances in Economic Theory* (London and New York: Cambridge University Press), 183-224.

MANUELLI, R. and PECK, J. (1988), "Sunspot-like Effects of Random Endowments" (GSB Research Paper No. 1022, Stanford University).

MASKIN, E. and TIROLE, J. (1987), "Correlated Equilibria and Sunspots", *Journal of Economic Theory*, **43**, 364-373.

PECK, J. and SHELL, K. (1985), "Market Uncertainty: Sunspot Equilibria in Imperfectly Competitive Economies" (CARESS Working Paper No. 85-21, University of Pennsylvania).

PECK, J. and SHELL, K. (1989), "On the Nonequivalence of the Arrow-Securities Game and the Contingent-Commodities Game", in Barnett, W., Geneke, J. and Shell, K. (eds.), *Economic Complexity: Chaos, Sunspots, Bubbles, and Nonlinearity* (London and New York: Cambridge University Press), 61-85.

PECK, J. and SHELL, K. (1990), "Liquid Markets and Competition", *Games and Economic Behavior*, **2**, 362-377.

PECK, J., SHELL, K. and SPEAR, S. E. (1989), "The Market Game: Existence and Structure of Equilibrium" (CAE Working Paper No. 89-15RR, Cornell University).

SHAPLEY, L. (1976), "Noncooperative General Exchange," in Lin, S. A. Y. (ed.), *Theory and Measurement of Economic Externalities* (New York: Academic Press), 155-175.

SHAPLEY, L. and SHUBIK, M. (1977), "Trade Using One Commodity as a Means of Payment", *Journal of Political Economy*, **85**, 937-968.

SHELL, K. (1977), "Monnaie et Allocation Intertemporelle" (CNRS Séminaire d'Econométrie Roy-Malinvaud, Paris, mimeo). (Title and abstract in French; text in English.)

SHELL, K. and WRIGHT, R. (1989), "Indivisibilities, Lotteries, and Sunspot Equilibria" (CAE Working Paper No. 89-25, Cornell University).

SHUBIK, M. (1973), "A Theory of Money and Financial Institutions: Commodity, Money, Oligopoly, Credit and Bankruptcy in a General Equilibrium Model", *Western Economic Journal*, **11**, 24-38.