The Role of Securities in the Optimal Allocation of Risk-bearing[1]

I INTRODUCTION

The theory of the optimal allocation of resources under conditions of certainty is well-known. In the present note, an extension of the theory to conditions of subjective uncertainty is considered.

Attention is confined to the case of a pure exchange economy; the introduction of production would not be difficult. We suppose I individuals, and S possible states of nature. In the sth state, amount x_{sc} of commodity c ($c=1, \ldots, C$) is produced. It is assumed that each individual acts on the basis of subjective probabilities as to the states of nature; let π_{is} be the subjective probability of state s according to individual i. Further, let x_{isc} be the amount of commodity claimed by individual i if state s occurs. These claims are, of course, limited by available resources, so that

$$(1) \qquad \sum_{i=1}^{I} x_{isc} = x_{sc}$$

assuming the absence of saturation of individuals' desires.

The problem of optimal allocation of risk-bearing is that of choosing the magnitudes x_{isc}, subject to restraints (1), in such a way that no other choice will make every individual better off. In Section 2, it is briefly argued that, if there exists markets for claims on all commodities, the competitive system will lead to an optimal allocation under certain hypotheses.

However, in the real world the allocation of risk-bearing is accomplished by claims payable in money, not in commodities. In Section 3, it is shown that the von Neumann-Morgenstern theorem enables us to conclude that, under certain hypotheses, the allocation of risk-bearing by competitive securities markets is in fact optimal.

In Section 4, it is shown that the hypotheses used in Sections 2 and 3 contain an important implication: *that the competitive allocation of risk-bearing is guaranteed to be viable only if the individuals have attitudes of risk-aversion.*[2]

II ALLOCATION OF RISK-BEARING BY COMMODITY CLAIMS

Let V_i ($x_{i11}, \ldots, x_{i1C}, x_{i21}, \ldots, x_{iSC}$) be the utility of individual i if he is assigned claims of amount x_{isc} for commodity c if state s occurs ($c = 1, \ldots, C$; $s = 1, \ldots, S$). This is exactly analogous to the utility function in the case of certainty, except that the number of variables has increased from C to SC. We may therefore achieve any optimal

[1] This paper was originally read at the Colloque sur les Fondements et Applications de la Theorie du Risque en Econometrie of the Centre Nationale de la Recherche Scientifique, Paris, France, on May 13, 1952 and appeared in French in the proceedings of the colloquium, published by the Centre Nationale under the title, *Économétric*, 1953. The research was carried out under contract Nour-225 (50) of the U.S. Office of Naval Research at Stanford University.

[2] Note added for this translation. Since the above was written I have come to the conclusion that this statement needs very severe qualification as explained in footnote [1], page 96.

allocation of risk-bearing by a competitive system. Let x^*_{isc} $(i = 1, \ldots, I;\ s = 1, \ldots, S;\ c = 1, \ldots, C)$ be any optimal allocation; then there exist a set of money incomes y_i for individual i, and prices $\bar{p}_{sc}$ for a unit claim on commodity c if state s occurs, such that if each individual i chooses values of the variables x_{isc} $(s = 1, \ldots, S;\ c = 1, \ldots, C)$ subject to the restraint

$$\sum_{s=1}^{S} \sum_{c=1}^{C} \bar{p}_{sc}\, x_{isc} = y_i \tag{2}$$

taking prices as given, the chosen values of the x_{isc}'s will be the given optimal allocation x^*_{isc} $(i = 1, \ldots, I;\ s = 1, \ldots, S;\ c = 1, \ldots, C)$.

The argument is a trivial reformulation of the usual one in welfare economics.[1] However, there is one important qualification; the validity of the theorem depends on the assumption (not always made explicitly) that the indifference surfaces are convex to the origin, or, to state the condition equivalently, that $V_i\,(x_{i11}, \ldots, x_{iSC})$ is a *quasi-concave* function of its arguments. [The function $f(x_1, \ldots, x_n)$ is said to be quasi-concave if for every pair of points $(x_1, \ldots, x_n)$ and $(x_1^1, \ldots, x_n^1)$ such that $f(x_1^1, \ldots, x_n^1) \geq f(x_1^2, \ldots, x_n^2)$ and every real number α,

$$0 \leqslant \alpha \leqslant 1;\ f(\alpha x_1^1 + (1-\alpha)x_1^2, \ldots, \alpha x_n^1 + (1-\alpha)x_n^2) \geq f(x_1^2, \ldots, x_n^2).$$

It is easy to see geometrically the equivalence between this definition and the convexity of the indifference surfaces.]

Theorem 1. If $V_i\,(x_{i11}, \ldots, x_{iSC})$ is quasi-concave for every i, then any optimal allocation of risk-bearing can be realized by a system of perfectly competitive markets in claims on commodities.

The meaning of the hypothesis of Theorem 1 will be explored in Section 4.

III ALLOCATION OF RISK-BEARING BY SECURITIES

In the actual world, risk-bearing is not allocated by the sale of claims against specific commodities. A simplified picture would rather be the following: securities are sold which are payable in money, the amount depending on the state s which has actuallyoccurred (this concept is obvious for stocks; for bonds, we have only to recall the possibility of default if certain states s occur); when the state s occurs, the money transfers determined by the securities take place, and then the allocation of commodities takes place through the market in the ordinary way, without further risk-bearing.

It is not difficult to show that any optimal allocation of risk-bearing can be achieved by such a competitive system involving securities payable in money. For the given optimal allocation, x^*_{isc}, let the prices $\bar{p}_{sc}$ and the incomes y_i be determined as in the previous section. For simplicity, assume there are precisely S types of securities, where a unit security of the sth type is a claim paying one monetary unit if state s occurs and nothing otherwise. Any security whatever may be regarded as a bundle of the elementary types just described.

Let q_s be the price of the sth security and p_{sc} the price of commodity c is state s occurs. Choose them so that

$$q_s p_{sc} = \bar{p}_{sc}. \tag{3}$$

An individual confronted with these prices has the same range of alternatives available as he did under the system described in Section 2, taking $q_s p_{sc}$ as equivalent to the price of a claim on commodity c in state s. He will plan to acquire the same claims, and therefore,

[1] See, for a simple exposition, O. Lange, " The Foundation of Welfare Economics ", *Econometrica*, Volume 10 (1942), pp. 215-228, or P.A. Samuelson, *Foundations of Economic Analysis*, Chapter VIII.

on the market for securities of the sth type, individual i will purchase sufficient securities of type s to realize the desired purchase of commodities if state s occurs, ie., he will purchase

(4) $$y_{is}^{*} = \sum_{c=1}^{C} p_{sc} x_{isc}^{*}$$

units of the sth type of security. His purchase of securities of all types is restricted by the restraint

$$\sum_{s=1}^{S} q_s y_{is} = y_i$$

the allocation y_{is}^{*} $(s = 1, \ldots, S)$ satisfies this restraint, as can be seen from (2), (3), and (4).

The total monetary stock available is $\sum_{i=1}^{I} y_i = y$. The net volume of claims payable when any state s occurs must therefore be precisely y or

(5) $$\sum_{i=1}^{I} y_{is} = y \qquad (s = 1, \ldots, S).$$

Substitute (4) into (5) and multiply both sides by q_s/y, then, from (3),

(6) $$q_s = \frac{\sum_{i=1}^{I} \sum_{c=1}^{C} \bar{p}_{sc} x_{isc}^{*} \qquad (s = 1, \ldots, S)}{y}.$$

The prices p_{sc} are then determined from (3).

With the prices q_s and $\bar{p}_{sc}$ thus determined, and the incomes y_i, the competitive system, operating first on the securities markets and then on the separate commodity markets, will lead to the allocation x_{isc}. For, as we have already seen, individual i will demand y_{is}^{*} of security s. Suppose state s occurs. He then has income y_{is} to allocate among commodities with prices $\bar{p}_{sc}$. Let $U_i\,(x_{is1}, \ldots, x_{isC})$ be a utility function of individual i for commodities, then he chooses a bundle so as to maximize U_i subject to the restraint

(7) $$\sum_{c=1}^{C} p_{sc} x_{isc} = y_{is}.$$

Let x_{isc}^{+} $(c = 1, \ldots, C)$ be the chosen commodity amounts. Since by (4), the quantities x_{isc}^{+} satisfy (7), it follows from the definition of a maximum that,

(8) $$U_i(x_{is1}^{+}, \ldots, x_{isC}^{+} \geq U_i(x_{iis1}^{*}, \ldots, x_{isC}^{*}).$$

The quantities x_{isc}^{+} are defined for all s.

By the von Neumann-Morgenstern theorem, the function U_i may be chosen so that

(9) $$V_i(X_{i11}, \ldots, X_{iSC}) = \sum_{s=1}^{S} \pi_{is} U_i(x_{is1}, \ldots, x_{isC}).$$

Suppose that in (8), the strict inequality holds for at least s for which $\pi_{is} > 0$. Then, by (9),

(10) $$V_i(x_{i11}^{+}, \ldots, x_{iSC}^{+}) > V_i(x_{i11}^{*}, \ldots, x_{iSC}^{*}).$$

On the other hand, if we multiply in (7) by q_s and sum over s, it is seen that the bundle of claims $(x_{i11}^{+}, \ldots, x_{iSC}^{+})$ satisfies restraint (2). But by construction the bundle $(x_{i11}^{*}, \ldots, x_{iSC}^{*})$ maximizes V_i subject to (2); hence (10) is a contradiction, and the equality holds in (8) for all states with positive subjective probability. If the *strict* quasi-concavity of U_i is assumed, as usual, the equality implies that $x_{isc}^{+} = x_{isc}^{*}$ for all c and all i and s for which $\pi_{is} < 0$. If $\pi_{is} = 0$, then obviously $x_{isc}^{+} = 0$ $(c = 1, \ldots, C)$, which implies that $y_{is} = 0$ and therefore $x_{isc}^{*} = 0$ $(c = 1, \ldots, C)$. Hence, once the state s occurs, individual i will in fact purchase the bundle prescribed under the optimal allocation.

Theorem 2. If $\sum_{s=1} \pi_{is} U_i(X_{is1}, \ldots, X_{isC})$ is quasi-concave, in all its variables, when any optimal allocation of risk-bearing can be achieved by perfect competition on the securities and commodity markets, where securities are payable in money.

Socially, the significance of the theorem is that it permits economizing on markets; only $S + C$ markets are needed to achieve the optimal allocation, instead of the SC markets implied in Theorem 1.

One might wonder if any loopholes have been left through arbitrage between securities and hold of money; in the allocation of securities, an individual has the option of holding cash instead and using the hoarding in the commodity allocation.

If we sum over s in (6) and use (2),

$$\sum_{s=1}^{S} q_s = \frac{\sum_{i=1}^{I} \sum_{s=1}^{S} \sum_{c=1}^{C} \bar{p}_{sc} x_{isc}^{*}}{y} = \frac{\sum_{i=1}^{I} y_i}{y} = 1. \tag{11}$$

A monetary unit is equivalent to a bundle of S unit securities, one of each type; to avoid arbitrage, then, such a bundle should have a unit price. This is insured by (11).

IV RISK-AVERSION AND THE COMPETITIVE ALLOCATION OF RISK-BEARING

What is the economic significance of the hypothesis that the utility-functions:

$$V_i = \sum_{s=1}^{S} \pi_{is} U_i$$

be quasi-concave? The easiest case to consider is that in which

$$S = 2, \ \pi_{is} = 1/2 \qquad (s = 1, 2).$$

Theorem 3. If $\frac{1}{2}[f(x_1, \ldots, x_C) + f(x_{C+1}, \ldots, x_{2C})]$ is quasi-concave in all its variables, then $f(x_1, \ldots, x_C)$ is a concave function.

[$f(x_1, \ldots, x_C)$ will be said to be concave if for every pair of points $(x_1^1, \ldots, x_C^1)$ and $(x_1^2, \ldots, x_C^2)$,

$$f(\tfrac{1}{2}x_1^1 + \tfrac{1}{2}x_1^2, \ldots, \tfrac{1}{2}x_C^1 + \tfrac{1}{2}x_C^2) \geq \tfrac{1}{2}[f(x_1^1, \ldots, x_C^1) + \tfrac{1}{2}f(x_1^2, \ldots, x_C^2).]$$

It is well known that a concave function is always quasi-concave but not conversely.]

Proof: Suppose $f(x_1, \ldots, x_C)$ is not concave. Then for some pair of points, $(x_1^1, \ldots, x_C^1)$ and $(x_1^2, \ldots, x_C^2)$,

$$f(\frac{x_1^1 + x_1^2}{2}, \ldots, \frac{x_C^2 + x_C^2}{2}) < \tfrac{1}{2}f(x_1^1, \ldots, x_C^1) + \tfrac{1}{2}f(x_1^2, \ldots, x_C^2). \tag{12}$$

Let

$$g(x_1, \ldots, x_{2C}) = \tfrac{1}{2}[f(x_1, \ldots, x_C) + f(x_{C+1}, \ldots, x_{2C})]. \tag{13}$$

Then obviously

$$g(x_1^1, \ldots, x_C^1, x_1^2, \ldots, x_C^2) = g(x_1^2, \ldots, x_C^2, x_1^1, \ldots, x_C^1). \tag{14}$$

By the hypothesis that g is quasi-concave, then,

$$g\,(\frac{x_1^1 + x_1^2}{2}, \ldots, \frac{x_C^1 + x_C^2}{2}, \frac{x_1^2 + x_1^1}{2}, \ldots, \frac{x_C^2 + x_C^1}{2}) \tag{14}$$

$$\geq g(x_1, \ldots, x_C^1, x_1^2, \ldots, x_C^2).$$

But from (13) and (12),

$$g\left(\frac{x_1^1+x_1^2}{2},\ldots,\frac{x_C^1+x_C^2}{2},\frac{x_1^2+x_1^1}{2},\ldots,\frac{x_C^2+x_C^1}{2}\right)$$

$$=\tfrac{1}{2}\left[f\left(\frac{x_1^1+x_1^2}{2},\ldots,\frac{x_C^1+x_C^2}{2}\right)+f\left(\frac{x_1^2+x_1^1}{2},\ldots\frac{x_C^2+x_C^1}{2}\right)\right]$$

$$=f\left(\frac{x_1^1+x_1^2}{2},\ldots,\frac{x_C^1+x_C^2}{2}\right)$$

$$<\tfrac{1}{2}[f(x_1^1,\ldots,x_C^1)+f(x_1^2,\ldots,x_C^2)]$$

$$=g(x_1^1,\ldots,x_C^1,x_1^2,\ldots,x_C^2),$$

which contradicts (14). Hence, $f(x_1, \ldots, x_C)$ must be concave.

In terms of the allocation of risk-bearing, Theorem 3 implies that if one wishes to insure the viability of the competitive allocation for all possible assignments of probabilities π_{is}, it must be assumed that the individual utility functions U_i must be concave. This condition, in turn, is obviously equivalent to the assumption of risk-aversion; for the conditions

$$U_i\left(\frac{x_1^1+x_1^2}{2},\ldots,\frac{x_C^1+x_C^2}{2}\right)\geq\tfrac{1}{2}[U_i(x_1^1,\ldots,x_C^1)+U_i(x_1^2,\ldots,x_C^2)]$$

means that an even gamble as between two bundles is never preferred to the arithmetic mean of those bundles.

The hypothesis of quasi-concavity of the utility function has here only been indicated as a sufficient, not a necessary condition for the viability of competitive allocation. However, without the assumption of quasi-concavity, some optimal allocations cannot be achieved by competitive means, and in general, there would be only very special cases in which any competitive equilibrium is achievable. Consider the following simple examples:

There are one commodity, two individuals, and two states. Both individuals have the same utility function.

(15) $$U_i(x)=x^2 \qquad (i=1,2);$$

this function is monotonic and hence quasi-concave, but not concave, since it implies risk-preference. Assume further that $\pi_{is}=\frac{1}{2}(i=1,2;\ s=1,2)$ then

(16) $$V_i(x_{i11},x_{i21})=\tfrac{1}{2}(x_{i11}^2+x_{i21}^2) \qquad (i=1,2).$$

Finally, suppose that

(17) $$x_{11}=1, \quad x_{21}=2.$$

It is easy to see that for any fixed set of prices on claims under alternative states, each individual will buy all of one claim or all of the other. Hence any optimal allocation in which both individuals possess positive claims in both states is unachievable by competitive means. Such optimal allocations do exist; we have only to choose the variables x_{is1} $(i=1,2;\ s=1,2)$ so as to maximise V_i subject to the restraints, implied by (17),

$$x_{111}+x_{211}=1, \quad x_{121}+x_{221}=2,$$

and the restraint $V_2=$ constant. If, for example, we fix $V_2=1$, we have the optimal allocation,

$$x_{111}=\frac{\sqrt{5}-1}{\sqrt{5}},\ x_{121}=\frac{2\sqrt{5}-1}{5},\ x_{211}=\frac{1}{\sqrt{5}},\ x_{221}=\frac{2}{\sqrt{5}}.$$

In fact, for the functions given by (16), a competitive equilibrium usually does not exist. Let y_i ($i = 1, 2$) be the incomes of the two individuals. Let p be the price of a unit claim for state 1, taking the unit claim in state 2 as *numéraire*. Then in a competitive market, individual i maximises V_i subject to

$$px_{i11} + x_{i21} = y_i.$$

He will then choose $x_{i11} = \frac{y_i}{p}$, $x_{i21} = 0$ if $p < 1$ and choose $x_{i11} = 0$, $x_{i21} = y_i$ if $p > 1$.

Hence, if $p \neq 1$, there will be zero demand, and hence disequilibrium, on one market. If $p = 1$, each individual will be indifferent between the bundles $(y_i, 0)$ and $(0, y_i)$. Except in the special case where $y_1 = 1$, $y_2 = 2$ (or *vice versa*), there is again no possible way of achieving equilibrium.[1]

K. J. ARROW.

[1] Though there is nothing wrong formally with the analysis of this last section, I now consider it misleading. If there are a large number of consumers, the income of each being relatively small, it has now been established by the important work of Farrell and Rothenberg, that the quasi-concavity of the indifference curves is unnecessary to the existence of competitive equilibrium; see M. J. Farrell, "The Convexity Assumption in the Theory of Competitive Markets", *Journal of Political Economy*, Volume 67 (1954), pp. 377-391, and J. Rothenberg, "Non-Convexity, Aggregation and Pareto Optimality", *Ibid.*, Volume 68 (1960), pp. 435-468.