Connections between Futures Market Economy and Money Market Economy

One good per period, \( \ell = 1 \), two periods, \( t = 1, 2 \).

**Futures Market:**

\[
\max_u u_h(x^1_h, x^2_h) \\
\text{s.t. } p^1 x^1_h + p^2 x^2_h = p^1 \omega^1_h + p^2 \omega^2_h
\]

Equilibrium is a price vector \( (p^1, p^2) \) such that

\[
\sum_h x^t_h = \sum_h \omega^t_h \text{ for } t = 1, 2.
\]

Define the interest factor \( R \) and the interest rate \( r \) in terms of the equilibrium commodity prices \( (p^1, p^2) \).
Money Market:

$$\max \ u_h(x_h^1, x_h^2)$$

\[\begin{align*}
    &s.t. \quad p^1 x_h^1 + p^m m_h^1 = p^1 \omega_h^1 \\
    &\quad p^2 x_h^2 + p^m m_h^2 = p^2 \omega_h^2
\end{align*}\]

Equilibrium \(\left(p^1, p^2, p^m, p^m\right)\) such that

$$\sum_h x_h^t = \sum_h \omega_h^t \text{ and } \sum_h m_h^t = 0 \text{ for } t = 1, 2.$$ 

0.

From the definitions of present prices, the interest factor and the interest rate, we have:

$$p^1 = 1, \quad (p^1, p^2) = (1, p^2).$$

$$p_2 = \frac{1}{R} = \frac{1}{1+r}.$$
1.

Prove that in equilibrium $p^{m_1} = p^{m_2} = p^m \geq 0$. This is a no-arbitrage-property result.

Answer:

Consumer h maximizes his financial wealth through arbitrage:

$$\max W_h = p^{m_1} m_h^1 + p^{m_2} m_h^2$$

s.t.

$$m_h^1 + m_h^2 = 0$$

$$m_h^2 = -m_h^1$$

$$W_h = (p^{m_1} - p^{m_2}) m_h^1.$$ 

If $p^{m_1} > p^{m_2}$, $m_h^1 > 0$.
Borrow (high) in period 1 and re-pay (low) in period 2.

If $p^{m_2} > p^{m_1}$, $m_h^1 > 0$.

If $p^{m_1} \neq p^{m_2}$, $W_h$ can be arbitrarily large by choosing $|m_h^1|$ arbitrarily large. This allows $x_h^1$ and $x_h^2$ to be arbitrarily large.

So,

$$\sum_h x_h^t > \sum_h \omega_h^t \text{ for } t = 1, 2.$$ 

Hence, $p^{m_1} = p^{m_2}$ is required for competitive equilibrium. Let

$$p^m = p^{m_1} = p^{m_2} \geq 0.$$
2.

Show that if, \((x^1_h, x^2_h), h = 1, \ldots, n\) solves the futures market problem, it also solves the money market problem.

**Proof:**

Excess demand is \(z'_h = x'_h - \omega'_h\). \(p^1 = 1\).

\[
\begin{align*}
z^1_h + p^2 z^2_h &= 0 \\
z^1_h &= -p^2 z^2_h \\

\end{align*}
\]

Define the scalar \(k\) by:

\[
k = z^1_h - p^2 z^2_h.
\]

Set

\[
k = p^m m^1_h = -p^m m^2_h.
\]

Hence,

\[
\begin{align*}
z^1_h + p^m m^2_h &= 0 \\
p^2 z^2_h + p^m m^2_h &= 0 \\
m^1_h + m^2_h &= 0.
\end{align*}
\]

Hence if \((x^1_h, x^2_h)\) is an equilibrium allocation in the futures market, it is also an equilibrium in the money market.
Show that if, \((x_h^1, x_h^2), \ h = 1, \ ..., \ n\) solves the money market problem with \(p^m > 0\), then it also solves the futures market problem.

\[
m_h^1 + m_h^2 = 0, \ m_h^2 = -m_h^1.
\]
\[
p^m > 0.
\]
\[
z_h^1 = -p^m m_h^1, \quad p^2 z_h^2 = p^m m_h^1.
\]
\[
\frac{z_h^1}{p^m} = -m_h^1, \quad \frac{p^2 z_h^2}{p^m} = m_h^1.
\]

\(m_h^1\) is a slack variable.

Hence, we have
\[
z_h^1 + p^2 z_h^2 = 0.
\]

Hence, if \((x_h^1, x_h^2)\) is an equilibrium allocation in the money market economy with \(p^m > 0\), then \((x_h^1, x_h^2)\) is also an equilibrium allocation in the corresponding futures market economy.