Price-Level Volatility, Optimal Taxation and Voting

April 11, 2016

Abstract

1. Introduction

2. The Economy

There is one period and one consumption good in the economy. There are three consumers, $h \in H = \{1, 2, 3\}$. The consumption set of each type of consumer is $X_h = \mathbb{R}_+$. The endowment of type h is $\omega_h \geq 0$. There exists income endowment inequality such that $\omega_1 < \omega_2 < \omega_3$. Consumer 1 is poor, and consumer 3 is the richest. Consumer 2 has the average income of the economy, i.e., $\omega_2 = (\omega_1 + \omega_2 + \omega_3)/3$. The income inequality provides an incentive for the government to use income-redistribution policies.

Three consumers have identical preferences which are represented by a vMN utility function $u_h : X_h \to \mathbb{R}$. We assume that

$$u_h(x) = \log(x_h).$$

These preferences ensure that the equilibrium is unique. We introduce sunspots in the economy. The idea of extrinsic uncertainty can be illustrated with two extrinsic states of nature $s = \alpha, \beta$, that occur with probability $\pi(\alpha), \pi(\beta), 0 < \pi(\alpha) < 1, \pi(\beta) = 1 - \pi(\alpha)$. Mr. h maximizes his expected utility

$$V_h = \pi(\alpha)\log(x_h(\alpha)) + \pi(\beta)\log(x_h(\beta)) \qquad \text{for} \quad h = 1, 2, 3.$$

The social welfare function W is the sum of the individual expected utilities:

$$W = V_1 + V_2 + V_3.$$

The maximum possible social welfare is $W = \log\left(\frac{\omega_1+\omega_2+\omega_3}{3}\right)$, which can be achieved when the tax-authority levels all individuals' expected utility values under money taxation. See CGKS (2015).

3. Money Taxation

We introduce endogenous taxes that are denominated in units of money with: $\tau \in \mathbb{R}^3$ where $\tau = (\tau_1, \tau_2, \tau_3)$. Each individual's lump-sum dollar tax is also independent of s, $\tau_h(\alpha) = \tau_h(\beta) = \tau_h$. If τ_h is negative, he is subsidized. If τ_h is zero, he is neither taxed nor subsidized. The tax and transfer plan is balanced, i.e., $\tau_1 + \tau_2 + \tau_3 = 0$.

We work in the traditional framework of economic policy formulation where consumers form price expectations and the policy maker then chooses the tax policy. Given these expectations an equilibrium outcome is realized. In equilibrium, the price expectations of consumers must be consistent with the equilibrium outcome: rational expectations must hold. Figure 1 is the time-line.

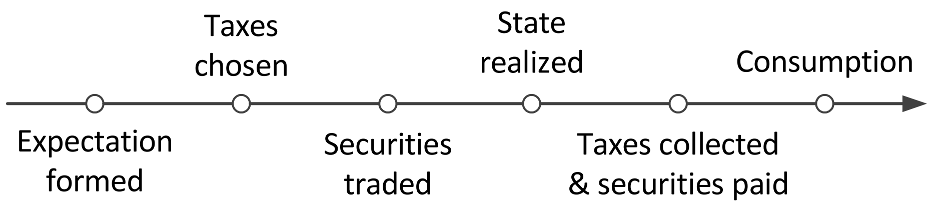

Figure 1: The time line

Here are three basic cases based on the pattern of the asset market restrictions: (U) unrestricted security market participation allowing for perfect risk-sharing among the 3 consumers, (I) Incomplete securities-market participation allowing for risk-sharing between 2 of the consumers but not the third, and (R) Fully restricted securities-market participation, in which none of the consumers can hedge against price-level fluctuations. CGKS (2015) show that the social welfare is lower as (1) the asset market is more restricted and (2) the price-level volatility is higher.

In the case of perfect risk-sharing, sunspots do not matter and the first-best social welfare is achieved. The most interesting cases are when some consumers are restricted and others are not. Consider, for example, the case in which Mr 2 and Mr 3 have access to the security market and Mr. 1 does not.

Let $p(s)$ be the ex-ante (accounting) price of chocolate delivered in state s and $p^m(s)$ be ex-ante (accounting) price of money delivered in state s. Then $P^m(s) = p(s)/p^m(s)$ is the chocolate price of money in s, while $1/P^m(s)$ is the money price of chocolate in s, or the general price level in s.

The problem of restricted consumer 1 is simple. He chooses $x_1(s) > 0$ to

$$\text{maximize } \log(x_1(s))$$

subject to

$$p(s)x_1(s) = p(s)\omega_1 - p^m(s)\tau_1$$

for $s = \alpha, \beta$.

Define the tax-adjusted endowment $\widetilde{\omega}_h(s) = \omega_h - P^m(s)\tau_h$. Then, *Mr* 1's budget constraint reduces to

$$x_1(s) = \widetilde{\omega}_1(s)$$

for $s = \alpha, \beta$. Mr 1 is passive: he consumes his tax-adjusted endowment in each state.

Mr 2 and Mr 3 trade in the securities market and the spot market. Each faces a single budget constraint. Mr h's problem is to choose $(x_h(\alpha), x_h(\beta)) > 0$ to

$$\text{maximize } V_h,$$

subject to

$$p(\alpha)x_h(\alpha) + p(\beta)x_h(\beta) = (p(\alpha) + p(\beta))\,\omega_h - (p^m(\alpha) + p^m(\beta))\,\tau_h, \tag{1}$$

for $h = 2, 3$. From the first-order conditions, we have

$$\frac{p(\beta)}{p(\alpha)} = \frac{\pi(\beta)\,x_2(\alpha)}{\pi(\alpha)\,x_2(\beta)} = \frac{\pi(\beta)\,x_3(\alpha)}{\pi(\alpha)\,x_3(\beta)}. \tag{2}$$

Market clearing implies

$$x_1(s) + x_2(s) + x_3(s) = \omega_1(s) + \omega_2(s) + \omega_3(s),$$

or simply

$$x_1(s) + x_2(s) + x_3(s) = \widetilde{\omega}_1(s) + \widetilde{\omega}_2(s) + \widetilde{\omega}_3(s), \tag{3}$$

for $s = \alpha, \beta$. Since $x_1(s) = \widetilde{\omega}_1(s)$, so we obtain

$$x_2(s) + x_3(s) = \widetilde{\omega}_2(s) + \widetilde{\omega}_3(s) \quad \text{for } s = \alpha, \beta. \tag{4}$$

Equation (4) defines the relevant tax-adjusted Edgeworth box (typically a proper rectangular). The set of equilibria is typically very large, but we focus on a sub-set in which volatilities can be ranked. We measure volatility by the mean-preserving spread parameter σ defined by

$$P^m(\alpha) = P^m - \frac{\sigma}{\pi(\alpha)}$$

$$P^m(\beta) = P^m + \frac{\sigma}{\pi(\beta)}$$

where P^m is the non-sunspot equilibrium chocolate price of dollars and σ belongs to $[0, \pi(\alpha)\,P^m)$. When $\sigma = 0$, the equilibrium allocations are not affected by sunspots (a non-sunspots economy). When $\sigma > 0$, the economy is a proper sunspots economy. State α is the inflationary state: a dollar buys less chocolate in state α than in state β. State β is the deflationary state: a dollar buys more chocolate in β than in α.

CGKS (2015) shows that as volatility σ increases, social welfare decreases when some consumers are restricted from asset-market participation. Under perfect risk-sharing, the maximum social welfare is always achieved for any level of volatility σ.

4. Who will be subsidized and taxed?

This section investigates whether each individual is taxed or subsidized when the government implements income-redistribution policy based on monetary taxes-and-subsidies. The taxed guy would be better off through monetary income-redistribution policy while the subsidized guy would be worse off. It is somewhat trivial that the rich (consumer 3) is taxed and the poor (consumer 1) is subsidized (see Lemma 1 for the proof). However, the middle-income guy can be subsidized or taxed based on the structure of asset-market-restriction. Because the poor one supports the income-redistribution policy and the rich one objects the policy, whether to implement the policy or not is determined by the middle-income guy. When individuals need to vote between no-policy vs. monetary redistribution policy, the no-policy (monetary distribution policy) will be finally selected if the middle-income guy is taxed (subsidized) under the monetary redistribution policy.

The following lemma shows that for any asset-market restriction structure and for any price-volatility level, under the monetary income-redistribution policy the rich one is always taxed and the poor one is always subsidized.

Lemma 1 *For any volatility level σ and for any asset-market restrictions (U,I,R), the rich one is taxed and the poor one is subsidized under the monetary income-redistribution policy.*

Proof. By contradiction, the rich (Mr 3) is not taxed under fully restricted securities-market participation. Denote $\widetilde{P}^m$ as a random variable such that $\widetilde{P}^m = \{P^m(\alpha), P^m(\beta); \pi(\alpha), \pi(\beta)\}$. The tax-authority's maximization problem can be expressed as

$$W = \max_{\tau} E \log\left(\omega_1 - \widetilde{P}^m \tau_1\right) + E \log\left(\omega_2 - \widetilde{P}^m \tau_2\right) + E \log\left(\omega_3 - \widetilde{P}^m \tau_3\right)$$

subject to

$$\tau_1 + \tau_2 + \tau_3 = 0.$$

Defining $V_3(\tau_3; R)$ as consumer 3's expected utility under fully restricted securities-market participation, the cost of taxation at $\tau_3 = 0$ is

$$-\frac{\partial V_3(0; R)}{\partial \tau_3} = \frac{P^m}{\omega_3}. \tag{5}$$

For any level of taxation $\tau_2 = 0$, the benefit of decreased taxation (subsidy) for Mr 2 is that

$$\frac{\partial V_2(0; R)}{\partial (-\tau_2)} = \frac{P^m}{\omega_2}. \tag{6}$$

Because $\omega_3 > \omega_2$, the cost is smaller than the benefit, which implies that the marginal increase in taxation for Mr 3 increases the social welfare. This contracts that the rich is not taxed. In the same way, we can prove that the poor one is always subsidized under fully restricted securities-market.

For the small increase/decrease in taxation at $\tau = 0$, the utility changes for all three cases of asset-market participation are identical. That is

$$\frac{\partial V_i(0; U)}{\partial \tau_i} = \frac{\partial V_i(0; I)}{\partial \tau_i} = \frac{\partial V_i(0; R)}{\partial \tau_i} \text{ for } i = 1, 2, 3.$$

because where $\tau = 0$, there is no uncertainty in tax-adjusted endowment and therefore, no need to do risk-sharing. Therefore, for the cases of perfect risk-sharing and partially risk-sharing, we can prove the rich is taxed and the poor is subsidized in the same way. ■

We will first check the simple case where all consumers are doing perfect risk-sharing in the market. Under the perfect risk-sharing, the maximum welfare, $W = \log\left(\frac{\omega_1+\omega_2+\omega_3}{3}\right)$, is achieved and the middle-income guy would not be taxed nor subsidized.

Proposition 1 *Under the perfect risk-sharing, the middle-income guy (Mr 2) is not taxed nor subsidized.*

Proof. Where the 3 consumers do perfect risk sharing, $p(\alpha)$ and $p(\beta)$ are invariant in σ:

$$\frac{p(\beta)}{p(\alpha)} = \frac{\pi(\beta)}{\pi(\alpha)}.$$

Each consumer chooses $x_h(\alpha) = x_h(\beta)$, because we have

$$\frac{x_h(\alpha)}{x_h(\beta)} = \frac{p(\beta)/\pi(\beta)}{p(\alpha)/\pi(\alpha)} = 1.$$

With $x_h(\alpha) = x_h(\beta)$, the equilibrium V_h can be expressed as

$$V_h = \log\{\omega_h - (P^m(\alpha) + P^m(\beta))\,\tau_h\}$$

and social welfare can be expressed as

$$W(U) = \max_{\tau_1,\tau_2,\tau_3} \sum_{h\in H} \log\{\omega_h - (P^m(\alpha) + P^m(\beta))\,\tau_h\}$$

$$\text{subject to } \tau_1 + \tau_2 + \tau_3 = 0.$$

By the first order conditions, we obtain

$$\frac{-(P^m(\alpha) + P^m(\beta))}{\omega_1 - (P^m(\alpha) + P^m(\beta))\,\tau_1} = \frac{-(P^m(\alpha) + P^m(\beta))}{\omega_2 - (P^m(\alpha) + P^m(\beta))\,\tau_2} = \frac{-(P^m(\alpha) + P^m(\beta))}{\omega_3 - (P^m(\alpha) + P^m(\beta))\,\tau_3},$$

which implies that

$$\begin{aligned} x_1(\alpha) &= x_1(\beta) = x_2(\alpha) = x_2(\beta) = x_3(\alpha) = x_3(\beta) \\ &= \frac{\omega_1 + \omega_2 + \omega_3}{3}, \end{aligned} \tag{7}$$

and therefore, we have $x_2(\alpha) = x_2(\beta) = \omega_2$. ∎

Now, we move to the case of partially restricted asset-market participation. There are three possible cases: (1) Mr 1 and 3 do risk-sharing but Mr 2 does not, (2) Mr 1 and Mr 2 (the poor one and middle-income guy) do risk-sharing and Mr 3 does not, and the most interesting case, Mr 2 and 3 (the middle income and the rich) do risk-sharing but Mr 1 does not.

The common fact in three cases is that optimization entails leveling the expected utilities among the group of consumers who have access to the hedging market, which is shown in the following Proposition.

Proposition 2 *(Proposition 2 in CGKS 2016) In a partially restricted market, if Mr h and Mr h′ do risk-sharing, we have* $x_h(\alpha) = x_{h'}(\alpha)$ *and* $x_h(\beta) = x_{h'}(\beta)$ *and, therefore,* $V_h = V_{h'}$

Proof. For the detailed proof, see CGKS (2015). ■

We first check the case where the poor one and the rich one do risk-sharing under the partially restricted asset-market participation.

Proposition 3 *When Mr 1 and 3 do risk-sharing and Mr 2 does not, Mr 2 is not taxed nor subsidized.*

Proof. From Proposition 2, we know that $V_1 = V_3$. Then, the maximization problem is that

$$\begin{aligned} \max_{\tau_2} V_1 + V_2 + V_3 &= \max_{\tau_2} 2V_1 + V_2 \\ &= \max_{\tau_2} 2E \log\left(\frac{\omega_1 + \omega_3 + \widetilde{P}^m \tau_2}{2}\right) + E \log\left(\omega_2 - \widetilde{P}^m \tau_2\right). \end{aligned}$$

By the first-order condition, we have

$$2E \frac{\widetilde{P}^m}{\omega_1 + \omega_3 + \widetilde{P}^m \tau_2} - E \frac{\widetilde{P}^m}{\omega_2 - \widetilde{P}^m \tau_2} = 0. \tag{8}$$

Because $\omega_2 = (\omega_1 + \omega_3)/2$, equation (8) can be expressed as

$$E \frac{\widetilde{P}^m}{\omega_2 + \widetilde{P}^m \frac{\tau_2}{2}} - E \frac{\widetilde{P}^m}{\omega_2 - \widetilde{P}^m \tau_2} = 0. \tag{9}$$

The solution for equation (9) is $\tau_2 = 0$. ■

In the two cases where the asset market is perfectly open to everyone and it is closed for the middle-income guy, the maximum social welfare is achieved and the middle-income guy is not taxed nor subsidized. However, under restricted asset-market participation, the middle-income guy can be taxed or subsidized, which is shown in the following section.

5. When is the middle-income guy taxed or subsidized?

In the section, we will investigate whether the middle-income guy is taxed or subsidized under the 2 scenarios of asset-market participation: (1) the poor one is restricted from asset-market participation, and (2) the rich one is restricted from asset-market participation. When Mr 1 (the poor) does not participate in the asset market, the price-level volatility resulting from the tax policy is directly connected to Mr 1's consumption volatility. Therefore, the monetary subsidy to Mr 1 becomes inefficient in increasing social welfare and, consequently, the tax authority decreases subsidies to Mr 1. The decreased Mr 1's subsidy enables Mr 2 to be subsidized, which is shown in the following proposition.

Proposition 4 *Where the middle income-guy (Mr 2) and the rich (Mr 3) do risk-sharing but the poor (Mr 1) does not, the middle income-guy is subsidized.*

Proof. Directly from following Lemmas 2 and 3. ■

The following lemma shows that the subsidy to Mr 1 is decreasing in volatility σ if Mr 1 is restricted from asset-market participation.

Lemma 2 *Where Mr 2 and Mr 3 do risk-sharing but Mr 1 does not, the optimal amount of subsidy to Mr 1 decreases.*

Proof. The maximization problem can be written as

$$W = \max_{s_1} E \log \left(\omega_1 + \widetilde{P}^m s_1 \right) + 2E \log \left(\frac{\omega_2 + \omega_3 - \widetilde{P}^m s_1}{2} \right) \tag{10}$$

where $s_1 (= \tau_1)$ represents the amount of subsidy to Mr. 1.

The first order condition of the maximization problem (10) is

$$E \frac{\widetilde{P}^m}{\omega_1 + \widetilde{P}^m s_1} - E \frac{2\widetilde{P}^m}{\omega_2 + \omega_3 - \widetilde{P}^m \tau_1} = 0. \tag{11}$$

Let's define $g(x, s_1)$ and $h(x, s_1)$ as

$$g(x, s_1) = \frac{x}{\omega + x s_1} \text{ and } h(x, s_1) = \frac{2x}{\omega_2 + \omega_3 - x s_1}.$$

Then, the first order condition in equation (10) can be expressed as

$$Eg\left(\widetilde{P}^m, s_1\right) - Eh\left(\widetilde{P}^m, s_1\right) = 0 \tag{12}$$

Implicitly differentiating equation (12) with respect to σ, we have

$$Eg_1\left(\widetilde{P}^m, s_1\right)\frac{\partial \widetilde{P}^m}{\partial \sigma} + Eg_2(x, s_1)\frac{ds_1}{d\sigma} - Eh_1\left(\widetilde{P}^m, s_1\right)\frac{\partial \widetilde{P}^m}{\partial \sigma} - Eh_2(x, s_1)\frac{ds_1}{d\sigma} = 0,$$

and, in turn,

$$\frac{ds_1}{d\sigma} = -\frac{Eg_1\left(\widetilde{P}^m, s_1\right)\frac{\partial \widetilde{P}^m}{\partial \sigma} - Eh_1\left(\widetilde{P}^m, s_1\right)\frac{\partial \widetilde{P}^m}{\partial \sigma}}{Eg_2(x, s_1) - Eh_2(x, s_1)}. \tag{13}$$

Where $s_1 > 0$, $g(x, s_1)$ is decreasing in s_1 and concave in x. $h(x, s_1)$ is increasing in s_1 and convex in x. Therefore, $Eg_2(x, s_1) < 0$, $Eg_1\left(\widetilde{P}^m, s_1\right)\frac{\partial \widetilde{P}^m}{\partial \sigma} < 0$, $Eh_2(x, s_1) > 0$, and $Eh_1\left(\widetilde{P}^m, s_1\right)\frac{\partial \widetilde{P}^m}{\partial \sigma} > 0$. Thus, we acquire

$$\frac{ds_1}{d\sigma} < 0,$$

which implies that the amount of subsidies to the poor one is decreasing in volatility. ■

The following lemma indicates that the middle-income guy will be subsidized if the poor one is restricted from asset-market participation. The main result of the following lemma can be directly used for the proof of Proposition 4.

Lemma 3 *Where the middle one (Mr 2) and the rich one (Mr 3) do risk-sharing but the poor one (Mr 1) does not, the middle one (Mr 2) is subsidized, i.e.,* $\tau_2 < 0$.

Proof. We know that where $\sigma = 0$, the tax-authority levels all three consumers' utilities and it is true that $P^m s_1 = \omega_3 - \omega_2$. Therefore, from Lemma 2 we have $P^m s_1 < \omega_3 - \omega_2$ when $\sigma > 0$. Because $V_2 = V_3$, the tax-adjusted wealth of Mr 2 should be the same as that of Mr 3. Hence, we have

$$\begin{aligned} & p(\alpha)\omega_2 + p(\beta)\omega_2 - p(\alpha)P^m(\alpha)\tau_2 - p(\beta)P^m(\beta)\tau_2 \\ = \; & p(\alpha)\omega_3 + p(\beta)\omega_3 - p(\alpha)P^m(\alpha)\tau_3 - p(\beta)P^m(\beta)\tau_3. \end{aligned} \tag{14}$$

Equation (14) can be expressed as

$$(\omega_3 - \omega_2) = \frac{p(\alpha)P^m(\alpha) + p(\beta)P^m(\beta)}{p(\alpha) + p(\beta)} (\tau_3 - \tau_2). \tag{15}$$

We have

$$\begin{aligned} \frac{p(\alpha)P^m(\alpha) + p(\beta)P^m(\beta)}{p(\alpha) + p(\beta)} &= \frac{\begin{array}{c}\pi(\alpha)(\omega_2 + \omega_3 - P^m(\beta)s_1) P^m(\alpha) \\ +\pi(\beta)(\omega_2 + \omega_3 - P^m(\alpha)s_1) P^m(\beta)\end{array}}{\omega_2 + \omega_3 - s_1} \\ &= \frac{(\omega_2 + \omega_3) P^m - P^m(\alpha)P^m(\beta)s_1}{\omega_2 + \omega_3 - [\pi(\alpha) P^m(\beta) + \pi(\beta) P^m(\alpha)] s_1} \\ &= \left(\frac{(\omega_2 + \omega_3) - \frac{P^m(\alpha)P^m(\beta)}{P^m} s_1}{(\omega_2 + \omega_3) - [\pi(\alpha) P^m(\beta) + \pi(\beta) P^m(\alpha)] s_1} \right) P^m. \end{aligned} \tag{16}$$

We want to show that $\frac{p(\alpha)P^m(\alpha)+p(\beta)P^m(\beta)}{p(\alpha)+p(\beta)} < P^m$ by proving that

$$\frac{P^m(\alpha)P^m(\beta)}{P^m} < [\pi(\alpha) P^m(\beta) + \pi(\beta) P^m(\alpha)] \tag{17}$$

Inequality (17) is equivalent to

$$P^m - \frac{\sigma}{\pi(\alpha)} + \frac{\sigma}{\pi(\beta)} - \frac{\sigma^2}{\pi(\alpha)\pi(\beta) P^m} < P^m - \frac{\pi(\beta)\sigma}{\pi(\alpha)} + \frac{\pi(\alpha)\sigma}{\pi(\beta)}.$$

Because $-\frac{1}{\pi(\alpha)}+\frac{1}{\pi(\beta)}=-\frac{\pi(\alpha)+\pi(\beta)}{\pi(\alpha)}+\frac{\pi(\alpha)+\pi(\beta)}{\pi(\beta)}=-\frac{\pi(\beta)}{\pi(\alpha)}+\frac{\pi(\alpha)}{\pi(\beta)}$ and $-\frac{\sigma^2}{\pi(\alpha)\pi(\beta)P^m}<0$, inequality (17) is true. From (17), we have

$$\frac{p(\alpha)P^m(\alpha)+p(\beta)P^m(\beta)}{p(\alpha)+p(\beta)}<P^m. \tag{18}$$

From inequalities (15) and (18), we have

$$(\omega_3-\omega_2)<P^m(\tau_3-\tau_2). \tag{19}$$

Then we plug $\tau_3=s_1-\tau_2$ into inequality (19),

$$2\tau_2<-\frac{(\omega_3-\omega_2)}{P^m}+s_1. \tag{20}$$

Because $P^m s_1<\omega_3-\omega_2$ (Lemma 2), from inequality (20) we have

$$2\tau_2<-\frac{(\omega_3-\omega_2)}{P^m}+s_1<-\frac{(\omega_3-\omega_2)}{P^m}+\frac{(\omega_3-\omega_2)}{P^m}=0$$

Therefore, $\tau_2<0$. ■

Now, consider that the rich one is restricted from the asset market participation. In this case, the price-level volatility is directly connected to Mr 3's consumption volatility. Therefore, increasing monetary taxes to Mr 3 is inefficient in increasing social welfare and, consequently, the tax authority would decrease the taxes to Mr 3 and decides to collect more taxes from Mr 2, which is shown in following Proposition.

Proposition 5 *Where the poor one (Mr 1) and the middle one (Mr 2) do risk-sharing but the rich one (Mr 3) does not, the middle one (Mr 2) is taxed.*

Proof. Directly from following Lemmas 4 and 5. ■

The following lemma shows that the tax to Mr 3 is decreasing in volatility σ when Mr 3 is restricted from asset-market participation.

Lemma 4 *Where Mr 1 and Mr 2 do risk-sharing but Mr 3 does not, the optimal τ_3 strictly decreases in σ.*

Proof. The maximization problem can be written as

$$W = \max_{\tau_3} 2E \log \left(\frac{\omega_1 + \omega_2 + \widetilde{P}^m \tau_3}{2} \right) + E \log \left(\omega_3 - \widetilde{P}^m \tau_3 \right) \tag{21}$$

The first order condition of the maximization problem (21) is

$$E \frac{2\widetilde{P}^m}{\omega_1 + \omega_2 + \widetilde{P}^m \tau_3} - E \frac{\widetilde{P}^m}{\omega_3 - \widetilde{P}^m \tau_3} = 0. \tag{22}$$

Let's define $g(x)$ and $h(x)$ as

$$g(x, \tau_3) = \frac{2x}{\omega_1 + \omega_2 + x\tau_3} \text{ and } h(x, \tau_3) = \frac{x}{\omega_3 - x\tau_3}.$$

Then, the first order condition in equation (22) can be expressed as

$$Eg\left(\widetilde{P}^m, \tau_3\right) - Eh\left(\widetilde{P}^m, \tau_3\right) = 0 \tag{23}$$

Implicitly differentiating equation (23) with respect to σ, we have

$$Eg_1\left(\widetilde{P}^m, \tau_3\right) \frac{\partial \widetilde{P}^m}{\partial \sigma} + Eg_2(x, \tau_3) \frac{d\tau_3}{d\sigma} - Eh_1\left(\widetilde{P}^m, \tau_3\right) \frac{\partial \widetilde{P}^m}{\partial \sigma} - Eh_2(x, \tau_3) \frac{d\tau_3}{d\sigma} = 0$$

$$\frac{d\tau_3}{d\sigma} = -\frac{Eg_1\left(\widetilde{P}^m, \tau_3\right) \frac{\partial \widetilde{P}^m}{\partial \sigma} - Eh_1\left(\widetilde{P}^m, \tau_3\right) \frac{\partial \widetilde{P}^m}{\partial \sigma}}{Eg_2(x, \tau_3) - Eh_2(x, \tau_3)}$$

$g(x, \tau_3)$ is decreasing in τ_3 and strictly concave in x. $h(x, \tau_3)$ is increasing in τ_3 and strictly convex in x. Therefore, $Eg_1\left(\widetilde{P}^m, \tau_3\right) \frac{\partial \widetilde{P}^m}{\partial \sigma} < 0$, $Eh_1\left(\widetilde{P}^m, \tau_3\right) \frac{\partial \widetilde{P}^m}{\partial \sigma} > 0$, $Eg_2(x, \tau_3) < 0$ and $Eh_2(x, \tau_3) > 0$. Thus, we acquire

$$\frac{d\tau_3}{d\sigma} < 0$$

■

The following lemma shows that the middle-income guy will be taxed if the rich one is restricted from asset-market participation. The main result of the lemma can be directly used for the proof of Proposition 5.

Lemma 5 *Where Mr 1 and Mr 2 do risk-sharing but Mr 3 does not, Mr 2 is taxed, i.e.,* $\tau_2 > 0$.

Proof. We know that where $\sigma = 0$, it is true that $P^m \tau_3 = \frac{\omega_3 - \omega_1}{2}$. Therefore, from Lemma 4 we have $P^m \tau_3 < \frac{\omega_3 - \omega_1}{2}$ if $\sigma > 0$. Because $V_1 = V_2$, the tax-adjusted wealth of Mr 1 should be the same as that of Mr 2. Hence, we have

$$\begin{aligned} & p(\alpha)\omega_1 + p(\beta)\omega_1 - p(\alpha)P^m(\alpha)\tau_1 - p(\beta)P^m(\beta)\tau_1 \\ = \; & p(\alpha)\omega_2 + p(\beta)\omega_2 - p(\alpha)P^m(\alpha)\tau_2 - p(\beta)P^m(\beta)\tau_2. \end{aligned} \tag{24}$$

Equation (24) can be expressed as

$$(\omega_2 - \omega_1) = \frac{p(\alpha)P^m(\alpha) + p(\beta)P^m(\beta)}{p(\alpha) + p(\beta)} (\tau_2 - \tau_1). \tag{25}$$

We also have

$$\begin{aligned} \frac{p(\alpha)P^m(\alpha) + p(\beta)P^m(\beta)}{p(\alpha) + p(\beta)} &= \frac{\begin{array}{c} \pi(\alpha)(\omega_1 + \omega_2 + P^m(\beta)\tau_3) P^m(\alpha) \\ + \pi(\beta)(\omega_1 + \omega_2 + P^m(\alpha)\tau_3) P^m(\beta) \end{array}}{\omega_1 + \omega_2 + \tau_3} \\ &= \frac{(\omega_1 + \omega_2) P^m + P^m(\alpha)P^m(\beta)\tau_3}{\omega_1 + \omega_2 + [\pi(\alpha) P^m(\beta) + \pi(\beta) P^m(\alpha)]\tau_3} \\ &= \left(\frac{(\omega_1 + \omega_2) + \frac{P^m(\alpha)P^m(\beta)}{P^m}\tau_3}{(\omega_1 + \omega_2) + [\pi(\alpha) P^m(\beta) + \pi(\beta) P^m(\alpha)]\tau_3} \right) P^m \end{aligned} \tag{26}$$

We want to show that $\frac{p(\alpha)P^m(\alpha) + p(\beta)P^m(\beta)}{p(\alpha) + p(\beta)} < P^m$ by proving that

$$\frac{P^m(\alpha)P^m(\beta)}{P^m} < [\pi(\alpha) P^m(\beta) + \pi(\beta) P^m(\alpha)] \tag{27}$$

Inequality (27) is equivalent to

$$P^m - \frac{\sigma}{\pi(\alpha)} + \frac{\sigma}{\pi(\beta)} - \frac{\sigma^2}{\pi(\alpha)\pi(\beta) P^m} < P^m - \frac{\pi(\beta)\sigma}{\pi(\alpha)} + \frac{\pi(\alpha)\sigma}{\pi(\beta)}.$$

Because $-\frac{1}{\pi(\alpha)}+\frac{1}{\pi(\beta)} = -\frac{\pi(\alpha)+\pi(\beta)}{\pi(\alpha)}+\frac{\pi(\alpha)+\pi(\beta)}{\pi(\beta)} = -\frac{\pi(\beta)}{\pi(\alpha)}+\frac{\pi(\alpha)}{\pi(\beta)}$ and $-\frac{\sigma^2}{\pi(\alpha)\pi(\beta)P^m} < 0$, inequality (27) is true. From (27), we know that

$$\frac{p(\alpha)P^m(\alpha)+p(\beta)P^m(\beta)}{p(\alpha)+p(\beta)} < P^m. \tag{28}$$

From inequalities (25) and (28), we have

$$(\omega_2 - \omega_1) < P^m (\tau_2 - \tau_1). \tag{29}$$

Then we plug $\tau_1 = -\tau_3 - \tau_2$ into inequality (29),

$$2\tau_2 > \frac{(\omega_2 - \omega_1)}{P^m} - \tau_3. \tag{30}$$

Because we have $P^m \tau_3 < \frac{\omega_2 - \omega_1}{2}$ (Lemma 4), from inequality (30) we have

$$\begin{aligned} 2\tau_2 &> \frac{(\omega_2 - \omega_1)}{P^m} - \tau_3 > \frac{(\omega_2 - \omega_1)}{P^m} - \frac{\omega_3 - \omega_1}{2P^m} \\ &= \frac{1}{P^m}\left(\omega_2 - \omega_1 - \frac{\omega_3 - \omega_1}{2}\right) = 0. \end{aligned}$$

Therefore, $\tau_2 > 0$. ■

In the table below, we summarize whether each individual is taxed or subsidized for the four cases of asset-market participation. The most interesting case is case 2 where the middle-income guy and the rich one do risk-sharing but the poor does not. In this case, the middle-income guy would be subsidized when the tax-authority implements monetary income-redistribution policy.

		Mr 1	Mr 2	Mr 3
Case 1	Perfect risk-sharing	*Taxed*	0	*Subsidized*
Case 2	poor-rich risk-sharing	*Taxed*	0	*Subsidized*
Case 3	middle-rich risk-sharing	*Taxed*	*Subsidized*	*Subsidized*
Case 4	poor-middle risk-sharing	*Taxed*	*Taxed*	*Subsidized*

Table 1. Individuals' taxes and subsidies under monetary income-redistribution

When the poor one and the middle one do risk-sharing, the middle income guy will be taxed and, consequently, would not support the monetary income-redistribution policy.

Proposition 6 *When voting monetary income-redistribution-policy vs. no-policy, no-policy can be chosen by the majority if the rich one is restricted from asset-market participation.*

Proof. See Proposition 5 and Table 1. ■

6. Volatility and Individuals' welfare

Under the monetary income-redistribution policy, CGKS (2015) showed that increased volatility definitely decreases social welfare. However, this does not mean that volatility decreases all individuals' welfare. There are three effects of increased volatility on individuals' welfare. First, assuming the optimal tax is fixed and asset prices are also fixed, price volatility directly decreases each individual's utility because each consumer's consumption is mean-preserving increasing in price volatility. Second, the volatility can change the asset prices, which also affect each individual's expected utility. Third, volatility can change the tax-authority's optimal tax plans, which can increase or decrease individuals' utilities. In the case where volatility decreases Mr h's optimal tax (or increases Mr h's optimal subsidy), his utility may even increase with increased volatility. In the case where volatility increases Mr h's optimal tax (or decreases Mr h's optimal subsidy), his utility would definitely decrease.

The interesting case is that the decreased-taxation (or increased-subsidy) effect dominates the other two effects and, consequently, the individuals' welfare increases in increased volatility. Among the four cases of asset-market participation, in this section, we focus on the most interesting case where the middle-income guy and the rich one do risk-sharing but the poor is restricted from asset-market participation.

Proposition 7 *With increased volatility, some individuals' welfare can increase even though social welfare always decreases.*

Proof. See the example below. ∎

For the proof for Proposition 7, we present an example where analytical solution is attainable. For better mathematical approach with a closed form of solution, we assume that Mr 1 is extremely poor, $\omega_1 = 0$. In addition, we assume that the probability of sunspots is equal to that of non-sunspots, i.e., $\pi(\alpha) = \pi(\beta) = 0.5$. We also assume that $P^m = 1$. Then, the tax authority has the following maximization problem:

$$\begin{aligned} W &= \max_{\tau_1} E \log\left(-\widetilde{P}^m \tau_1\right) + 2E \log\left(\frac{(\omega_2+\omega_3)+\widetilde{P}^m\tau_1}{2}\right) \qquad (31) \\ &= \max_{s_1} \log(s_1) + E\log\left(\widetilde{P}^m\right) - 2\log 2 \\ &\quad + \log\left((\omega_2+\omega_3) - \left(P^m - \frac{\sigma}{2}\right)s_1\right) + \log\left((\omega_2+\omega_3) - \left(P^m + \frac{\sigma}{2}\right)s_1\right) \end{aligned}$$

where $s_1 = -\tau_1$.

From the first-order condition of the maximization problem (31), we have

$$s_1 = \frac{(\omega_2+\omega_3)}{3}\left(\frac{8-2\sqrt{4+3\sigma^2}}{4-\sigma^2}\right). \qquad (32)$$

Plugging equation (32) into $V_2 (= V_3)$, we obtain

$$\begin{aligned} V_2 &= V_3 \qquad (33) \\ &= -\log 2 + \log\frac{\omega_2+\omega_3}{3} + \\ &\quad \frac{1}{2}\log\left(3 - \left(1-\frac{\sigma}{2}\right)\left(\frac{8-2\sqrt{4+3\sigma^2}}{4-\sigma^2}\right)\right) \\ &\quad \times \left(3 - \left(1+\frac{\sigma}{2}\right)\left(\frac{8-2\sqrt{4+3\sigma^2}}{4-\sigma^2}\right)\right) \end{aligned}$$

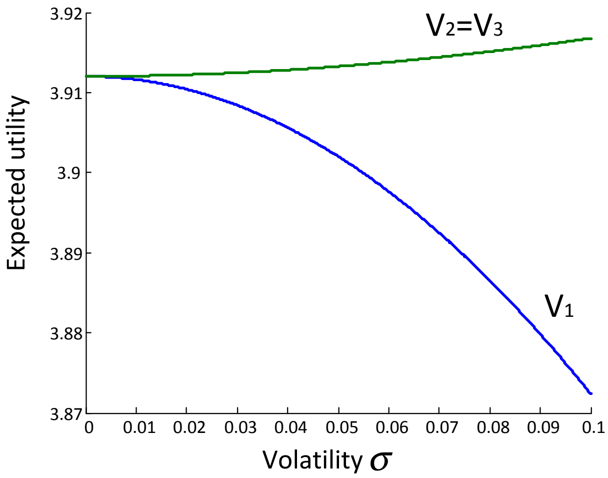

Figure 2: Individuals' welfare and volatility where $(\omega_1, \omega_2, \omega_3) = (0, 50, 100)$

From (33), we can prove that their utilities strictly increase with σ, which implies that the direct negative effect of increased volatility is overwhelmed by the positive effect of decreased taxes or increased subsidies.

The poor's expected utility (V_1) is

$$V_1 = \frac{1}{2}\log\left(1 - \frac{\sigma^2}{4}\right) + \log\left(\frac{8 - 2\sqrt{4 + 3\sigma^2}}{4 - \sigma^2}\right) + \frac{(\omega_2 + \omega_3)}{3}.$$

V_1 is strictly decreasing in σ. As in CGKS (2015), social welfare is always strictly decreasing in σ. With increased volatility, two individuals' welfare among three voters increases. This implies that for a higher level of price volatility, the possibility for monetary income-redistribution policy to be selected is ironically higher.

Individuals' expected utilities where $(\omega_1, \omega_2, \omega_3) = (0, 50, 100)$ are plot in Figure 2. Where there is no price-level volatility i.e., $\sigma = 0$, the tax-authority levels all individuals' welfare. As volatility σ increases, the poor one becomes worse off, while the middle-income guy and the rich one become better off.

7. Commodity Taxation

In this section, we introduce commodity taxation for income redistribution. The effect of commodity taxation is not affected by price-level volatility. We assume that the cost of commodity taxation is an iceberg cost and show that as the ice-berg cost increases, the poor one becomes worse off but the rich one becomes better off. The middle-income guy is not taxed nor subsidized.

We assume that when the tax authority makes a net transfer of x units of chocolate from one consumer to another, δx units of chocolate are lost to "melting". The melting rate is $0 < \delta < 1$. If Mr h is taxed, i.e., $\tau_h^c > 0$, he owes the tax authority τ_h^c in chocolate. If Mr h is subsidized, i.e., $\tau_h^c < 0$, he will receive $(1-\delta)\,\tau_h^c$ units of chocolate from the tax authority. Mr h's consumption is then

$$\omega_h - \max(0, \tau_h^c) - \min(0, (1-\delta)\,\tau_h^c) \tag{34}$$

With the 3 consumers, the tax-authority solves the following maximization problem:

$$W = \max_{\tau_1^c, \tau_2^c, \tau_3^c} \sum_{h=1,2,3} \log\left[\omega_h - \max(0, \tau_h^c) - \min(0, (1-\delta)\,\tau_h^c)\right] \tag{35}$$

$$\text{subject to } \tau_1^c + \tau_2^c + \tau_3^c = 0$$

For the higher level of the iceberg cost, the tax authority decreases the amount of taxes and, also, subsidies and, consequently, the subsidized poor one becomes worse off while the taxed rich one becomes better off.

Proposition 8 *As δ increases, Mr 1's utility decreases; Mr 2' utility is constant; Mr 3' utility increases.*

Proof. CGKS (2015) show that (a) if $\delta < (\omega_3 - \omega_1)/\omega_3$, Mr 1 is taxed and Mr 3 is subsidized, and (b) if $\omega_2 = (\omega_1 + \omega_2 + \omega_3)/3$, Mr 2 is not taxed nor subsidized. Therefore, the social welfare maximization problem is

$$\max_{\tau_3^c} \log(\omega_1 + (1-\delta)\tau_3^c) + \log(\omega_2) + \log(\omega_3 - \tau_3^c) \tag{36}$$

We need to show that as δ increases, τ_3^c decreases and also $(1-\delta)\tau_3^c$ decreases.

The first order condition from (36) is

$$\frac{(1-\delta)}{\omega_1 + (1-\delta)\tau_3^c} + \frac{-1}{\omega_3 - \tau_3^c} = 0,$$

which is equivalent to

$$\tau_3^c = -\frac{1}{2}\frac{\omega_1}{1-\delta} + \frac{\omega_3}{2}. \tag{37}$$

From equation (37), we know that the optimal τ_3^c strictly decreases in δ. This means that the rich becomes better off as the iceberg cost is increasing. The poor becomes worse off with increase the iceberg cost because $(1-\delta)\tau_3^c$ is decreasing in δ. ■

The utility for Mr 1 is

$$V_1 = \log\left(\omega_1 + (1-\delta)\left(-\frac{1}{2}\frac{\omega_1}{1-\delta} + \frac{\omega_3}{2}\right)\right) = \log\left(\frac{1}{2}(1-\delta)\omega_3 + \frac{\omega_1}{2}\right).$$

Because the middle-income guy is not taxed nor subsidized, the consumption is the same as the endowment. Therefore, Mr 2's utility is

$$V_2 = \log \omega_2.$$

The utility for Mr 3 is

$$V_3 = \log\left(\omega_3 + \frac{1}{2}\frac{\omega_1}{1-\delta} - \frac{\omega_3}{2}\right) = \log\left(\frac{1}{2}\frac{\omega_1}{1-\delta} + \frac{\omega_3}{2}\right).$$

8. Voting on commodity vs. monetary taxation

We have investigated the individual' welfare impact of monetary taxation and commodity taxation, separately. We now analyze the political economy of taxes that is voting on taxes. We assume that each individual can vote for either monetary taxation or commodity taxation. We adopt the simplest voting mechanism of majority voting with three types. We show that the tax regime chosen by majority of voters can be different from what the tax-authority wants. In both monetary and commodity taxation regimes, increased cost (that can be either volatility or iceberg cost) strictly decreases the social welfare (see CGKS 2015). Because individuals' welfare changes differently from the social welfare, it is possible that majority of individuals can choose socially undesirable taxation system.

This section shows that if the asset markets are complete and perfect, the socially undesirable taxation system would never be chosen by the majority. However, if some of the consumers are restricted from asset market participation, majority can choose a socially inefficient taxation system, which will be shown by a specific example in this section.

We define "efficient voting regime" as the case where the tax system supported by the majority results in higher social welfare than the other taxation system. In the same way, we define "inefficient voting regime" as the case where the tax system chosen by the majority results in lower social welfare than socially desired tax system. In the case where one individual prefers monetary taxation, one prefers commodity taxation, and the third one is indifferent between two tax systems, we assume that the tax-authority chooses the tax system resulting in higher social welfare; "weak efficient voting regime" is achieved.

Under perfect risk-sharing, we know that the tax-authority can level all individual utilities regardless of the level of volatility. Thus, the maximum social welfare is achieved (See CGKS (2015).) Therefore, with a positive iceberg cost, the social welfare under monetary taxation is strictly higher than that under commodity taxation. The following proposition also shows that weak efficient voting regime is achieved in the perfect risk-sharing market.

Proposition 9 *Under perfect risk sharing, the majority chooses monetary taxation and weak efficient voting regime is achieved.*

Proof. Under monetary taxation, the tax-authority levels all individual's utilities. Therefore, we have

$$V_1^M = V_2^M = V_3^M = \log \omega_2.$$

Under the commodity taxation, the individual's utilities from Proposition 8 are

$$V_1^C = \log\left(\frac{1}{2}(1-\delta)\omega_3 + \frac{\omega_1}{2}\right), V_2^C = \omega_2, V_3^C = \log\left(\frac{1}{2}\frac{\omega_1}{1-\delta} + \frac{\omega_3}{2}\right).$$

If $\delta > 0$, we have

$$\frac{1}{2}(1-\delta)\omega_3 + \frac{\omega_1}{2} < \frac{\omega_3}{2} + \frac{\omega_1}{2} = \omega_2.$$

and

$$\frac{1}{2}\frac{\omega_1}{1-\delta} + \frac{\omega_3}{2} > \frac{\omega_3}{2} + \frac{\omega_1}{2} = \omega_2.$$

Therefore, the poor prefers monetary taxation, the middle-income is indifferent, the rich prefers commodity taxation. ■

Under the perfect risk-sharing, weak efficient voting regime would be achieved. However, with restricted asset-market participation, efficient voting regime would not be guaranteed. Especially, we focus on the case where the poor one is restricted from asset market participation. In this case, monetary subsidies to the poor one are less efficient in the tax authority's perspective because the poor's consumption volatility is directly correlated with the price-level volatility. Decreased subsidy to the poor by the asset market restriction provides a chance for the middle-income guy to get subsidized. Therefore, the middle-income guy would vote for monetary taxation,

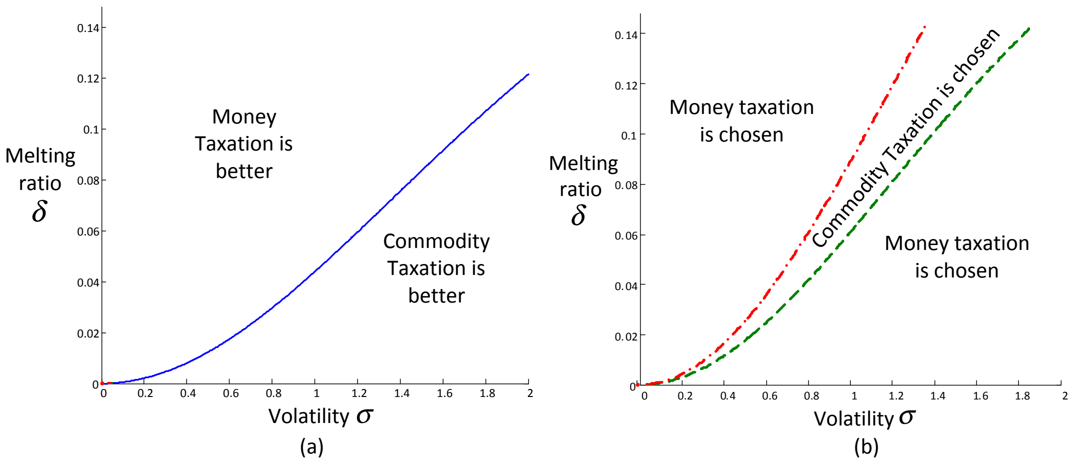

Figure 3: Money taxation vs. Commodity taxation where $(\omega_1, \omega_2, \omega_3) = (40, 55, 70)$

which increases the possibility that the monetary taxation is chosen by the majority.

Proposition 10 *When some of the consumers are restricted from asset-market participation, the voting regime can be inefficient.*

Proof. When the rich one and the middle-income guy do risk-sharing but the poor one cannot, from Proposition 4, we know that the middle-income one is subsidized. This necessarily means that the middle-income guy's utility is higher under the monetary taxation than commodity taxation. In the case where the poor one is extremely poor, $\omega_1 = 0$ in the example in Section 7, the poor one's expected utility under monetary taxation is V_1^M is strictly greater than the utility under commodity taxation V_1^C. Therefore, the poor and the middle-income guy would vote for monetary taxation for any volatility level and any level of iceberg cost. This means that it is possible that social welfare under commodity taxation is higher but the majority choose monetary taxation. ■

We consider the numerical example where the poor one is restricted from asset-market participation. In the numerical example, the three consumers'

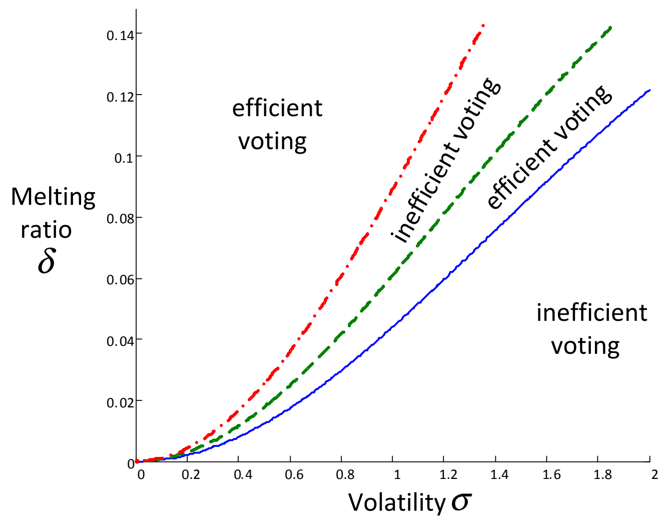

Figure 4: Efficient and inefficient voting regimes where $(\omega_1, \omega_2, \omega_3) = (40, 55, 70)$

endowment income is $(\omega_1, \omega_2, \omega_3) = (40, 55, 70)$. Figure 3(a) shows regions where the monetary taxation or commodity taxation is socially preferred. CGKS (2015) show that social welfare under commodity taxation strictly decreases in the ice-berg cost δ and that under money taxation also strictly decreases in volatility σ. Therefore, there exists a constant $\overline{\delta}$ for any given volatility level σ such that if $\delta > \overline{\delta}$ ($\delta < \overline{\delta}$), money taxation (commodity taxation) is socially preferred. Figure 3(b) shows the regions for the monetary taxation or commodity taxation chosen by the majority. From 3(a) and (b), we can derive the regions, "inefficient voting region" and "efficient voting region", which are described in Figure 4.