Bank Runs: The Pre-Deposit Game*

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Abstract

We analyze in some detail the full *pre-deposit* game in a simple, tractable, but very rich, banking environment. How does run-risk affect the optimal deposit contract? If there is a run equilibrium in the post-deposit game, then the optimal contract in the pre-deposit game tolerates runs for small probabilities. In some cases, the optimal contract becomes – as one would expect – strictly more conservative as the run-probability increases (until it switches to the best run-proof contract), and the equilibrium allocation is not a mere randomization over the equilibrium allocations from the post-deposit game. In the other cases, the allocation *is* a mere randomization over the equilibria from the post-deposit game. In first cases (the more intuitive ones), the incentive constraint does not bind. In the second cases, the incentive constraint does bind.

Keywords: bank runs, constrained efficient allocation, deposit contract, impulse demand, pre-deposit game, post-deposit game, run probability, strategic complementarity, sunspots, unconstrained efficient allocation

JEL classification numbers: G21; E44

1 Introduction

Bryant (1980) and Diamond and Dybvig (1983) – hereafter DD – introduced the modern literature on panic-based bank runs. The bank deposit contract is a mechanism designed to improve the welfare of depositors facing an uncertain impulse demand (i.e., when they become impatient). Since the impulse demand itself is not directly observable, it is uninsurable in the market. The deposit contract facilitates some "insurance" by specifying early and late withdrawals such that they are Bayesian incentive compatible (hereafter BIC): depositors with different liquidity needs correctly self-select their types.[1] Besides the "good" Bayes-Nash equilibrium in which the depositors self-select, DD show that there is also a "bad" equilibrium, the self-fulfilling bank run. When a bank run occurs, depositors attempt to withdraw early independent of their liquidity needs. Bank runs are possible because, the "good" equilibrium is only BIC rather than dominant-strategy incentive compatible (hereafter DSIC).[2] If the patient depositor expects that a bank run will take place, he will choose early withdrawal.

But given the two equilibria of the post-deposit game, the consumers will never deposit in the bank if they anticipate the run: a probability-one bank-run will not be an equilibrium for the *pre-deposit* game. DD recognized this problem and offered sunspots as an answer.[3] Peck and Shell (2003) – hereafter PS – provided the formal analysis of *sunspot equilibrium* in the *pre-deposit* game.[4]

[1]That is, an impatient depositor chooses early withdrawal and a patient depositor chooses late withdrawal if he expects that the other patient depositors will also choose late withdrawal.

[2]In other words, for a patient depositor, choosing late withdrawal is weakly implementable rather than strongly implementable.

[3]DD say "This could happen if the selection between the bank run equilibrium and the good equilibrium depended on some commonly observed random variable in the economy. This could be a bad earnings report, a commonly observed run at some other bank, a negative government forecast, or *even sunspots* [emphasis ours]."

Postlewaite and Vives (1987) show how bank runs can be seen as a prisoner's dilemma-type situation in which there is a unique equilibrium that involves a positive probability of a run.

[4]See also Cooper and Ross (1998) and Ennis and Keister (2006). These two papers analyze how banks respond to the possibility of runs in their design of deposit contracts and in their investment decisions. Gu (2011) analyzes noisy sunspots and bank runs.

PS showed that a *sunspot-driven* run can be an equilibrium in the *pre-deposit* game as long as (1) the *post-deposit* game has both a run equilibrium and a non-run equilibrium, and (2) the run probability is below a threshold level. PS employ a 2-depositor numerical example to calculate the optimal deposit contract as a function of the run probability. The numerical example is in their proof for Proposition 3 (Page 114). In PS, the banking contract is characterized by the scalar c, the withdrawal of the first depositor in line in period 1. The optimal c is denoted by $c^*(s)$, which is a function of the exogenous, sunspot run-probability s.[5] In the PS calculated example, $c^*(s)$ is a step function: If the probability s is less than the threshold probability s_0, the optimal contract $c^*(s)$ tolerates runs and is a constant. If the probability s is greater than s_0, the optimal contract is the best run-proof contract.

In the present paper, we ask: Why doesn't the optimal contract become strictly more conservative as the run probability increases (until runs are no longer tolerated)? In other words, shouldn't we expect that $c^*(s)$ is strictly decreasing in s until the contract switches to the best run-proof contract? If yes, in which economies will we have this property and in which economies is $c^*(s)$ a step function? Are there other (perhaps mixed) cases? These issues are important to banks and regulators.[6] Contracts and regulations could well be different based on the financial sector's level of fragility or

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These three papers focus on simple deposit contracts, while PS allows for partial or full suspension of convertibility. The PS model is more general, but we adopt their 2-depositor example.

In contrast to DD, there is also intrinsic uncertainty (as opposed to extrinsic uncertainty in the form of sunspots) in the PS model: (1) The aggregate number of impatient consumers is uncertain. It could be $0, 1$ or 2. This is important because it does not allow the bank to know with certainty that a run is underway. (2) Sequential service is taken seriously (Wallace (1988)). When 2 depositors withdraw early, their positions in the queue are random. (3) Partial suspension of convertibility is feasible for the bank. Typically the first in line in period 1 will be allowed a larger withdrawal than the second in line.

[5] Like other papers in the literature, PS assume that in equilibrium the bank chooses the contract that maximizes the *ex-ante* expected utility of depositors. This assumption can be justified when banks compete for deposits, there is no entry cost for banks, and the depositors are *ex-ante* identical.

[6] It is also an important question in the theory of sunspot equilibrium. Some practitioners confuse sunspot equilibrium (SSE) with randomizations over certainty equilibria (CE). Not all SSE are randomizations over CE. Not all randomizations over CE are SSE. See Shell (2008).

pessimism as measured by s.[7] To answer these questions and to keep the analysis tractable, we employ the general 2-depositor banking example from PS,[8] but instead of relying solely on a single numerical example, we provide the global comparative statics of the optimal contract.

We analyze the game backward and start the analysis from the *post-deposit* game for an arbitrary contract c. We characterize the conditions on c which allow for a run equilibrium and a non-run equilibrium in the post-deposit game. The optimal contract c^* is the BIC contract which maximizes the expected utility of the depositor in the *pre-deposit* game.[9]

For "unusual" values of the parameters, the set of BIC contracts is the same as the set of DSIC contracts. Hence, bank runs are not relevant for the optimal contract. The analysis of the *pre-deposit* game is similar to that of the corresponding *post-deposit* game. Analysis of c^* for the "unusual" parameters is in the Appendix.

For the "usual" values of the parameters – which is our focus in the present paper – the set of DSIC contracts is a strict subset of the set of BIC contracts. For the "usual" parameters, c^* is affected by s unless c^* is DSIC. The shape of the function $c^*(s)$ depends on further specification of the remaining parameters.

To further examine $c^*(s)$, we divide the "usual" parameter space into three cases: (1) the *unconstrained efficient allocation* is DSIC;[10] (2) it is

[7] For example, an economy experiencing hyperinflation, hyperdeflation, or episodes of bank runs might be thought to have high s banking sectors.

[8] Green and Lin (2000), Andolfatto, Nosal and Wallace (2007) and Nosal and Wallace (2009) analyze a model similar to PS. The main differences among the models are on the amount of information that a depositor has at the time of making his withdrawal decision. Ennis and Keister (2009) show that the PS assumptions on marginal utilities are not necessary for the qualitative results in PS. Ennis and Keister (2009) also study the Green-Lin model under a more general specification of the distribution of types across agents. See Ennis and Keister (2010) for a good survey on this part of the literature.

[9] A contract is BIC if, for this contract, there exists a non-run Bayes-Nash equilibrium in the *post-deposit* game. A contract is DSIC if the non-run equilibrium is also the unique Bayes-Nash equilibrium in the *post-deposit* game. By definition, the DSIC contracts form a subset of the BIC contracts.

[10] The *unconstrained efficient allocation* is the best allocation that can be attained when agent types (patient or impatient) are observable. In other words, the allocation maximizes the ex ante expected utility of agents without imposing incentive compatibility, but it is still subject to the sequential service and resource feasibility constraints. See Ennis and

not DSIC but it is BIC; or (3) it is not even BIC. In the first case, c^* is the contract supporting the *unconstrained efficient allocation* and it is not affected by s since the "good equilibrium" is DSIC.[11] In the second case, c^* balances the trade-off between the non-run and run equilibria. As s increases, the trade-off changes continuously and the expected utility of the depositor is more dependant on the run equilibrium. Hence the optimal contract $c^*(s)$ is continuous and strictly decreasing until it switches to the best run-proof contract. In the third case, the trade-off is constrained by the incentive-compatibility constraint (hereafter ICC) which implies that c^* is merely a BIC contract, not a DSIC contract. When s is small, the ICC binds and the optimal contract does not respond to changes in s since the binding ICC is independent of s. In this case, a larger run probability does not induce a more conservative contract since the binding ICC has already forced $c^*(s)$ to be more conservative than it would have been absent the ICC.

In Section 2, we introduce the notation and the formal set-up. In Section 3, we analyze the post-deposit game and characterize the non-run equilibrium and the run equilibrium in the post-deposit game. We identify the "usual" and "unusual" values of parameters.

In Section 4, we analyze the *pre-deposit* game and provide comparative statics for the optimal contract for the "usual" parameter values. We focus in Section 4 on comparative statics with respect to the impulse demand parameter. Holding the other parameters constant, the impulse demand parameter determines whether we are in Case 1 or Case 2 or Case 3. In the Appendix, we provide the proofs of our results. In the Online Appendix, we provide the comparative statics with respect to the other parameters.

Keister (2010). The associated contract is sometimes called the "first-best contract".

[11] Note the difference between the parameters with "unusual" values and Case 1. With "unusual" parameter values, the non-run equilibrium and run equilibrium cannot co-exist for *any* contract c. But for Case 1, the non-run equilibrium and run equilibrium can co-exist for some contracts, but these contracts are not optimal.

2 The Environment

The notation is from PS. There are two consumers and three periods: 0, 1 and 2. In period 0, each consumer is endowed with y units of the consumption good.[12] Let c^1 and c^2 denote the *withdrawals* of the depositor in period 1 and 2 respectively. The impatient consumers derive utility only from period-1 consumption, while the patient consumers derive utility only from period-2 consumption. The patient consumers can store consumption goods costlessly across the two periods. The impatient and patient consumers, respectively, have utility functions $u(c^1)$ and $v(c^1 + c^2)$, where

$$u(x) = A\frac{(x)^{1-b}}{1-b} \text{ where } A > 0, \tag{1}$$

$$v(x) = \frac{(x)^{1-b}}{1-b}. \tag{2}$$

A is the strength of the impulse demand of an impatient consumer. We analyze this parameter in detail in section 3. The parameter b, larger than 1, is the coefficient of relative risk aversion of a consumer. Consumers are identical in period 0. In period 1, each consumer becomes either impatient with probability p or patient with probability $1-p$. Types are uncorrelated and private information. Since the number of consumers is finite, the aggregate number of patient depositors is stochastic. In period 1, each depositor also observes a sunspot variable δ distributed uniformly on $[0,1]$. Besides the storage technology, there is another investment technology. Investing one unit of period 0 consumption yields $R > 1$ units if harvested in period 2 and yields 1 unit if harvested in period 1.

The sequential service constraint is part of the physical environment. A depositor visits the bank only when he makes a withdrawal. When a depositor learns his type and makes his withdrawal decision, he does not know his position in the bank queue. If more than one depositor chooses to withdraw, a depositor's position in the queue is random; in particular, positions in the queue are equally probable.

[12] There are no endowments in periods 1 and 2.

3 The Post-Deposit Game

The post-deposit game is a static game between the two depositors with incomplete information. The incomplete information is due to the fact that "type" is private information. The action set of each depositor is $\{E, L\}$, where E (L) stands for early (late) withdrawal in period 1. An impatient depositor always chooses E. Hence the relevant strategy of a depositor is (E, E) or (L, E), where the first (second) element in the parenthesis stands for the action chosen by the depositor when he is patient (impatient). The pay-off matrix is (The first letter in each cell represents the payoff to the row player and the second number represents the payoff to the column player):

	(E, E)	(L, E)
(E, E)	T_1, T_1	T_3, T_2
(L, E)	T_2, T_3	T_4, T_4

where,

$$T_1 = (1-p)\frac{v(c)+v(2y-c)}{2} + p\frac{u(c)+u(2y-c)}{2},$$

$$T_2 = (1-p)v[(2y-c)R] + p\frac{u(c)+u(2y-c)}{2},$$

$$\begin{aligned} T_3 = {} & (1-p)[(1-p)v(c) + p\tfrac{v(c)+v(2y-c)}{2}] \\ & +p[(1-p)u(c) + p\tfrac{u(c)+u(2y-c)}{2}], \end{aligned}$$

$$\begin{aligned} T_4 = {} & (1-p)[(1-p)v(yR) + pv[(2y-c)R]] \\ & +p[(1-p)u(c) + p\tfrac{u(c)+u(2y-c)}{2}]. \end{aligned}$$

3.1 Run Equilibrium in the Post-Deposit Game

A run equilibrium in the *post-deposit* game is defined as a Bayes-Nash equilibrium in which both depositors choose (E, E), which requires that $T_1 > T_2$, or equivalently, that $c \in [0, 2y]$ satisfies[13]

[13] As in other papers in the literature, we assume that a patient depositor chooses early withdrawal if he *strictly* prefers the period-1 withdrawal to the period-2 withdrawal. And he chooses period-2 withdrawal if he *weakly* prefers to do so.

$$[v(c) + v(2y - c)]/2 > v[(2y - c)R]. \quad (3)$$

Whether there is $c \in [0, 2y]$ satisfying inequality (3) depends on the parameters b and R. This is because, from inequality (3), the patient depositor's comparison between the early and late withdrawals depends on: (1) his attitude toward the risk of being the second in line when participating in runs, and (2) the productivity of the investment technology R. Given R, the more risk-averse the patient depositor, the less preferable is it for the patient depositor to run on the bank. Given his attitude toward risk, the more productive the investment, the less preferable is it for the patient depositor to run on the bank because more resource will be left in the last period. Since $v(c) = (c^{1-b} - 1)/(1 - b)$, a patient depositor's attitude toward risk is characterized by b. Hence the existence of $c \in [0, 2y]$ satisfying inequality (3) restricts the parameters b and R and we have the following lemma.[14]

Lemma 1 *If $b < 1 + \ln 2/\ln R$, there is a run equilibrium in the post-deposit game exists if and only if c satisfies the inequality*

$$c > c^{early} = 2y/[(2/R^{b-1} - 1)^{1/(b-1)} + 1]. \quad (4)$$

If $b \geq 1 + \ln 2/\ln R$, there is no run equilibrium in the post-deposit game for any $c \in [0, 2y]$.

If $b < 1 + \ln 2/\ln R$, c^{early} is the level of c *beyond* which a patient depositor chooses early withdrawal if he expects that the other depositor – if patient – will also choose early withdrawal.

3.2 Non-Run Equilibrium in the Post-Deposit Game

A non-run equilibrium in the *post-deposit* game is defined as a Bayes-Nash equilibrium in which both depositors choose (L, E), which requires $T_4 \geq T_3$, or equivalently, that $c \in [0, 2y]$ satisfies

[14] y is not important since it only changes the scale of the economy.

$$(1-p)v(yR)+pv[(2y-c)R] \geq (1-p)v(c)+p[v(c)+v(2y-c)]/2. \quad (5)$$

Inequality (5) is also the Incentive Compatibility Constraint (ICC).

Lemma 2 *If $b < 1+\ln 2/\ln R$, there is a non-run equilibrium in the post-deposit game if and only if c satisfies the inequality*

$$c \leq c^{wait}, \quad (6)$$

where c^{wait} is the level of c such that (5) holds as an equality.

Thus c^{wait} is the level of c *below* which a patient depositor chooses late withdrawal if he expects that the other patient depositor will also choose late withdrawal. Thus, if $b < 1+\ln 2/\ln R$, the set of BIC contracts is $[0, c^{wait}]$.

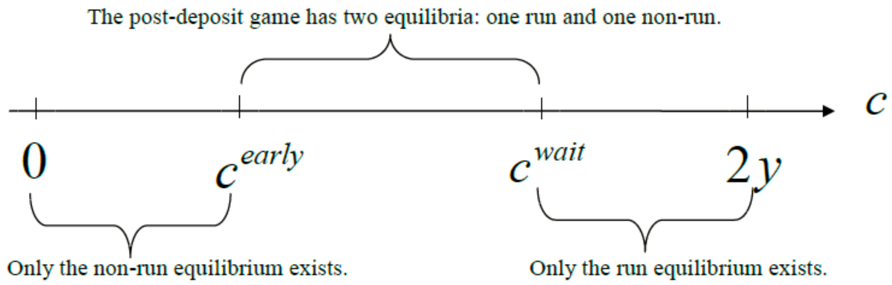
3.3 Equilibria in the Post-Deposit Game for an arbitrary contract

From the analysis above, we know that if $b < 1+\ln 2/\ln R$, c^{early} and c^{wait} are well-defined and they are two critical thresholds in the contract space. Furthermore, if $c^{early} < c^{wait}$, the set of DSIC contracts (i.e., $[0, c^{early}]$) is a strict subset of BIC contracts (i.e., $[0, c^{wait}]$). The *post-deposit* game has a unique non-run equilibrium for $c \in [0, c^{early}]$, two equilibria (one non-run equilibrium and one run equilibrium) for $c \in (c^{early}, c^{wait}]$, and a unique run equilibrium for $c \in (c^{wait}, 2y]$. (See Figure 1.) The interval $(c^{early}, c^{wait}]$ is the region of c for which the patient depositors' withdrawal decisions exhibit strategic complementarity.

In other words, for $c \in [0, c^{early}]$, we have $T_2 \geq T_1$ and $T_4 > T_3$. (L, E) is the dominant strategy for each depositor. The *post-deposit* game is "prisoners' dilemma" type, but with Pareto efficiency. For $c \in (c^{early}, c^{wait}]$, we have $T_2 < T_1$ and $T_4 \geq T_3$. The decisions of the two depositors are strategic complements: both run and non-run are Bayes-Nash equilibria. The *post-deposit* game is "stag hunt" type. For $c \in (c^{wait}, 2y]$, we have $T_2 < T_1$ and $T_4 < T_3$.

(E, E) is the dominant strategy for each depositor. The *post-deposit* game is "prisoners' dilemma" type.

Figure 1. Equilibrium in the Post-Deposit Game



The following gives the parameters b and R for which we have $c^{early} < c^{wait}$.

Lemma 3 *$c^{early} < c^{wait}$ if and only if*

$$b < \min\{2, 1 + \ln 2 / \ln R\} \tag{7}$$

We call the part of parameter space where b and R satisfy (7) "usual" since the set of DSIC contracts is a strict subset of the set of BIC contracts. From PS, we know that the *pre-deposit* game has a run equilibrium only if the *post-deposit* game has both a non-run equilibrium and a run equilibrium.[15] Given the "usual" values of b and R, we know that a run-equilibrium exists in the *pre-deposit* game only if the optimal deposit contract belongs to $(c^{early}, c^{wait}]$. In the next section, we will solve the optimal deposit contract. Before we discuss the optimal contract, we give a numerical example in which b and R are "usual".

Example 1 *The parameters are*

$$b = 1.01; p = 0.5; y = 3; R = 1.5.$$

These parameters will be fixed throughout the examples. We see that b and R satisfy (7). Hence c^{early} must be strictly smaller than c^{wait}. Indeed, we

[15] Proposition 2 in PS.

have $c^{early} = 4.155955$ *and* $c^{wait} = 4.280878$*. Therefore, whenever a contract* c *is larger than* 4.155955 *and smaller than or equal to* 4.280878*, both a run equilibrium and a non-run equilibrium exist in the post-deposit game.*

For completeness, we take a look at the *post-deposit* game when b and R are "unusual" (i.e., they do not satisfy (7)). They are "unusual" since, under these values of parameters, the set of BIC contracts is the same as the set of DSIC contracts.[16] According to the Revelation Principle,[17] when we search for the optimal contract we only have to focus on c which is BIC. Therefore, under the "unusual" parameters, the optimal contract must also be DSIC and bank runs are not relevant. We discuss the optimal contract under these "unusual" parameters in the Appendix. For the rest of the paper, we just focus on the economy with "usual" values of b and R.

4 The Optimal Contract for the Pre-Deposit Game

The *pre-deposit* game is a sequential game. In the first-stage (period 0), the bank chooses the contract c and the consumers choose whether to deposit or not. If the consumers choose to deposit,[18] they move on to the second stage (period 1) which is the *post-deposit* game. We have shown that, when the values of b and R are "usual", for any $c \in (c^{early}, c^{wait}]$ we have multiple equilibria in the *post-deposit* game. We assume – as in PS – that bank runs are *sunspot-driven*. Hence whether a run equilibrium exists in the *pre-deposit* game depends on whether in the first stage the contract chosen by the bank (i.e., the optimal contract) belongs to the set $(c^{early}, c^{wait}]$. To characterize

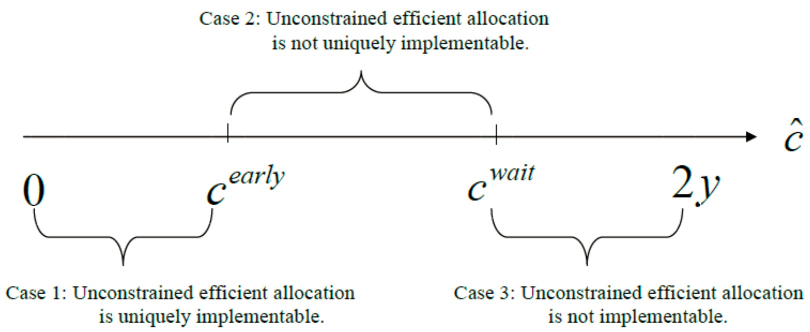
[16] It is so because any BIC contract is also DSIC. To see this, we know that for the "unusual" values of parameters we either have $2 \leq b < 1 + \ln 2/\ln R$ or $b \geq 1 + \ln 2/\ln R$. For the former case $c^{early} \geq c^{wait}$ and thus the set of DSIC contracts is also $[0, c^{wait}]$. For the latter case, a run-equilibrium does not exist for any feasible contract according to Lemma 1. Thus any BIC contract must be DSIC.

[17] Myerson (1979)

[18] Consumers always choose to deposit in equilibrium since the contract chosen by the bank can at least mimic the autarky allocation.

the optimal contract, we divide the "usual parameters" into three cases depending on $\hat{c}$, the contract supporting the *unconstrained efficient allocation*. These three cases are: $\hat{c} \leq c^{early}$ (Case 1), $\hat{c} \in (c^{early}, c^{wait}]$ (Case 2), and $\hat{c} > c^{wait}$ (Case 3). We next characterize the parameters for each case. To be more specific, when b and R are "usual", we show that each case corresponds to certain range of the parameter A, the impulse multiplier in the impatient consumer's utility function.

Figure 2. Three Cases



4.1 The Impulse Parameter A and the 3 Cases

The contract $\hat{c}$ supports the *unconstrained efficient allocation*, where $\hat{c}$ is defined by

$$\hat{c} = \arg\max_{c \in [0,2y]} \widehat{W}(c),$$

where

$$\widehat{W}(c) = p^2[u(c)+u(2y-c)]+2p(1-p)[u(c)+v[(2y-c)R]]+2(1-p)^2 v(yR). \quad (8)$$

$\widehat{W}(c)$ is the depositor's expected utility if the types of the depositors are publicly observable *ex-post*.[19] Given the CRRA utility function, we have

$$\hat{c} = \frac{2y}{\{p/(2-p)+2(1-p)/[(2-p)AR^{b-1}]\}^{1/b}+1}. \quad (9)$$

[19] $\widehat{W}(c)$ is also the depositor's expected utility in the non-run equilibrium of the *pre-deposit* game.

From (9), we see that $\widehat{c}$ is increasing in A. Therefore, we can write $\widehat{c}$ as an increasing function of A, $\widehat{c}(A)$. That is, when the "impulse demand" is stronger, the *unconstrained efficient allocation* allows larger first-period withdrawal. From equation (9), we also have

$$\lim_{A \to 0} \widehat{c}(A) = 0$$

and

$$\lim_{A \to \infty} \widehat{c}(A) = \frac{2y}{[p/(2-p)]^{1/b} + 1}.$$

From equations (4) and (5), we know that neither c^{early} nor c^{wait} depends on A. This is intuitive: c^{early} and c^{wait} are determined by the patient depositor's utility which is independent of A.

Hence if A is sufficiently small, we have Case 1. Furthermore, if

$$\frac{2y}{[p/(2-p)]^{1/b} + 1} \leq c^{early},$$

only Case 1 obtains.

If

$$c^{early} < \frac{2y}{[p/(2-p)]^{1/b} + 1} \leq c^{wait},$$

there is a unique level of A, denoted by A^{early}, such that

$$\widehat{c}(A^{early}) = c^{early}. \tag{10}$$

Hence if $A \leq A^{early}$, we have Case 1. If $A > A^{early}$, we have Case 2. Case 3 does not exist.

If

$$c^{wait} < \frac{2y}{[p/(2-p)]^{1/b} + 1},$$

there is a unique level of A, denoted by A^{wait}, such that

$$\widehat{c}(A^{wait}) = c^{wait}. \tag{11}$$

We now have all three cases: if $A \leq A^{early}$, we are in Case 1; if $A^{early} < A \leq A^{wait}$, we are in Case 2; if $A > A^{wait}$, we are in Case 3.[20]

Example 2 *In Example 1, we have shown that* $c^{early} = 4.155955$ *and* $c^{wait} = 4.280878$. *Now we calculate the thresholds of A for each case. It is easy to check that* $c^{wait} < \frac{2y}{[p/(2-p)]^{1/b}+1}$ *for the specified parameter values. Therefore, all three cases exist. We have* $A^{early} = 6.217686$ *and* $A^{wait} = 10.27799$. *Hence if* $A \leq 6.217686$, *we are in Case 1; If* $6.217686 < A \leq 10.27799$, *we are in Case 2; If* $A > 10.27799$, *we are in Case 3.*

In the Online Appendix, we show how the other parameters – namely p, R and b – affect $\widehat{c}$. Unlike A, these parameters also affect the values of c^{early} and/or c^{wait}, and $\widehat{c}$ is not monotonic in the parameters. The analysis is slightly more complicated in these cases, but once the parameter values are given, we can readily compute the values of $\widehat{c}$, c^{early} and c^{wait} to determine which case is applicable.

4.2 The Optimal Contract

In this subsection, we focus on the parameter values of b and R satisfying condition (7). We will describe the optimal contract c^* for the three cases specified above.

For an arbitrary contract $c \in [0, 2y]$, we have one or two equilibria in the *post-deposit* game depending on whether c belongs to $(c^{early}, c^{wait}]$ or not. As equation (8) shows, the depositor's expected utility in the non-run equilibrium is $\widehat{W}(c)$. Let $W^{run}(c)$ denote the depositor's expected utility in the run equilibrium (if it exists). It is given by

$$\begin{aligned} W^{run}(c) &= p^2[u(c) + u(2y-c)] + p(1-p)[u(c) + v(2y-c) + v(c) + u(2y-c)] \\ &\quad + (1-p)^2[v(c) + v(2y-c)]. \end{aligned} \tag{12}$$

[20]It is easy to see that c^{early} does not depend on p and that we have $y < c^{early} < c^{wait} < Ry$. $\lim_{p\to 1} \frac{2y}{[p/(2-p)]^{1/b}+1} = y$ and $\lim_{p\to 0} \frac{2y}{[p/(2-p)]^{1/b}+1} = 2y$. Hence we know that for sufficiently large p, we have $\frac{2y}{[p/(2-p)]^{1/b}+1} \leq c^{early}$. If $R < 2$, we have $c^{wait} < \frac{2y}{[p/(2-p)]^{1/b}+1}$ for sufficiently small p. For intermediate values of p, we have $c^{early} < \frac{2y}{[p/(2-p)]^{1/b}+1} \leq c^{wait}$.

When $c \leq c^{early}$, only the non-run-equilibrium exists and the depositor's ex-ante expected utility is simply $\widehat{W}(c)$. When $c^{early} < c \leq c^{wait}$, both the run-equilibrium and the non-run-equilibrium exist in the *post-deposit* game. If the run probability is s, the depositor's ex-ante expected utility is $(1 - s)\widehat{W}(c) + sW^{run}(c)$. When we have $c^{wait} < c \leq 2y$, only the run-equilibrium exists and therefore, no consumer would want to deposit in this bank. Hence when we consider the optimal contract, $[0, c^{wait}]$ is the relevant choice interval for c.

Let $c^*(s)$ denote the optimal contract which maximizes the depositor's ex-ante expected utility in the *pre-deposit* game given the run probability s.[21] We have

$$c^*(s) = \arg \max_{c \in [0, c^{wait}]} W(c; s),$$

where

$$W(c; s) = \begin{cases} \widehat{W}(c) & \text{if } c \leq c^{early}. \\ (1-s)\widehat{W}(c) + sW^{run}(c) & \text{if } c^{early} < c \leq c^{wait}. \end{cases} \tag{13}$$

Case 1 *The unconstrained efficient allocation is DSIC, i.e., we have* $\widehat{c} \leq c^{early}$.

Since the *unconstrained efficient allocation* is strongly implementable, it is straightforward to see that the optimal contract for the pre-deposit game supports the *unconstrained efficient allocation*

$$c^*(s) = \widehat{c}$$

and that a bank run does not occur in equilibrium. Other contracts cannot be optimal because they either deliver lower welfare in the non-run equilibrium or, to make things worse, they may also support a run equilibrium. The next is a numerical example for Case 1.

[21] At $c^*(s; A)$, consumers must weakly prefer depositing to autarky. This is because the deposit contract can always mimic the autarky allocation by setting $c^*(s; A)$ equal to y. Hence the participation constraint is not an issue for $c \in [0, c^{wait}]$

Example 3 *In Example 2, we have seen that as long as* $A \leq A^{early} =$ 6.217686, *we have Case 1 for this economy. Let* A *be equal to* 1. *We have* $c^*(s) = \widehat{c} = 3.004012$ *for any* $s \in [0,1]$. *Since* $c^*(s) < c^{wait} = 4.280878$, *a bank run is not an equilibrium.*

As we have discussed in section 4.1, when A is larger than A^{wait}, we have Case 2:

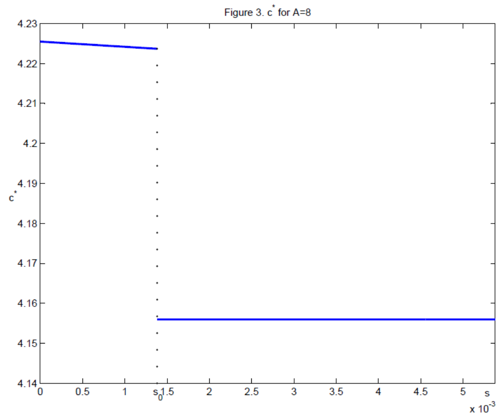
Case 2 *The unconstrained efficient allocation is BIC but not DSIC, i.e., we have* $c^{early} < \widehat{c} \leq c^{wait}$.

The optimal contract for the *pre-deposit* game cannot be $\widehat{c}$ except for the degenerate case when $s = 0$. This is because a run equilibrium exists at $\widehat{c}$. How much the optimal contract deviates from $\widehat{c}$ depends on s, which changes the trade-off between the expected utilities over the 2 *post-deposit* game equilibria. We show in Proposition 1 how the optimal contract to the *pre-deposit* game changes with the probability s.

Proposition 1 *In Case 2, the optimal contract* $c^*(s)$ *satisfies: (1) If* s *is larger than the threshold probability* s_0 *(specified in equation (21) in the proof), the optimal contract is run-proof,* $c^*(s) = c^{early}$. *(2) If* s *is smaller than* s_0, *the optimal contract* $c^*(s)$ *tolerates runs and it is a strictly decreasing function of* s. *We have that* $c^*(s) \leq \widehat{c}$ *(with equality if and only if* $s = 0$*).*

From Proposition 1, we see that in Case 2 the contract supporting the *unconstrained efficient allocation* also supports the run equilibrium from the *post-deposit* game. Except for the degenerate case of the run probability s being zero, this contract cannot be optimal since it delivers very low welfare in the run equilibrium. The optimal contract should optimize the ex-ante trade-off between the depositors' welfares in the run and the non-run equilibria. Intuitively: The trade-off depends on the run probability s. For positive s, a more conservative contract, still tolerating runs or eliminating runs completely, is desirable. If s is larger than the threshold probability s_0, eliminating runs is less costly (in terms of ex-ante welfare) than tolerating runs; hence the optimal contract is the best run-proof contract c^{early} in which the non-run equilibrium is implemented uniquely. On the other hand, if s is smaller than s_0, tolerating runs is less costly. Furthermore, $c^*(s)$ will be strictly decreasing in s since, as s increases, the ex-ante welfare leans more towards the welfare in the run equilibrium. The sunspot equilibrium allocation in this case is *not* a mere randomization over the *unconstrained efficient allocation* and the corresponding *run allocation*.

Example 4 *In Example 2, we have seen that as long as* $6.217686 < A \leq 10.277988$, *we are in Case 2. Set* $A = 8$. *We have* $s_0 = 1.382358 \times 10^{-3}$. *If* $s > s_0$, *the optimal contract is run-proof and* $c^*(s) = c^{early} = 4.155955$. *If* $s < s_0$, *the optimal contract tolerates runs and* $c^*(s)$ *is strictly decreasing in* s *with* $c^*(0) = \widehat{c} = 4.225479$. *If* $s = s_0$, *both the run-proof contract* (c^{early}) *and the run-tolerating contract are optimal. Figure 3 is a plot of* $c^*(s)$ *for this case.*



As we have discussed in section 4.1, we shift from Case 2 to Case 3 for even larger values of A:

Case 3 *The unconstrained efficient allocation is not BIC, i.e., we have* $c^{wait} < \widehat{c}$.

In this case, the *unconstrained efficient allocation* is not implementable since $\widehat{c}$ cannot satisfy the ICC. Like Case 2, the optimal contract for the *pre-deposit* game also involves the trade-off between the two *post-deposit* game equilibria, but the trade-off is constrained by the ICC.[22] This changes how $c^*(s)$ tolerates runs. To provide sufficient incentives for the patient depositors to choose late withdrawal, the ICC requires that c be not too large. Hence the binding ICC forces $c^*(s)$ to be more conservative than it would be without the ICC. If the ICC binds, then for small s when runs are still tolerated, c^* is independent of s.

Proposition 2 *In Case 3: (1) If s is larger than the threshold probability s_1 (specified in equation (26) in the proof in the Appendix), we have $c^*(s) = c^{early}$ and the optimal contract is run-proof. (2) If s is smaller than s_1, the optimal contract $c^*(s)$ tolerates runs and it is a weakly decreasing function of s. Furthermore, we have $c^*(s) = c^{wait}$ (i.e., ICC binds) for at least part of the run tolerating range of s.*

At least when s is sufficiently small, the ICC binds. So for Case 3, when s is sufficiently small, $c^*(s)$ does not change with s since the ICC does not depend on s. The ICC may bind for just part of the run-tolerating range of s (the first sub-case) or the whole run-tolerating range of s (the second sub-case). The PS numerical example belongs to the second sub-case. When the ICC binds, the allocation supported by the optimal contract *is* a simple randomization over the *constrained efficient allocation*[23] and the corresponding run allocation. The *unconstrained efficient allocation* is never supported by $c^*(s)$ since it is not implementable.

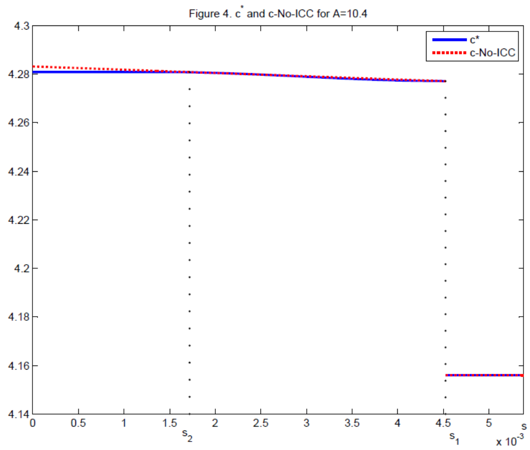
We next provide two examples of the optimal contract for Case 3, which correspond to the two sub-cases.

Example 5 *In Example 2, we have seen that as long as $A > 10.277988$, we have Case 3. Let A be equal to 10.4 and we are in the first sub-case of*

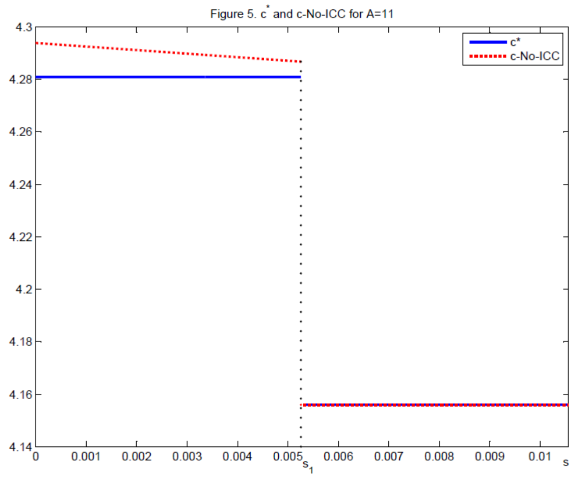
[22] In Case 2, the ICC cannot bind. To see this, firstly we know that a contract c which is larger than the contract supporting the *unconstrained efficient allocation* makes the run equilibrium more devastating. Secondly, it does not improve welfare at the non-run equilibrium. Hence to find the optimal contract, we need only to focus on contracts which are more conservative than the contract supporting the *unconstrained efficient allocation.* So binding incentive compatibility cannot occur in Case 2.

[23] Ennis and Keister (2010) define the *constrained efficient allocation* as "a (contingent) consumption allocation to maximize the ex ante expected utility of agents subject to incentive compatibility, sequential service, and resource feasibility constraints". We use the same definition.

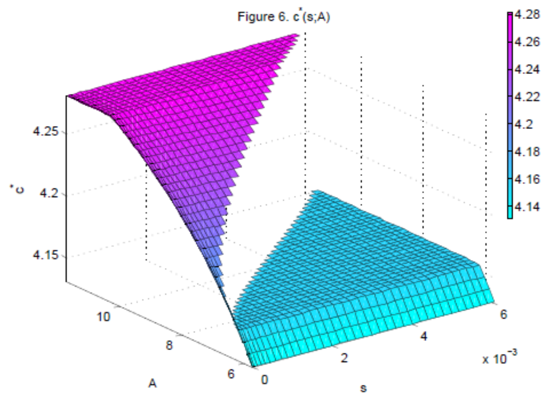
Case 3. *If* s *is larger than* $s_1 = 4.524181 \times 10^{-3}$*, the optimal contract is run-proof and* $c^*(s) = c^{early} = 4.155955$*. If* s *is smaller than* s_1*, the optimal contract* $c^*(s)$ *tolerates runs and it is a weakly decreasing function of* s*. To be more specific, if* s *is smaller than* $s_2 = 1.719643 \times 10^{-3}$*, the ICC binds and* $c^*(s) = c^{wait} = 4.280878$*; If* $s_2 \leq s < s_1$*, the ICC does not bind and* $c^*(s)$ *is strictly decreasing in* s*. To show the effect of the binding ICC, we plot in Figure 4 the hypothetical optimal contract in which the ICC is removed (labelled as c-No-ICC) as well as* c^**. The hypothetical curve is infeasible if – as we assume – types are private information.*



Example 6 *Let* A *be equal to* 11*. We are in the second sub-case of Case 3. If* s *is larger than* $s_1 = 5.281242 \times 10^{-3}$*, the optimal contract is run-proof and* $c^*(s) = c^{early} = 4.155955$*. If* $s < s_1$*, the optimal contract tolerates runs and* $c^*(s) = c^{wait} = 4.280878$ *since ICC binds. When* $s = s_1$*, both the run-proof contract (*c^{early}*) and the run-tolerating contract (*c^{wait}*) are optimal. Again we plot both* c^* *and the hypothetical optimal contract c-No-ICC in Figure 5.*



To show how the optimal contract changes with both s and A, we plot c^* versus s and A in Figure 6. Given the parameters we have used in previous examples, we have: if $A \leq A^{early} = 6.217686$, we are in Case 1 and the optimal contract doesn't tolerate runs and it is equal to $\widehat{c}$ the value of which depends on A. If $6.217686 < A \leq 10.277988$, we are in Case 2 and the optimal contract is strictly decreasing in s until it levels off at the best run-proof contract $c^{early} = 4.155955$. If $10.277988 < A$, we are in Case 3 and the ICC binds when s is small. The ICC may bind either in part of the run-tolerating range of s (the first sub-case) or the whole run-tolerating range of s (the second sub-case).



5 Summary

Our analysis is in terms of a very simple 2-person banking economy. It is very tractable: the deposit contract (optimal or otherwise) is completely characterized by the scalar c in the post-deposit game and the function $c^*(s)$ in the pre-deposit game. The simple model is, however, very rich: It takes sequential service seriously and allows for partial suspension of convertibility.

Understanding the pre-deposit game is essential to understanding bank runs. "Runs" in the DD post-deposit game are not runs in the truly sequential pre-deposit game.

Peck and Shell (2003) showed that sunspot-driven bank runs can be equilibria in the *pre-deposit* game of the corresponding DD-type *post-deposit* banking model. The optimal contract to the *pre-deposit* game in the PS example is a step-function of the run-probability: the optimal contract tolerating runs does not change with the run-probability until the probability reaches the threshold at which the optimal contract switches to the best run-proof contract.

In this paper, the general form of the optimal contract to the *pre-deposit* game is analyzed for different parameters. We start the analysis (backward in time) with the *post-deposit* game. For the "usual" parameter values, the *post-deposit* game can be in different forms depending on the contract chosen by the bank: For $c \in [0, c^{early}]$, the *post-deposit* game is "prisoners' dilemma" type with Pareto efficiency, and the unique equilibrium is non-run. For $c \in (c^{early}, c^{wait}]$, the decisions of the two depositors are strategic complements and the *post-deposit* game is "stag hunt" type. Hence both the run and non-run equilibria are Bayes-Nash. For $c \in (c^{wait}, 2y]$, the *post-deposit* game is "prisoners' dilemma" type, and the unique equilibrium is a bank-run.

For the "unusual" parameter values, the decisions of the two depositors *cannot* be strategic complements (for any contract!). In fact, they become strategic substitutes for some contracts: a patient depositor withdraws early if he believes the other depositor will withdraw late and vice versa. This implies that the optimal contract for the *pre-deposit* game *never* tolerates runs. This is why we focus on the "usual" parameter values and leave the "unusual" parameter values for the Appendix.

To solve the optimal contract for the *pre-deposit* game, the set of the "usual" parameter values is divided into three cases: the *unconstrained efficient allocation* (1) is DSIC, (2) BIC, but not DSIC, or (3) not even BIC. We characterize the parameters for each case. Bank runs matter in the last two cases. In both of these cases, the optimal contract switches to being run-proof when the run probability is sufficiently large. When runs are tolerated, whether or not the optimal contract becomes strictly more conservative as the run probability differs between the two cases: In Case 2, the ICC doesn't bind since the *unconstrained efficient allocation* is (weakly) implementable. As a result of balancing the trade-off between the run equilibrium and non-run equilibrium in the *post-deposit* game, the optimal contract adjusts continuously and becomes more conservative as the run probability increases, i.e., $c^*(s)$ is strictly decreasing in the run probability s. However, in Case 3, the ICC binds for small run-probabilities, which forces the contract to be more conservative than it would have been without the incentive constraint. Hence, for Case 3, the optimal contract does not change with s until the ICC no longer binds.

The implication of identifying the cases of the optimal contract is that *how* bank runs are tolerated can be complicated. As the economy's level of fragility or pessimism (indicated by the probability s) changes, how the banking contract and bank regulation should respond is different for different cases.

Our paper makes a contribution to the wider literature on sunspot equilibrium. This is yet another example in which not all sunspot equilibria are mere randomizations over certainty equilibria. See Shell (1987, 2008) and Shell and Smith (1992).

6 Appendix

6.1 Proof of Lemma 1

Proof. Inequality (3) holds if and only if

$$\frac{-(c^{1-b})/2 + (2y-c)^{1-b}(R^{1-b} - 1/2)}{(b-1)} > 0.$$

For $c \in [0, 2y]$ to satisfy the above inequality, it is necessary that $(R^{1-b} - 1/2) > 0$, which can be re-written as

$$b < 1 + \ln 2 / \ln R. \tag{14}$$

When b and R satisfy condition (14), define c^{early} to be the value of c such that inequality (3) holds as an equality. We have

$$c^{early} = 2y/[(2/R^{b-1} - 1)^{1/(b-1)} + 1].$$

Inequality (3) is equivalent to

$$c \in (c^{early}, 2y]. \tag{15}$$

■

6.2 Proof of Lemma 2

Proof. The difference between the left-hand side and the right hand side of inequality (5) is a continuous function of c. If $b < 1+\ln 2/\ln R$, the difference is decreasing in c. It changes from $+\infty$ when $c = 0$ to $-\infty$ when $c = 2y$. Hence there is a unique $c \in (0, 2y)$ which solves the equation

$$pv[(2y-c^{wait})R]+(1-p)v(yR) = p[v(c^{wait})+v(2y-c^{wait})]/2+(1-p)v(c^{wait}).$$

So when b and R satisfy the condition $b < 1 + \ln 2/\ln R$, inequality (5) is equivalent to

$$c \in [0, c^{wait}]. \tag{16}$$

■

6.3 Proof of Lemma 3

Proof. If condition (14) holds, c^{wait} and c^{early} are well defined. To get the condition on b and R implying that the inequality

$$c^{wait} > c^{early}. \tag{17}$$

holds, we merely need to replace c in inequality (5) by c^{early}. This results in

$$\frac{2/R}{(2/R^{b-1} - 1)^{1/(b-1)} + 1} < 1. \tag{18}$$

When b and R satisfy condition (14), $(2/R^{b-1} - 1)^{1/(b-1)}$ is decreasing in b. Hence inequality (18) is equivalent to

$$b < 2 \tag{19}$$

To summarize: the set of c satisfying both conditions (3) and (5) is non-empty if and only if b and R satisfy both inequality (14) and inequality (19), which results in condition (7). ■

6.4 Proof of Proposition 1

Proof. Since we have $\widehat{W}(c) > W^{run}(c)$, $W(c; s)$ is not continuous at c^{early} if $s > 0$. We study the two regions $[0, c^{early}]$ and $(c^{early}, c^{wait}]$ separately, and compare the maximum values of $W(c; s)$ in these two regions.

For $c \in [0, c^{early}]$, $W(c; s)$ is strictly increasing in c since $c^{early} < \widehat{c}$. Hence the maximum value of $W(c; s)$ over $[0, c^{early}]$ is achieved at c^{early}. Therefore the best run-proof contract is $c = c^{early}$.

For $c \in (c^{early}, c^{wait}]$, the maximum value of $W(c; s)$ may not be achievable because $(c^{early}, c^{wait}]$ is not closed. To fix this problem, we define a function $\widetilde{W}(c;s)$ on $[c^{early}, c^{wait}]$ by

$$\widetilde{W}(c;s) = (1-s)\widehat{W}(c) + sW^{run}(c).$$

When $c \in (c^{early}, c^{wait}]$, $\widetilde{W}(c;s) = W(c;s)$. When $c = c^{early}$, $\widetilde{W}(c;s) < W(c;s)$. Let $\widetilde{c}(s)$ be defined by

$$\widetilde{c}(s) = \arg \max_{c \in [c^{early}, c^{wait}]} \widetilde{W}(c;s).$$

We have

$$\widetilde{c}(s) = \max\{\frac{2y}{\gamma^{1/b}+1}, c^{early}\}, \tag{20}$$

where

$$\gamma = \frac{s(1-p)(pA + 1 - p\frac{2}{R^{b-1}}) + (p^2 A + (1-p)p\frac{2}{R^{b-1}})}{s(1-p)(1-pA) + p(2-p)A}.$$

It can be shown that $\widetilde{c}(s)$ is continuous in s. Furthermore, $\widetilde{c}(s)$ is strictly decreasing in s when s is small such that $\widetilde{c}(s) > c^{early}$.[24] We also have $c^{early} = \widetilde{c}(1) < \widetilde{c}(0) = \widehat{c}$. $\widetilde{W}(\widetilde{c}(s);s)$ is continuous in s and it is also strictly decreasing in s since $\widehat{W}(c) > W^{run}(c)$. Furthermore, we have

$$\widetilde{W}(\widetilde{c}(0);0) = \widehat{W}(\widehat{c}) > \widehat{W}(c^{early})$$

and

$$\widetilde{W}(\widetilde{c}(1);1) = W^{run}(c^{early}) < \widehat{W}(c^{early}).$$

Hence there is a unique $s_0 \in (0,1)$ such that

$$\widetilde{W}(\widetilde{c}(s_0);s_0) = \widehat{W}(c^{early}). \tag{21}$$

Obviously, we have $\widetilde{c}(s_0) > c^{early}$.

Hence if $s < s_0$, we have $c^*(s) = \widetilde{c}(s)$. The optimal contract $c^*(s)$ tolerates runs and it is a strictly decreasing function of s. We have $c^{early} < c^*(s) \leq \widehat{c}$

[24] It is easy to check that if $\widetilde{c}(s;A) > c^{early}$, $\widetilde{c}(s;A)$ is strictly decreasing in s because $AR^{b-1} > 1$. $AR^{b-1} > 1$ must hold in Case 2. To see why: it is trivial to establish that c^{early} must be larger than y. Hence in Case 2, we have $\widehat{c}(A) > y$, which leads to $AR^{b-1} > 1$.

(with equality if and only if $s = 0$).

If $s > s_0$, $c^*(s) = c^{early}$. The optimal contract is run-proof.

If $s = s_0$, $\widetilde{W}(\widetilde{c}(s); s) = \widehat{W}(c^{early})$. So both the run-proof contract (c^{early}) and the run-tolerating contract $(\widetilde{c}(s_0))$ are optimal. ■

6.5 Proof of Proposition 2

Proof. The proof is similar to that for Proposition 1. The only difference is that the ICC may bind. As before, we analyze separately the two regions $[0, c^{early}]$ and $(c^{early}, c^{wait}]$ separately, and compare the maximum values of $W(c; s)$ in these two regions.

For $c \in [0, c^{early}]$, it is easy to see that $W(c; s)$ is strictly increasing. Hence, as in Case 2, the best run-proof contract is $c = c^{early}$.

For $c \in (c^{early}, c^{wait}]$, the maximum value of $W(c; s)$ may not be achievable because $(c^{early}, c^{wait}]$ is not closed. To fix this problem and characterize the possibly binding ICC, we define the function $\overline{W}(c; s)$ on $[c^{early}, 2y]$ by:

$$\overline{W}(c; s) = (1 - s)\widehat{W}(c) + sW^{run}(c).$$

When $c \in (c^{early}, c^{wait}]$, we have $\overline{W}(c; s) = W(c; s)$. When $c = c^{early}$, we have $\overline{W}(c; s) < W(c; s)$. Let $\overline{c}(s)$ be defined by

$$\overline{c}(s) = \arg \max_{c \in [c^{early}, 2y]} \overline{W}(c; s).$$

We have that

$$\overline{c}(s) = \frac{2y}{\eta^{1/b} + 1}, \tag{22}$$

where

$$\eta = \frac{s(1 - p)(pA + 1 - p\frac{2}{R^{b-1}}) + (p^2 A + (1 - p)p\frac{2}{R^{b-1}})}{s(1 - p)(1 - pA) + p(2 - p)A}.$$

By using the same argument as that in Proposition 2, we can show that $\overline{c}(s)$ is continuous in s. Furthermore, $\overline{c}(s)$ is strictly decreasing in s when s is small such that $\overline{c}(s) > c^{early}$. We also have $c^{early} = \overline{c}(1) < \overline{c}(0) = \widehat{c}$. Note that in Case 3, we have $c^{wait} < \widehat{c}$. Hence there is a unique level of $s \in (0, 1)$, denoted by s_2, such that

$$\bar{c}(s_2) = c^{wait}. \tag{23}$$

That is, s_2 is the threshold run probability below which the ICC binds. Next, we need to check, when $s = s_2$, whether the optimal contract still tolerates runs. To do that, we define s_4 by

$$s_4 = \frac{\widehat{W}(c^{wait}) - \widehat{W}(c^{early})}{\widehat{W}(c^{wait}) - W^{run}(c^{early})}. \tag{24}$$

Obviously, we have $s_4 \in (0,1)$. There will be two sub-cases depending on whether the optimal contract still tolerates runs when when $s = s_2$.

In the first sub-case of Case 3, we have $s_4 > s_2$, that is, at the threshold run probability which makes the ICC just become non-binding, the optimal contract still tolerates runs. Now we need to determine the threshold run probability beyond which the optimal contract switches to being run-proof. That threshold level is s_3 which is defined by

$$\overline{W}(\bar{c}(s_3); s_3) = \widehat{W}(c^{early}). \tag{25}$$

Using the same argument as in Proposition 2, we know that $\overline{W}(\bar{c}(s); s)$ is continuous and strictly decreasing in s. Therefore, s_3 is unique. Since $s_4 > s_2$, we know that $s_3 > s_2$. The contract $c^*(s)$ satisfies the following: When $s < s_2$, the ICC binds and $c^*(s) = c^{wait}$ since we have

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s) > \widehat{W}(c^{early}).$$

When $s_2 \le s < s_3$, the ICC no longer binds and $c^*(s) = \bar{c}(s)$ since we have

$$W(\bar{c}(s); s) = \overline{W}(\bar{c}(s); s) > \widehat{W}(c^{early}).$$

When $s = s_3$, both $\bar{c}(s)$ and c^{early} are optimal since

$$W(\bar{c}(s); s) = \overline{W}(\bar{c}(s); s) = \widehat{W}(c^{early}).$$

When $s > s_3$, $c^*(s) = c^{early}$ since

$$W(\overline{c}(s); s) = \overline{W}(\overline{c}(s); s) < \widehat{W}(c^{early}).$$

To summarize, if $s_4 > s_2$ we have

$$c^*(s) = \begin{cases} c^{wait} \text{ if } s < s_2 \\ \overline{c}(s) \text{ if } s_2 \leq s \leq s_3 \\ c^{early} \text{ if } s_3 \leq s. \end{cases}$$

In the second sub-case of Case 3, we have $s_4 \leq s_2$, that is, at the run probability which makes the ICC just become non-binding, the optimal contract does not tolerate runs. Hence the optimal contract will switch to the best run-proof contract (c^{early}) when the ICC still binds. $c^*(s)$ satisfies the following property: When $s < s_4$, the ICC binds and $c^*(s) = c^{wait}$ since we have

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s) > \widehat{W}(c^{early}).$$

When $s = s_4$, both c^{wait} or c^{early} are optimal since we have

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s_4) = \widehat{W}(c^{early}).$$

When $s_4 < s < s_2$, we have $c^*(s) = c^{early}$. This is because the ICC binds and

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s) < \widehat{W}(c^{early}).$$

When $s_2 \leq s$, $c^*(s; A)$ is still equal to c^{early}. This is because the ICC no longer binds and

$$W(\overline{c}(s); s) = \overline{W}(\overline{c}(s); s) < \overline{W}(\overline{c}(s_2); s_2) = \overline{W}(c^{wait}; s_2) < \widehat{W}(c^{early}).$$

To summarize, if $s_4 \leq s_2$, we have

$$c^*(s) = \begin{cases} c^{wait} \text{ if } s \leq s_4 \\ c^{early} \text{ if } s \geq s_4. \end{cases}$$

We can see, in both of the two sub-cases, $c^*(s)$ switches to run-proof if the run probability is larger than the threshold. Let s_1 denote that threshold run probability and we can have

$$s_1 = \begin{cases} s_3 \text{ if } s_4 > s_2 \\ s_4 \text{ if } s_4 \leq s_2. \end{cases} \tag{26}$$

■

6.6 The Optimal Contract for b and R with "unusual" values.

6.6.1 The Post-Deposit Game

For b and R with "unusual" values (i.e., condition (7) is not satisfied), we either have

$$2 \leq b < 1 + \ln 2 / \ln R \tag{27}$$

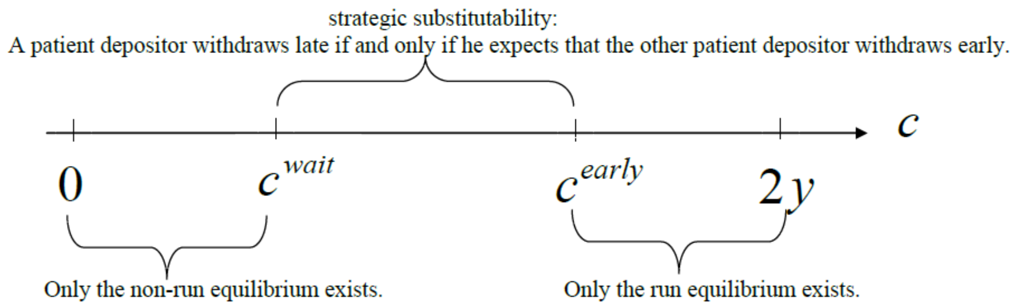
or

$$b \geq 1 + \ln 2 / \ln R. \tag{28}$$

For b and R satisfying inequality (27), we have $c^{wait} \leq c^{early}$. (This can be seen directly from the proof of Lemma 3). In contrast to the "usual" parameter values, now the order of c^{early} and c^{wait} is reversed. Thus, compared to "usual" parameter values, the *post-deposit* game is in different form for a given contract (See Figure 8.): From the pay-off matrix of the deposit game, we see that for $c \in [0, c^{wait}]$, we have $T_2 > T_1$ and $T_4 \geq T_3$. (L, E) is the dominant strategy for each depositor. The *post-deposit* game is "prisoners' dilemma" type, but with Pareto efficiency. For $c \in (c^{early}, 2y]$, we have $T_2 < T_1$ and $T_4 < T_3$. (E, E) is the dominant strategy for each depositor. The *post-deposit* game is "prisoners' dilemma" type. For $c \in (c^{wait}, c^{early}]$, we have $T_2 \geq T_1$ and $T_4 < T_3$.The *post-deposit* game is the "chicken game" type. And the interval $(c^{wait}, c^{early}]$ is the region of c for which the *post-deposit* game is the "chicken game" type and the patient depositors' withdrawal decisions exhibit strategic substitutability (rather than strategic complements):

A patient depositor withdraws late if and only if he expects that the other depositor – if patient – to withdraw early.[25] Thus, in contrast to the "usual" parameter values for which the set of DSIC contracts is a strict subset of the set of BIC contracts, now the two sets are the same and both of them are $[0, c^{wait}]$.

Figure 8. Equilibrium in the Post-Deposit Game



For b and R satisfying inequality (28), from the proof of Lemma 1 we can see that there is no run equilibrium for any contract $c \in [0, 2y]$ in the *post-deposit* game. Therefore any BIC contract is also a DSIC contract and the set of BIC contracts and the set of DSIC contracts are still the same.

6.6.2 The Optimal Contract for the Pre-Deposit Game

According to the Revelation Principle, to find c^* for the pre-deposit game, we only have to focus on the BIC contracts. As we have seen, for the "unusual" parameter values, a BIC contract is also a DSIC contract. Hence, bank runs are not relevant for the optimal contract c^* and c^* maximizes the expected welfare of the depositors at the non-run equilibrium:

$$c^* = \arg\max_{c} \widehat{W}(c) \tag{29}$$

s.t. c satisfies ICC (i.e. condition (5)).

[25] It is well-known that a mixed strategy equilibrium also exists in the "chicken game". We did not put too much emphasis on the analysis of the *post-deposit* game here since *not* all of the *post-deposit* games are relevant for the optimal contract for the *pre-deposit* game. According to the Revelation Principle, the optimal contract must be BIC. Thus, we only need to focus on the contract $c \in [0, c^{wait}]$. In other words, the bank will never choose a contract which leads to a "chicken game" in the *post-deposit* game.

For b and R satisfying inequality (27), we know that c satisfies (5) if and only if $c \leq c^{wait}$. Hence the solution to the problem (29) is:

$$c^* = \min\{\widehat{c}(A), c^{wait})\}.$$

For b and R satisfying inequality (28), c^{wait} is not well-defined. From the proof of Lemma 2, we know that the difference between the left-hand side and the right hand side of inequality (5) is no longer decreasing in c. Let us denote that difference by $Diff(c)$. $Diff(c)$ is strictly decreasing in c for $c \in [0, \overline{c^{wait}}]$ and strictly increasing in c when $c \in [\overline{c^{wait}}, 2y]$, where

$$\overline{c^{wait}} = \frac{2y}{[\frac{1-p/2}{-p(R^{1-b}-1/2)}]^{-1/b} + 1}.$$

Furthermore, $Diff(0) = +\infty$ and $Diff(2y) = +\infty$. Therefore, if $Diff(\overline{c^{wait}}) \geq 0$, (5) holds for any $c \in [0, 2y]$. If $Diff(\overline{c^{wait}}) < 0$, (5) holds for

$$c \in [0, c^{wait1}] \cup [c^{wait2}, 2y], \tag{30}$$

where $c^{wait1} < c^{wait2}$ and they are the two solutions for $Diff(c) = 0$. Hence if $Diff(\overline{c^{wait}}) \geq 0$ or $Diff(\overline{c^{wait}}) < 0$ and $\widehat{c}(A)$ satisfies condition (30), the ICC doesn't bind and the solution to the problem (29) is

$$c^* = \widehat{c}(A).$$

If $Diff(\overline{c^{wait}}) < 0$ and $\widehat{c}(A)$ doesn't satisfy condition (30), the ICC binds and c^* is equal to c^{wait1} or c^{wait2} depending on which one delivers higher expected welfare at the non-run equilibrium $\widehat{W}(c)$.

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