Diamond and Dybvig Bank Runs: Worked Example

Economics 4905: Financial Fragility and the Macroeconomy

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Setup

Probability of impatience is $\lambda = 30\%$.

Utility is $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, where $\gamma = 1.01$.

Costless storage.

$R = 2$

Each individual has endowment $\omega = 100$.

If the illiquid asset is harvested early the rate of return is zero, if harvested late the rate of return $(R-1)$ is 100%.

Let (d_1, d_2) be the deposit contract.

Expected Utility, W

The depositor's expected utility W as a function of early consumption c_1 and late consumption c_2 will be

$W = \lambda u(c_1) + (1 - \lambda)u(c_2)$

$W = \frac{\lambda c_1^{1-\gamma}}{1-\gamma} + \frac{(1-\lambda)c_2^{1-\gamma}}{1-\gamma} = \frac{0.3c_1^{-(0.01)}}{-(0.01)} + \frac{0.7c_2^{-(0.01)}}{-(0.01)}$

Preference for Consumption Smoothing

The smoothed allocation $(\bar{c}, \bar{c})$, where $\bar{c} = \lambda c_1 + (1-\lambda)c_2$, is preferred to (c_1, c_2) if $c_1 \neq c_2$.

Proof:

$u'(c) = \frac{(1-\gamma)c_1^{-\gamma}}{1-\gamma} = c^{-\gamma} > 0$

$u''(c) = -\gamma c^{-\gamma-1} < 0$

So $u(c)$ is strictly concave (the consumer is risk-averse). Concave functions lie above their chords (Jensen's inequality):

$u(\lambda c_1 + (1-\lambda)c_2) > \lambda u(c_1) + (1-\lambda)u(c_2)$ when $c_1 \neq c_2$,

So $W(\bar{c}, \bar{c}) > W(c_1, c_2)$ where $\bar{c} = \lambda c_1 + (1-\lambda)c_2$.

The Resource Constraint

The resource constraint RC will be

$$(1-\lambda)d_2 \leq (\omega - \lambda d_1)R \quad \Rightarrow \quad 0.70d_2 \leq 2(100 - 0.30d_1)$$

Where d_t is the withdrawal allowed in period $t = 1, 2$.

Depositors cannot withdraw more than what is left in the bank in the second period (although the remaining deposits will have grown with a return factor R).

The left-hand side of the inequality is the funds to be withdrawn in period 2. The right-hand side is the resources available in period 2. If the inequality is violated, the bank is insolvent.

Incentive Compatibility

The incentive compatibility constraint (ICC) is $d_1 \leq d_2$
If the inequality does not hold, then there will not be sufficient incentive to withdraw late. Everyone will attempt to withdraw early; the depositors do not self-select correctly.

The Optimal Deposit Contract

Now we can solve for the so-called "optimal deposit contract" to the post-deposit game, i.e., provide the numerical values for (c_1^*, c_2^*). The bank will design a contract to maximize $W(d_1, d_2)$ while constrained by the resources such that $(1-\lambda)d_2 = (\omega - \lambda d_1)R$.

$$\max\{W(d_1, d_2)\} \;\Rightarrow\; \max\{\lambda u(d_1) + (1-\lambda)u(d_2)\}$$

$$\text{Subject to } (1-\lambda)d_2 - (\omega - \lambda d_1)R = 0$$

A Note on Lagrange Optimization

Lagrangian optimization is used to find the points of a maximum or a minimum of a function along a boundary (i.e., a constraint). In our case, due to the monotonic but diminishing returns of consumption to utility, this will lead to a maximization.

We could also re-write one of our input variables with the constraint and plug it back into our original function, then set the derivative to zero. However, while the result will be equivalent, the math may become somewhat messier, especially in higher-dimensional cases.

Lagrange Optimization in General

Say we wish to find the maximum of $U : \mathbb{R}^n \to \mathbb{R}$ along a constraint $C(x) = 0$, a level set, where $x \in R^n$. We need only find points where the gradient of the function is a scalar multiple of the gradient of the constraint.

$$\nabla U = -\delta \nabla C \quad \text{or} \quad \nabla U + \delta \nabla C = 0$$

For convenience, we often define a Lagrangian as $\mathscr{L}(x) = U(x) + \delta C(x)$, such that the points x that maximize U will satisfy $\nabla \mathscr{L}(x) = 0$. This means $\frac{\partial \mathscr{L}}{\partial x_i} = 0 \;\; \forall x_i$ and $\frac{\partial \mathscr{L}}{\partial \delta} = 0$ if x maximizes U along the constraint.

Setting Up the Lagrangian

The Lagrangian is

$$\mathscr{L}(d_1, d_2) = [\lambda u(d_1) + (1-\lambda)u(d_2)] - \delta[(1-\lambda)d_2 - (\omega - \lambda d_1)R]$$

The first order constraints conditions become

$$\frac{\partial \mathscr{L}}{\partial d_1} = \lambda u'(d_1) - \delta\lambda R = 0$$

$$\Rightarrow \quad \lambda u'(d_1) = \delta\lambda R \;\Rightarrow\; u'(d_1) = \delta R$$

And

$$\frac{\partial \mathscr{L}}{\partial d_2} = (1-\lambda)u'(d_2) - \delta(1-\lambda) = 0$$

$$\Rightarrow \quad (1-\lambda)u'(d_2) = \delta(1-\lambda) \;\Rightarrow\; u'(d_2) = \delta$$

Dividing out the Lagrange Multiplier

Thus:

$$\frac{u'(d_1)}{u'(d_2)} = \frac{\delta R}{\delta} = R$$

And recalling that $u'(c) = c^{-\gamma}$,

$$\frac{d_1^{-\gamma}}{d_2^{-\gamma}} = \left(\frac{d_1}{d_2}\right)^{-\gamma} = \left(\frac{d_2}{d_1}\right)^{\gamma} = R$$

So $\frac{d_2}{d_1} = R^{\frac{1}{\gamma}}$.

Plugging Back into the Constraint

If the RC requires $(1-\lambda)d_2 = (\omega - \lambda d_1)R$, then $d_2 = \frac{(\omega-\lambda d_1)R}{(1-\lambda)}$. Therefore:

$$\frac{(\omega - \lambda d_1)R}{d_1(1-\lambda)} = R^{\frac{1}{\gamma}} \;\Rightarrow\; (\omega - \lambda d_1) = R^{\frac{1}{\gamma}-1} d_1(1-\lambda)$$

Dividing through by d_1,

$$\frac{\omega}{d_1} - \lambda = R^{\frac{1}{\gamma}-1}(1-\lambda) \;\Rightarrow\; \frac{1}{d_1} = \frac{R^{\frac{1}{\gamma}-1}(1-\lambda)+\lambda}{\omega}$$

So

$$c_1^* = d_1^* = \frac{\omega}{R^{\frac{1}{\gamma}-1}(1-\lambda)+\lambda}$$

And $c_2^* = d_2^*$ may be found using $d_2^* = R^{\frac{1}{\gamma}} d_1^*$.

Numerical Results

Plugging in our parameter values,

$$d_1^* = \frac{\omega}{R^{\frac{1}{\gamma}-1}(1-\lambda)+\lambda} = \frac{100}{2^{\left(\frac{1}{1.01}-1\right)}(0.70)+0.30} = 100.48$$

$$d_2^* = R^{\frac{1}{\gamma}} d_1^* = 2^{\frac{1}{1.01}}(100.48) = 199.59$$

So the optimal deposit contract is $(d_1^*, d_2^*) = (100.48, 199.59)$.

Running on the Bank

Is there a bank run to the "so-called optimal deposit contract" for this bank?

Yes, as $d_1^* > \omega = 100$. If all depositors (not just the impatient ones, but the patient ones, too) decide to withdraw from the bank in the first period, then the bank will be unable to provide enough funds to pay everyone.

Let the exogenous run probability be s. Then depositors will only deposit at the bank if their expected utility from doing so is greater than the utility they would receive in autarky, such that

$$E_s[W] = (1-s)W_{no-run} + sW_{run} \geq W_{Autarky}$$

Initial Deposit Decisions and Sequential Service

If there is a run, then each depositor will only receive d_1^* with probability $\frac{\omega}{d_1^*}$ if there is a total suspension of convertibility, so

$$W_{run} = \frac{\omega}{d_1^*} u(d_1^*)$$

While in the no-run equilibrium, individuals will expect to receive $W_{no-run} = W(d_1^*, d_2^*)$.
If $W_{run} < W_{Autarky}$ and if s is made sufficiently large, $E_s[W] < W_{Autarky}$ and the depositor will choose not to deposit (and instead accept the illiquid autarky scenario). However, for low enough s, depositors may tolerate the possibility of runs.

Interpreting the Post-Deposit Game

Now we know there exist 2 equilibria in the post-deposit game, one non-run, the other run. One of the equilibria is the "good" non-run equilibrium. Because the post-deposit game is Baysian Incentive Compatible but not Dominant Strategy IC, the other equilibrium is a bad one, the run-equilibrium.

The so-called "optimal contract" for the post-deposit game does not lead to a uniquely implementable equilibrium. This is because the "optimal contract" is based on unconstrained optimization in which a depositor's type is public knowledge, or at least known by the bank.

This would imply a source of financial fragility... but will depositors choose to deposit at the bank to begin with?