Economics 4905 Fall 2016
Cornell University
Financial Fragility and the Macroeconomy

Problem Set 4 Solutions

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The Overlapping Generations Model

The model is set up as follows:

- 2-period lives
- 1 commodity per period, $\ell = 1$
- Stationary environment
- 1 person per generation

Where

$$\omega_0^1 = B > 0 \text{ for } t = 0$$
$$(\omega_t^t, \omega_t^{t+1}) = (A, B) >> 0 \text{ for } t = 1, 2, ...$$
$$u_0(x_0^1) = D \log x_0^1 \text{ for } t = 0$$
$$u_t(x_t^t, x_t^{t+1}) = C \log x_t^t + D \log x_t^{t+1} \text{ for } t = 1, 2, ...$$

Define

$$z^t = \omega_t^t - x_t^t \text{ and } z^{t+1} = x_t^{t+1} - \omega_t^{t+1}$$

Solve for

a. The offer curve, OC

b. The set of equilibrium money prices, $\mathscr{P}^m$

c. The steady-states

d. The full dynamic analysis, including the stability of steady states

For each of the following cases:

1. $A = B = 1,\ C = 1,\ D = 5,$
   and $m_0^1 = 1$ for $s = 0$ and $m_s^t = 0$ otherwise.

2. $A = B = 1,\ C = 1,\ D = 5,$
   and $m_0^1 = 4$, $m_1^2 = 6$, and $m_s^t = 0$ otherwise.

3. $A = B = 2,\ C = 4,\ D = 1,$
   and $m_0^1 = 1$, $m_s^t = 0$ otherwise

4. $A = 10, B = 1,\ C = 5,\ D = 1,$
   and $m_0^1 = 1$, $m_s^t = 0$ otherwise

Solutions:

By solving the optimization problem

$$\arg \max_{x_t^t, x_t^{t+1}} \{u_t(x_t^t, x_t^{t+1})\}$$

$$\text{s.t. } p^t x_t^t + p^{t+1} x_t^{t+1} \leq p^t \omega_t^t + p^{t+1} \omega_t^{t+1}$$

We may derive the offer curve in general as

$$z^{t+1} = \frac{BCz^t}{AD - (C + D)z^t}$$

Where $z^{t+1} = x_t^{t+1} - \omega_t^{t+1}$ and $z^t = \omega_t^{t+1} - x_t^{t+1}$.

Case 1.

1.a. Plugging into the offer curve equation above, we get

$$z^{t+1} = \frac{z^t}{5 - 6z^t}$$

1.c. The steady states will be where $z = \frac{z^t}{5-6z^t}$. Thus, $z = 0$ will be the non-monetary steady state, while the second steady-state may be found as

$$z = \frac{z}{5-6z} \quad \Rightarrow \quad 1 = \frac{1}{5-6z} \quad \Rightarrow \quad 5 - 6z = 1 \quad 6z = 4 \quad \Rightarrow \quad z = \frac{4}{6} = \frac{2}{3}$$

The monetary steady state is thus $\bar{z} = \frac{2}{3}$.

1.b. The set of equilibrium money prices is thus

$$\mathscr{P}^m = \left[0, \frac{2}{3}\right]$$

1.d. If $0 < p^m < \frac{2}{3}$, then z^t is declining, and the bubble fades away through inflation. $z = 0$ is a stable steady state, in which money is worthless. $z = \bar{z}$ is an unstable steady state. If $z > \bar{z}$, hyperinflation ensues and the bubble bursts in finite time. We may note that this is a Samuelson case.

Case 2.

2. We may once again enter in our parameters to obtain

$$\frac{z^t}{5 - 6z^t}$$

Since $m_0^1 = 4$ and $m_1^2 = 6$, we may note that $m_0^1 + m_1^2 = 10$. Mr. 1. exchanges 10 dollars for chocolate in period 2. We may treat Mr. 1 as Mr. 0 in the previous example. All purchases made in excess of the endowment must be financed with money.

$$z^t = (m_0^1 + m_1^2)p^m \leq \omega_{t-1}^t$$

So

$$\bar{z} = 10\bar{p}^m$$

Since $\bar{z} = \frac{2}{3}$, we get that $\bar{p}^m = \frac{2}{30} = \frac{1}{15}$. The set of equilibrium money prices is thus

$$\mathscr{P}^m = \left[0, \frac{1}{15}\right]$$

There are two steady states, where $p^m = 0$ and where $\bar{p}^m = \frac{1}{15}$. The non-monetary autarky equilibrium is stable, but not Pareto optimal. The monetary equilibrium is Pareto Optimal, but unstable. As noted in Case 1, the economy is a Samuelson one, and the dynamics are the same as in the preceding problem.

Case 3.

3.a. Here, we may find the offer curve as

$$\frac{8z^t}{2-5z^t}$$

3.c. As usual, $z=0$ is a stable (non-monetary) equilibrium. If $z \neq 0$, however, the fixed point of the system will be

$$z = \frac{8z}{2-5z} \quad \Rightarrow 1 = \frac{8}{2-5z} \quad \Rightarrow \quad 2-5z=8 \quad \Rightarrow \quad 5z=-6$$

Since our proposed steady state is $z=-\frac{6}{5} \notin \mathbb{R}_+$, we do not have a monetary steady state. Autarky is thus Pareto Optimal and stable, and we are in a Ricardo economy.

3.b. $\mathscr{P}^m = \{0\}$

3.d. The non-monetary steady state where $p^m=0$ is unstable, unique, and Pareto Optimal. Trajectories originating away from it will be deflationary.

Case 4.

4.a. The offer curve will be

$$z^{t+1} = \frac{5z^t}{10-6z^t}$$

4.c. The non-monetary equilibrium will be $z=z^t=z^{t+1}=0$. If $z \neq 0$, then

$$z = \frac{5z}{10-6z} \quad \Rightarrow \quad 1 = \frac{5}{10-6z} \quad \Rightarrow \quad 10-6z=5 \quad \Rightarrow \quad 6z=5$$

Therefore the monetary equilibrium will be $\bar{z}=\frac{5}{6}$.

4.b. Since $m_0^1=1$, $\bar{z}=m_0^1\bar{p}^m$, such that $\frac{5}{6}=\bar{p}^m$. Thus, $\mathscr{P}^m = \left[0, \frac{5}{6}\right]$.

4.d. We are once more in the Samuelson case, where the non-monetary equilibrium is stable but not Pareto Optimal, while the monetary equilibrium will be Pareto Optimal but unstable. If $\bar{p}^m \in (0, \frac{5}{6})$, then z^t declines asymptotically to zero. If $p^m > \frac{5}{6}$, then z^t increases until the bubble bursts in finite time.

Further Remarks

i. We may note that the three Samuelson cases exhibit the same qualitative dynamics.

ii. In cases (1) and (2), near $z = 0$, $\lim_{z^t \to 0} \frac{z^{t+1}}{z^t} = \frac{1}{5} < 1$, so the excess demand trajectory is tending toward zero and we are in a Samuelson case. The same happens in case (4), where near zero $\lim_{z^t \to 0} \frac{z^{t+1}}{z^t} = \frac{5}{10} = \frac{1}{2} < 1$.

iii. The Ricardo case results in a non-monetary, no-trade case.

iv. In case (3), we may see that in arbitrarily small neighborhoods about the origin $\lim_{z^t \to 0} \frac{z^{t+1}}{z^t} = \frac{8}{2} = 4 > 1$. Hence, we are in the Ricardo case.