Economics 6130
Cornell University
Fall 2016
Macroeconomics, I - Part 2

Problem Set #2 Solutions

1 Overlapping Generations

Consider the following OLG economy:

2-period lives.
1 commodity per period, $l = 1$.
Stationary endowments:

$$\omega_0^1 = B > 0 \text{ for } t = 0$$
$$(\omega_t^t, \omega_t^{t+1}) = (A, B) > 0 \text{ for } t = 1, 2, ...$$

Stationary preferences:

$$u_0(x_0^1) = D \log(x_0^1) \text{ for } t = 0$$
$$u_t(x_t^t, x_t^{t+1}) = C \log(x_t^t) + D \log(x_t^{t+1}) \text{ for } t = 1, 2, ...$$

Passive fiscal policy:

$$m_0^1 = 2 \qquad m_t^s = 0 \text{ otherwise}$$

Goods price of money is $p^m \geq 0$.

For each of the following cases: Calculate the offer curve for Mr. $t \geq 1$. Precisely plot (use graph paper if necessary) the offer curve in excess demand space $(x_t^t - \omega_t^t, x_t^{t+1} - \omega_t^{t+1})$ for Mr. $t \geq 1$. Plot the reflected offer curve, and analyze the global dynamics.

(a) A = 10, B = 12, C = 1, D = 0.98

(b) A = 15, B = 10, C = 2, D = 3

(c) A = 40, B = 30, C = 0.5, D = 0.5

(d) A = 8 B = 4, C = 1.9, D = 0.95

Is there a pattern?

Derive the conditions on the MRS for a "Samuelson" verses a "Classical" (or "Ricardo") economy and relate them to the above.

Solution: The problem facing an agent born in date $t \geq 1$ is

$$\max_{(x_t^t, x_t^{t+1}, x_t^{t,m}, x_t^{t+1,m})} C \log x_t^t + D \log x_t^{t+1}$$
$$\text{s.t. } p^t x_t^t + p^{t+1} x_t^{t+1} + p^m (x_t^{t,m} + x_t^{t+1,m}) \leq p^t A + p^{t+1} B$$

Taking first order conditions and dividing through implies that $\frac{p^t}{p^{t+1}} = \frac{C x_t^{t+1}}{D x_t^t}$. We then divide the budget constraint of a date t household by p^{t+1} and plug this in to get that

$$x_t^{t+1} - B = \frac{C x_t^{t+1}}{D x_t^t}(A - x_t^t)$$

For the rest of the problem, let z^{t+1} be the excess demand of an old household. Let z^t be the excess demand of a young household, and $s^t = -z^t$ be the excess supply of a young household. Plugging this in above yields that

$$z^{t+1} = -\frac{C(z^{t+1} + B)}{D(z^t + A)} z^t$$

Solving this for z^{t+1} yields the offer curve:

$$z^{t+1} = \frac{-CBz^t}{(D + C)z^t + DA}$$

To solve for the inverse offer curve we use that $s^t = -z^t$, which yields

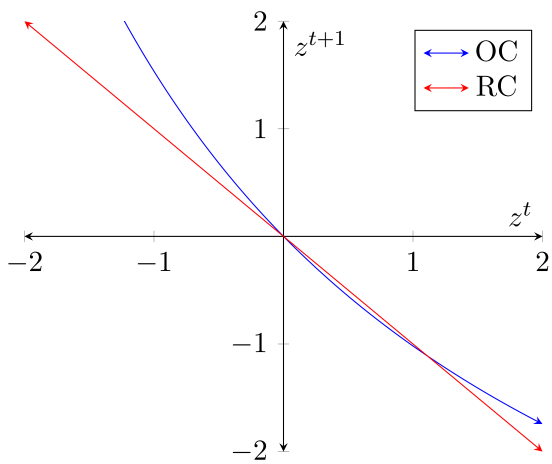
$$z^{t+1} = \frac{CBs^t}{DA - (D + C)s^t}$$

Below we calculate and plot for each case, the offer curve in excess demand space, the reflected offer curve, and the resource constraint.

(a) A = 10, B = 12, C = 1, D = 0.98

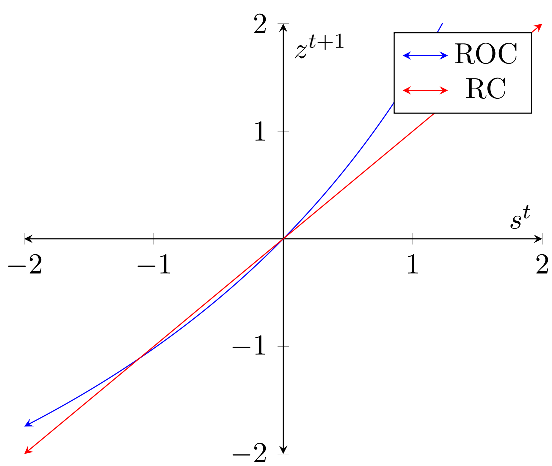
The offer curve:

$$z^{t+1} = \frac{-12z^t}{1.98z^t + 9.8}$$



The reflected offer curve:

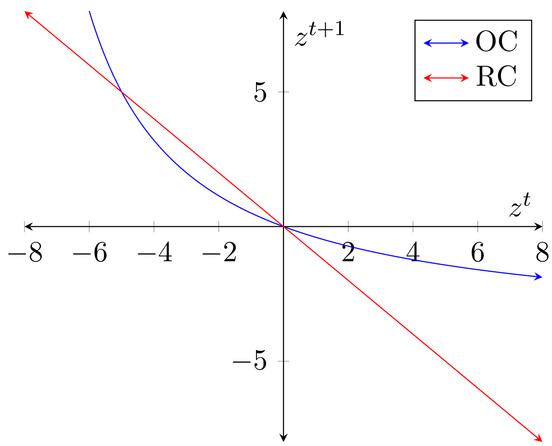
$$z^{t+1} = \frac{12s^t}{9.8 - 1.98s^t}$$



(b) A = 15, B = 10, C = 2, D = 3

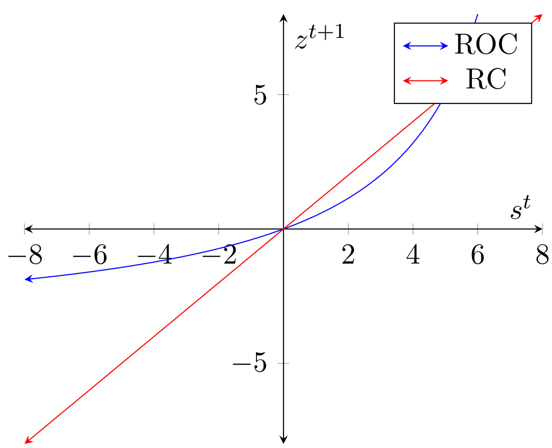
The offer curve:

$$z^{t+1} = \frac{-20z^t}{5z^t + 45}$$



The reflected offer curve:

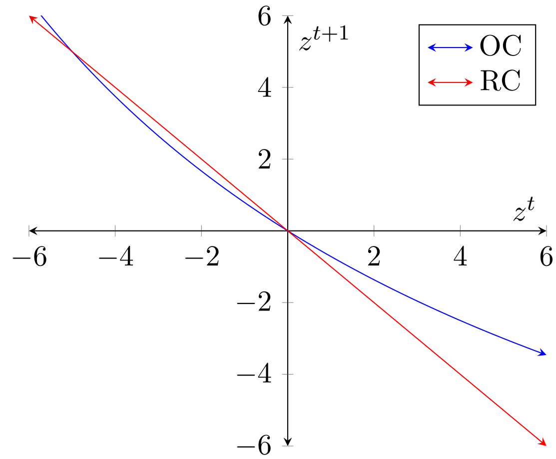
$$z^{t+1} = \frac{20s^t}{45 - 5s^t}$$



(c) A = 40, B = 30, C = 0.5, D = 0.5

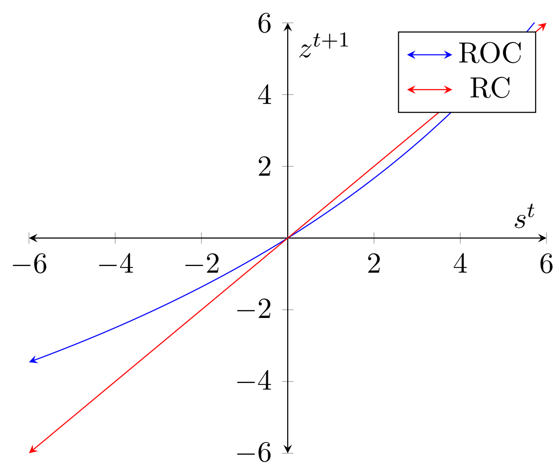
The offer curve:

$$z^{t+1} = \frac{-15z^t}{z^t + 20}$$



The reflected offer curve:

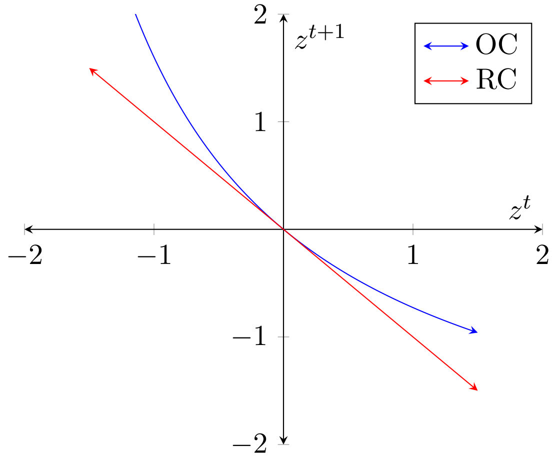
$$z^{t+1} = \frac{15s^t}{20 - s^t}$$



(d) A = 8 B = 4, C = 1.9, D = 0.95

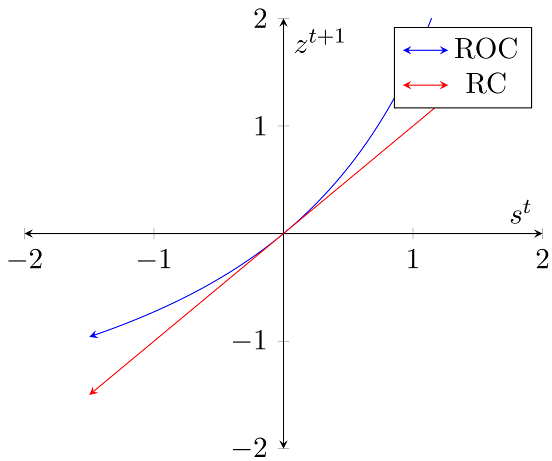
The offer curve:

$$z^{t+1} = \frac{-7.6z^t}{2.85z^t + 7.6}$$



The reflected offer curve:

$$z^{t+1} = \frac{7.6s^t}{7.6 - 2.85s^t}$$



The global dynamics we get are that of Ricardo in (a) and (d). We get the global dynamics of Samuelson in (b) and (c). The equation we get from the MRS is that we have Ricardo whenever $CB \geq DA$, and we get Samuelson whenever $CB < DA$.

2 Overlapping Generations

Consider the following OLG economy:

Pure exchange, 2-period lives, one consumer per generation.

$$u_0(x_0^1) = x_0^1$$
$$\omega_0^1 = 1, \text{ for } t = 0,$$

$$u_t(x_t^t, x_t^{t+1}) = x_t^t + x_t^{t+1}$$
$$(\omega_t^t, \omega_t^{t+1}) = (1, 1) \text{ for } t = 1, 2, ...$$

Money transfers:

$$m_0^1 = 2, m_1^1 = -1,$$
$$m_1^2 = 1, m_t^s = 0 \text{ otherwise.}$$

(a) What is the non-monetary equilibrium allocation? What are the prices? What are the interest rates?

(b) Derive the reflected offer curve for consumer $t = 1, 2, ...$

(c) Derive the set of equilibrium money prices.

(d) Draw the phase diagram and show the full evolution of this economy (depending on the price of money).

(e) What is the Pareto optimal allocation associated with the above (money) tax-transfer policy?

(f) Find an alternative tax-transfer policy and associated allocation which is not Pareto optimal but in which everyone is strictly better off than they would be in autarky.

(g) Find an alternative tax-transfer policy and associated allocation which is Pareto optimal and in which everyone is strictly better off than they would be in the non-monetary equilibrium.

Solution:

(a) When there is no money, the initial old problem is

$$\max_{x_0^1} x_0^1$$
$$\text{s.t. } p^1 x_0^1 \leq p^1$$

And the problem of a person born in date t is

$$\max_{(x_t^t, x_t^{t+1})} x_t^t + x_t^{t+1}$$
$$\text{s.t. } p^t x_t^t + p^{t+1} x_t^{t+1} \leq p^t + p^{t+1}$$

Normalizing the price of money in date 1 to be equal to 1, we get that the initial old choose $x_0^1 = 1$. Now looking at the date one budget constraint we must have that $x_1^1 = 1$, which from the budget constraint of the person born in date 1 implies that $x_1^2 = 1$. And we see that this argument goes on ad infinitum. So our equilibrium allocation is $(c_0^1, \{c_t^t, c_t^{t+1}\}) = (1, \{1, 1\}) \, \forall t$. And from the first order conditions of the date t generation (since we are at an interior solution) we can get the prices, $p^t = 1 \, \forall t$. Which implies the interest rate is $1 + r_t = 1$.

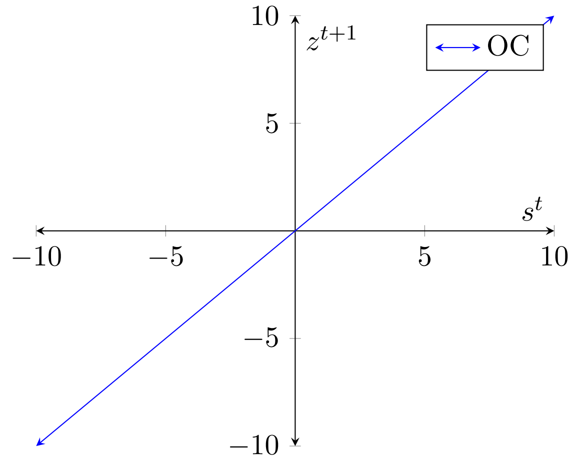
(b) The problem of the initial old is now

$$\max_{x_0^1} x_0^1$$
$$\text{s.t. } x_0^1 + p^m x_0^{1,m} \leq 1 + 2p^m$$

And the problem of generation t is now

$$\max_{(x_t^t, x_t^{t+1})} x_t^t + x_t^{t+1}$$
$$\text{s.t. } p^t x_t^t + p^{t+1} x_t^{t+1} + p^m (x_t^{t,m} + x_t^{t+1,m}) \leq p^t + p^{t+1} + p^m (m_t^t + m_t^{t+1})$$

We know that $p^m (x_t^{t,m} + x_t^{t+1,m}) = x_0^{1,m} = 0$ in equilibrium. And for every date t we note also that $(m_t^t + m_t^{t+1}) = 0$.
First order conditions again imply that $p^t = 1 \, \forall t$. We note that the solution for the initial old is now $x_0^1 = 1 + 2p^m$ which implies by the resource constraint that $x_1^1 = 1 - 2p^m$. We continue this ad infinitum as in part (a) to drive at the solutions $(x_{t-1}^t, x_t^t) = (1 + 2p^m, 1 - 2p^m) \, \forall t$. We now get that the reflected offer curve from knowing $x_t^{t+1} = 2 - x_t^t$, which implies our reflected offer curve, $z^{t+1} = s^t$. Graphed below



(c) The equilibrium set of money prices must be such that no individuals consumption is negative. This implies we must look at the consumption of the young. So we need that $x_t^t = 1 - 2p^m \geq 0$. Which implies our equilibrium set of money prices is $p^m \in [0, \frac{1}{2}]$.

(d) The phase diagram is as above, noting that the offer curve in this case is exactly equal to the resource constraint. Thus, for any price of money in the equilibrium set, this picks out a stationary point on the phase diagram.

(e) The Pareto optimal allocation associated with the above policy is when $p^m = \frac{1}{2}$, corresponding to the allocation $x_0^1 = 2$ and $(x_t^t, x_t^{t+1}) = (0, 2)$. The utility of the initial old generation is maximized ($u_0 = 2$, up from $u_0 = 1$ in autarky), and every date-t generation is made no worse off.

(f) Consider the tax and transfer policy such that the young in date t are taxed an amount of goods $\tau = \frac{1}{2}(1 - (\frac{1}{2})^t)$ and the transfer is $-\tau$ to the old. This allocation will make everyone strictly better off, but will not be Pareto Optimal, as we will see in part (g).

(g) We're looking for a solution similar to (f) but Pareto Optimal this time, so we're looking for a tax-transfer system that converges to $\tau = 1$. Such a tax on the young in date t is $\tau = 1 - (\frac{1}{2})^t$ and the transfer to the old of $-\tau$. Every generation is strictly better off than they would be in the non-monetary equilibrium (i.e., autarky). The utility of the date-t generation is now

$$\begin{aligned}
u_t &= \left(1 - 1 + \left(\frac{1}{2}\right)^t\right) + \left(1 + 1 - \left(\frac{1}{2}\right)^{t+1}\right) \\
&= 2 + \left(\frac{1}{2}\right)^t - \left(\frac{1}{2}\right)^{t+1}
\end{aligned}$$

which is an improvement from our solution to (f), where

$$\begin{aligned}
u_t &= \left(1 - \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^t\right) + \left(1 + \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^{t+1}\right) \\
&= 2 + \frac{1}{2}\left(\left(\frac{1}{2}\right)^t - \left(\frac{1}{2}\right)^{t+1}\right),
\end{aligned}$$

and to autarky, where $u_t = 2$. This also underscores that our solution to (f) is strictly Pareto improving relative to autarky.