

Economics 4905
Fall 2016
Cornell University
Financial Fragility and the Macro-economy

Prelim #2 Solutions

Answer each of the 2 questions. The prelim is designed to be done in 60 minutes, but you are permitted to use up to 75 minutes. Do not consult anyone or anything during the prelim. Do not use calculators, computers, or any other electronic devices. Use black ink or black pencil, if possible. Leave all personal items in places designated by the proctor. Show your work. You do not need to simplify your arithmetic.

1. Overlapping Generations, I (30 minutes)

Given

$$u_0(x_0^1) = \beta x_0^1, \quad \omega_0^1 = 1 \text{ for } t = 0$$

$$u_t(x_t^t, x_t^{t+1}) = x_t^t + \beta x_t^{t+1}, \quad (\omega_t^t, \omega_t^{t+1}) = (1, 1) \text{ for } t = 1, 2, \dots$$

Where $\beta \in \mathbb{R}$ is a scalar. Let p^t be the present price of the commodity delivered at date t , i.e. $p^1 = 1$. Let $p^{m,t}$ be the present price of money at date t (in terms of the period 1 commodity). In each of the following three cases,

- i. Solve for the reflected, translated offer curve (OC).
- ii. Graph the OC.
- iii. Provide the phase diagram and the full dynamic analysis, including the steady states and their stability.
- iv. Calculate \mathcal{P}^m , the set of equilibrium money prices.

[Hint: Calculus is not the best tool for analyzing the linear model]

Case A: $\beta = \frac{1}{2}$, $m_0^1 = 1$, $m_t^s = 0$ otherwise.

Case B: $\beta = 2$, $m_0^1 = 2$, $m_t^s = 0$ otherwise.

Case C: $\beta = 2$, $m_0^1 = 0$, $m_1^1 = m_1^2 = 1$, $m_t^s = 0$ otherwise.

Solution:

- i. For all of the cases, the solution to i) will be the same. It is possible to solve the problem in multiple ways; all should yield the same result. The fastest derivation may be done by drawing the indifference curve through the endowment, thereby finding the MRS. The Offer Curve is then the union of the indifference curve and the parts of the axes above the IC. To find the reflected offer curve, we then need only to re-write the original offer curve in terms of the excess demand of the old generations.

An indifference curve of the utility function will be a level set such that

$$u_t(x_t^t, x_t^{t+1}) = \bar{u} = x_t^t + \beta x_t^{t+1}$$

To draw the curve *through* the endowment, we may then set $\bar{u} = u(\omega_t^t, \omega_t^{t+1})$ to get

$$x_t^t + \beta x_t^{t+1} = \omega_t^t + \beta \omega_t^{t+1} \Rightarrow \beta(x_t^{t+1} - \omega_t^{t+1}) = (\omega_t^t - x_t^t) \Rightarrow \beta z^{t+1} = z^t$$

Where $z^{t+1} = x_t^{t+1} - \omega_t^{t+1}$ and $z^t = (\omega_t^t - x_t^t)$. The indifference curve through the origin is thus represented in terms of (z^{t+1}, z^t) as

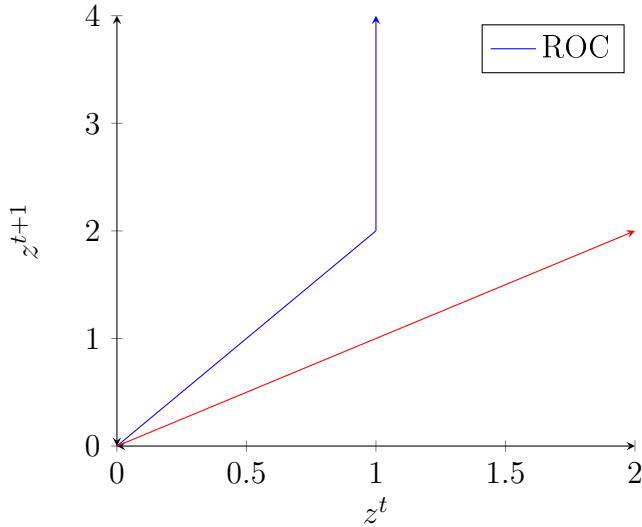
$$z^{t+1} = \frac{1}{\beta} z^t$$

The offer curve of the old generation is the the union of the indifference curve with the parts of the axes above the IC. This is equivalent to noting that $z^t \leq \omega_t^t = 1$, as the old generation cannot consume more than the young generation is able to offer them. Thus, due to the constraint of physical resources, a vertical line will interrupt the reflected offer curve where $z^t = 1$.

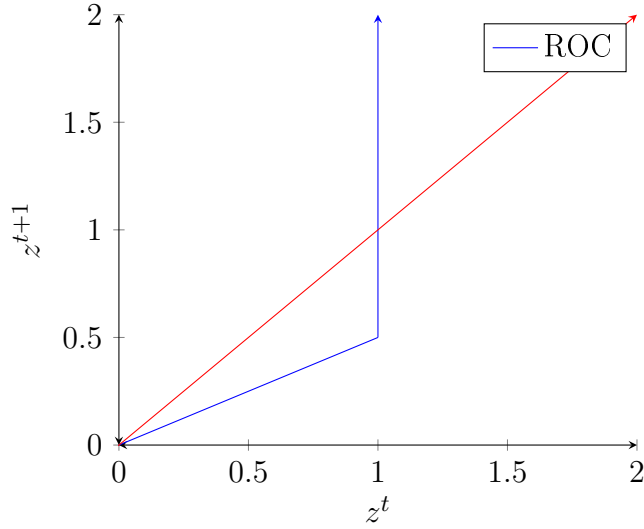
Solving the problem via Lagrangian optimization or by setting the MRS equal to the price ratio will yield exactly the same answer, although the calculation will be somewhat longer.

To make one final comment on the notation, we may observe that $z^t = -s_t^t = z_{t-1}^t$, while $z^{t+1} = z_t^{t+1}$. The excess demand of the old generation is equal to the excess supply of the young generation for each time period t , as the market will clear.

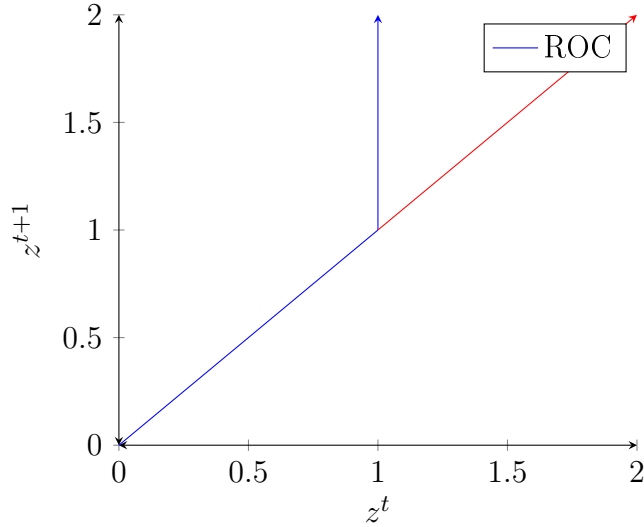
- ii. For Case A, since $\beta = \frac{1}{2}$ the reflected offer curve is $z^{t+1} = 2z^t$.



In contrast, for Cases B and C, the reflected offer curve is $z^{t+1} = \frac{1}{2}z^t$, as $\beta = 2$.



While this wasn't a case, we may also note that if $\beta = 1$ such that $z^{t+1} = z^t$, the graph will be as follows:



iii. and iv. We must either assert or prove a no-arbitrage profits condition such that $p^{m,t} = p^{m,t+1} = p^m \geq 0$.

In Case A, as $MRS = \frac{1}{1/2} = 2$ such that $r = 1 > 0$, we must be in the Ricardo case. There exists one stationary equilibrium, such that $z^t = z^{t+1} = 0$, the autarky state. For $z^1 \in (0, \infty)$, the money bubble bursts in finite time. For $z^1 = 0$ such that $z^t = z^{t+1} = 0$, the money bubble never forms. $p^m = 0$ is Pareto optimal, and $\mathcal{P}^m = \{0\}$.

In Case B, $MRS = \frac{1}{2}$ so that $r < 0$, implying we are in the Samuelson case. From there, we may solve to find the monetary steady state as $z^t = z^{t+1} = \frac{1}{2}$. Any $z^t > 1$ is infeasible, as each old generation can buy only up to one unit of consumption from their progeny. Thus,

the initial old generation will purchase goods in excess of their endowment only what is available on the market:

$$z^1 = p^m m_0^1 \leq \omega_1^1 \Rightarrow 2p^m \leq 1 \Rightarrow p^m \leq \frac{1}{2}$$

Therefore $\bar{p}^m = \frac{1}{2}$ and the set of equilibrium money prices is $\mathcal{P}^m = [0, \frac{1}{2}]$. This $\bar{p}^m = \frac{1}{2}$ is Pareto optimal, but it is *not* stationary. $p^m \in (0, \frac{1}{2})$ is not optimal, but it is Pareto-superior to the non-monetary autarky; if the economy starts in autarky ($\bar{p} = 0$), the money bubble will again never form. If $p^m \in (0, \frac{1}{2}]$, the money will inflate, price levels will rise, and the bubble will fade away asymptotically.

In Case C, the problem is quite similar to that described in Case B. Since $m_0^1 = 0$ but $m_1^1 = m_1^2 = 1$, we may treat the Ur Son generation Mr. 1 as we treated the Ur Father (the Mr. 0 generation) in the problem above. Any $z^t > 1$ is once more infeasible, so when the Ur Son generation is old,

$$\begin{aligned} z^1 &= p^m (m_1^1 + m_1^2) \leq \omega_2^2 \\ 2p^m &\leq 1 \Rightarrow p^m \leq \frac{1}{2} \end{aligned}$$

The equilibrium set of money prices is therefore again $\mathcal{P}^m = [0, \frac{1}{2}]$. As the offer curve is the same for Case B, the possible dynamics of the system remain the same as in the prior case.

2. Overlapping Generations, II (30 minutes)

$$u_0(x_0^1) = 2 \log x_0^1, \quad \omega_0^1 = 100 \text{ for } t = 0$$

$$u_t(x_t^t, x_t^{t+1}) = \log x_t^t + 2 \log x_t^{t+1}, \quad (\omega_t^t, \omega_t^{t+1}) = (100, 100) \text{ for } t = 1, 2, \dots$$

Let p^t be the present price of the commodity, $p^1 = 1$. $p^{m,t}$ is the present price of money in terms of the period 1 commodity. Additionally, let $m_0^1 = 1$ and $m_t^s = 0$ otherwise.

- a) Calculate the marginal rate of commodity substitution (MRS) at the endowment (100, 100).
- b) What is the rate of interest at the endowment?
- c) Is the economy Samuelson, or is it Ricardo? Why?
- d) Derive the reflected, translated offer curve (OC).
- e) Graph the OC.
- f) Conduct the complete dynamic analysis in the phase diagram, including:
 - i. Calculation of steady states.
 - ii. Calculation of the set of money prices \mathcal{P}^m .
 - iii. Stability analysis.
 - iv. Inflation.
 - v. Deflation.
 - vi. Welfare.

Solutions:

- a) We may find the MRS by finding the partial derivatives of utility with respect to consumption in each time period, although there are other methods that may work just as well.

$$MRS = -\frac{dx_t^{t+1}}{dx_t^t} = \frac{\frac{\partial u}{\partial x_t^t}}{\frac{\partial u}{\partial x_t^{t+1}}} = \frac{\frac{1}{x_t^t}}{\frac{2}{x_t^{t+1}}} = \frac{x_t^{t+1}}{2x_t^t}$$

Which at the endowment (100, 100) is

$$MRS(\omega_t^t, \omega_t^{t+1}) = \frac{100}{200} = \frac{1}{2}$$

- b) As $MRS = \frac{p^t}{p^{t+1}} = R$, it holds that $R = \frac{1}{2}$ such that $r = R - 1 = -\frac{1}{2}$.

- c) Since $r < 0$, the slope of the offer curve of Agent t is $\frac{1}{2} < 1$. We are thus in the Samuelson case.
- d) Rather than doing the entire optimization problem from scratch, we may recall from b) that the price ratio between periods will be $\frac{p^t}{p^{t+1}} = \frac{1}{2}$. Substituting back into the budget constraint,

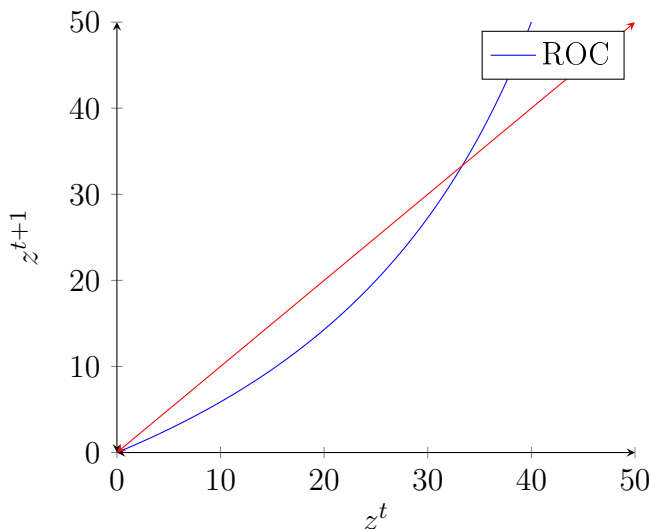
$$\begin{aligned}
 p^t x_t^t + p^{t+1} x^{t+1} &= p^t \omega_t^t + p^{t+1} \omega^{t+1} \Rightarrow p^{t+1} (x_{t+1}^t - \omega_{t+1}^t) = p^t (\omega_t^t - x_t^t) \\
 p^{t+1} z^{t+1} &= p^t z^t \\
 z^{t+1} &= \frac{p^t}{p^{t+1}} z^t \Rightarrow z^{t+1} = \frac{x_t^{t+1}}{2x_t^t} z^t = \frac{z^{t+1} + \omega_t^{t+1}}{2\omega_t^t - 2z_t^t} z^t \\
 \frac{z^{t+1}}{z^{t+1} + \omega_t^{t+1}} &= \frac{z^t}{2\omega_t^t - 2z_t^t} \Rightarrow \frac{z^{t+1} + \omega_t^{t+1}}{z^{t+1}} = \frac{2\omega_t^t - 2z_t^t}{z^t} \Rightarrow 1 + \frac{\omega_t^{t+1}}{z^{t+1}} = \frac{2\omega_t^t}{z^t} - 2 \\
 \frac{\omega_t^{t+1}}{z^{t+1}} &= \frac{2\omega_t^t}{z^t} - 3 \Rightarrow \frac{1}{z^{t+1}} = \frac{2\omega_t^t - 3z^t}{\omega_t^{t+1} z^t} \\
 z^{t+1} &= \frac{\omega_t^{t+1} z^t}{2\omega_t^t - 3z^t}
 \end{aligned}$$

The (reflected) offer curve is therefore

$$z^{t+1} = \frac{100z^t}{200 - 3z^t}$$

Students who memorized the format of the answer as $z^{t+1} = \frac{BCz^t}{AD - (C+D)z^t}$ were given full credit, although students who derived it algebraically received extra credit.

- e) The graph of the ROC is as follows:



f.i) At a steady state, $z^{t+1} = z^t$. One steady state will be $z^t = 0$. The other will satisfy

$$z = \frac{100z}{200 - 3z} \Rightarrow 200 - 3z = 100 \Rightarrow z = \frac{100}{3}$$

$z^t = 0$ will be a non-monetary, non-Pareto optimal steady-state. $z^t = \frac{100}{3}$, however, will be a Pareto Optimal fixed point.

f.ii) We may either show or assert a no-arbitrage profits condition, such that $p^{m,t} = p^{m,t+1} = p^m \geq 0 \forall t$. The set of equilibrium money prices is then

$$\mathcal{P}^m = \left[0, \frac{100}{3}\right]$$

f.iii) $z^t = 0$ is a stable no-monetary (autarky) steady state; trajectories that begin in the neighborhood of $z^0 \in (0, \frac{100}{3})$ will tend asymptotically to it. $z^t = \frac{100}{3}$ is in contract unstable.

f.iv) If $0 < p^m < \frac{100}{3}$, the money bubble will fade away through hyperinflation.

f.v) If $p^m > \frac{100}{3}$, then the bubble will burst in finite time through hyperdeflation. Note that neither hyperinflation nor hyperdeflation will occur if individuals avoid “disequilibrium” paths.

f.vi) $p^m = \frac{100}{3}$ is Pareto Optimal, while $p^m = 0$ is not. $p^m \in (0, \frac{100}{3})$ is not a steady state and is not Pareto Optimal, but it is Pareto Superior to $p^m = 0$.