Economics 4905 Fall 2016
Cornell University
Financial Fragility and the Macroeconomy

Practice Questions for Prelim 2 and Solutions

1. Overlapping Generations, Part I

Consider the following overlapping generations (OLG) economy:

- 2-period lives
- 1 commodity per period, $\ell = 1$
- Stationary endowments
$$\omega_0^1 = B > 0 \text{ for } t = 0$$
$$(\omega_t^t, \omega_t^{t+1}) = (A, B) >> 0 \text{ for } t = 1, 2, ...$$
- Stationary preferences
$$u_0(x_0^1) = D \log x_0^1 \text{ for } t = 0$$
$$u_t(x_t^t, x_t^{t+1}) = C \log x_t^t + D \log x_t^{t+1} \text{ for } t = 1, 2, ...$$
- 1 person per generation
- Passive fiscal policy
$$m_0^1 = 2, \quad m_0^s = 0 \text{ otherwise}$$
- Goods price of money is $p^m \geq 0$.

For each of the following cases, calculate the offer curve for Mr. $t \geq 1$. Then, find and plot the reverse-offer curve for Mr. $t \geq 1$ in excess demand space (z^t, z^{t+1}), or equivalently in the $(x_t^t - \omega_t^t, x_t^{t+1} - \omega_t^{t+1})$ domain. Plot the reflected offer curve, and analyze the global dynamics.

a) $A = 10$, $B = 12$, $C = 1$, $D = 0.98$

b) $A = 15$, $B = 10$, $C = 2$, $D = 3$

c) $A = 40$, $B = 30$, $C = 0.5$, $D = 0.5$

d) $A = 8$, $B = 4$, $C = 1.9$, $D = 0.95$

Is there a pattern? Derive the conditions on the MRS for a "Samuelson" versus a "Classical" (or "Ricardo") economy and relate them to your answers.

Solution:

The problem facing an agent born at time $t \geq 1$ is

$$\arg \max_{(x_t^t, x_t^{t+1})} \{u_t(x_t^t, x_t^{t+1})\}$$

$$\text{s.t. } p^t x_t^t + p^{t+1} x_t^{t+1} + p^m (x_t^{t,m} + x_t^{t+1,m}) \leq p^t \omega^t + p^{t+1} \omega^{t+1}$$

The agent thus acts to maximize their lifetime utility given the budget constraint of their lifetime wealth. More explicitly, the problem may be posed as

$$\arg \max_{(x_t^t, x_t^{t+1})} \{C \log x_t^t + D \log x_t^{t+1}\}$$

$$\text{s.t. } p^t x_t^t + p^{t+1} x_t^{t+1} + p^m (x_t^{t,m} + x_t^{t+1,m}) \leq p^t A + p^{t+1} B$$

We may proceed to write the Lagrangian

$$\mathscr{L} = u_t(x_t^t, x_t^{t+1}) + \lambda[p^t x_t^t + p^{t+1} x_t^{t+1} + p^m (x_t^{t,m} + x_t^{t+1,m}) - (p^t A + p^{t+1} B)]$$

The First Order Conditions of this optimization imply

$$\frac{\partial \mathscr{L}}{\partial x_t^t} = \frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^t} + \lambda p^t = 0 \;\Rightarrow\; \frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^t} = -\lambda p^t$$

$$\frac{\partial \mathscr{L}}{\partial x_t^{t+1}} = \frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^{t+1}} + \lambda p^{t+1} = 0 \;\Rightarrow\; \frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^{t+1}} = -\lambda p^{t+1}$$

Dividing these terms through, we may find the inter-temporal marginal rate of substitution (MRS) as

$$MRS = \frac{\frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^t}}{\frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^{t+1}}} = \frac{-\lambda p^t}{-\lambda p^{t+1}} = \frac{p^t}{p^{t+1}}$$

We may now find the MRS. Recalling that $u_t(x_t^t, x_t^{t+1}) = C \log x_t^t + D \log x_t^{t+1}$, we may note that

$$\frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^t} = \frac{C}{x_t^t} \quad \text{and} \quad \frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^{t+1}} = \frac{D}{x_t^{t+1}}$$

So

$$MRS = \frac{\frac{C}{x_t^t}}{\frac{D}{x_t^{t+1}}} = \frac{Cx_t^{t+1}}{Dx_t^t}$$

$$\Rightarrow \frac{Cx_t^{t+1}}{Dx_t^t} = \frac{p^t}{p^{t+1}}$$

This is, of course, just setting the MRS equal to the price ratio between periods, which is another way to solve this particular type of optimization problem. We may also set the growth rate of prices as

$$\frac{p^t}{p^{t+1}} = R^t = (1 + r^t)$$

Where R^t is the interest rate factor and r^t is the interest rate.

Returning to the main problem, the goal remains to solve for a dynamical system to explain consumption. If $\frac{p^t}{p^{t+1}} = \frac{Cx_t^{t+1}}{Dx_t^t}$, then since the budget constraint holds that

$$p^t x_t^t + p^{t+1} x_t^{t+1} + p^m(x_t^{t,m} + x_t^{t+1,m}) = p^t A + p^{t+1} B$$

We may divide through by p^{t+1} to get

$$\frac{p^t}{p^{t+1}} x_t^t + x_t^{t+1} = \frac{p^t}{p^{t+1}} A + B$$

$$\Rightarrow \frac{Cx_t^{t+1}}{Dx_t^t} x_t^t + x_t^{t+1} = \frac{Cx_t^{t+1}}{Dx_t^t} A + B$$

$$\frac{Cx_t^{t+1}}{D} + x_t^{t+1} = \frac{Cx_t^{t+1}}{Dx_t^t} A + B \quad \Rightarrow \quad x_t^{t+1} - B = \frac{ACx_t^{t+1}}{Dx_t^t} - \frac{Cx_t^{t+1}}{D}$$

$$x_t^{t+1} - B = \frac{Cx_t^{t+1}}{D} \left(\frac{A}{x_t^t} - 1 \right) \quad \Rightarrow \quad x_t^{t+1} - B = \frac{Cx_t^{t+1}}{D} \left(\frac{A - x_t^t}{x_t^t} \right)$$

Thus

$$x_t^{t+1} - B = \frac{Cx_t^{t+1}}{Dx_t^t} \left(A - x_t^t \right)$$

We may then define the excess demand of an agent born in period t at time t as $z^t = (x_t^t - \omega^t)$. Conversely, the excess supply will be $s^t = -z^t \ \forall t$, as markets will clear. For an old household (born at time t but now in period $t+1$), this will mean $z^{t+1} = (x_t^{t+1} - \omega^{t+1})$. Thus, in this context,

$$z^t = (x_t^t - A) \ \text{ and } \ z^{t+1} = (x_t^{t+1} - B)$$

We may note that in consequence, $x_t^t = z^t + A$ and $x_t^{t+1} = z^{t+1} + B$. Substituting this all in,

$$z^{t+1} = \frac{C(z^{t+1} + B)}{D(z^t + A)}(-z^t)$$

$$\frac{z^{t+1}}{z^{t+1} + B} = \frac{-Cz_t}{D(z^t + A)} \quad \Rightarrow \quad \frac{z^{t+1} + B}{z^{t+1}} = \frac{D(z^t + A)}{-Cz_t}$$

$$\frac{B}{z^{t+1}} = -\frac{D(z^t + A)}{Cz_t} - 1 = -\frac{D(z^t + A) + Cz_t}{Cz_t}$$

We at last arrive at the offer curve (OC) for an agent born at time t:

$$z^{t+1} = -\frac{BCz_t}{Dz^t + AD + Cz_t}$$

To clean up the expression a little, we may then write

$$z^{t+1} = \frac{-BCz_t}{(C + D)z^t + AD}$$

To find the reflected offer curve (ROC), we may use the fact that $s^t = -z^t$. This is how the offer curve is often written in class; sometimes we set z^t as the excess supply of agent t in time t and z^{t+1} as the excess demand of agent t at time $t + 1$. For clarity here, however, we can keep s^t as excess supply and z^t as excess demand.

$$z^{t+1} = \frac{BCs_t}{AD - (C + D)s^t}$$

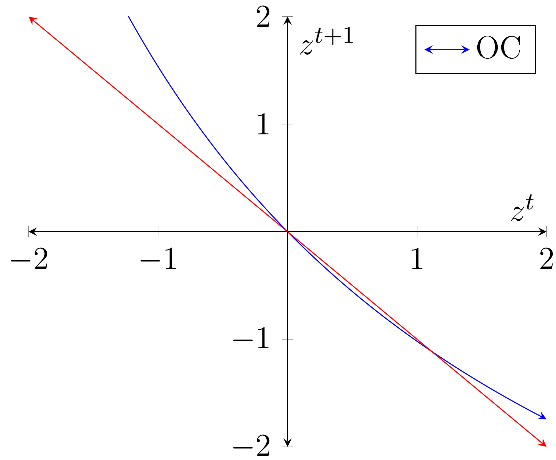
Using these equations, we can plot the graphs of the offer curve in the required cases.

Part a)

If $A = 10$, $B = 12$, $C = 1$, and $D = 0.98$, then the offer curve is

$$z^{t+1} = \frac{-12z^t}{1.98z^t + 9.8}$$

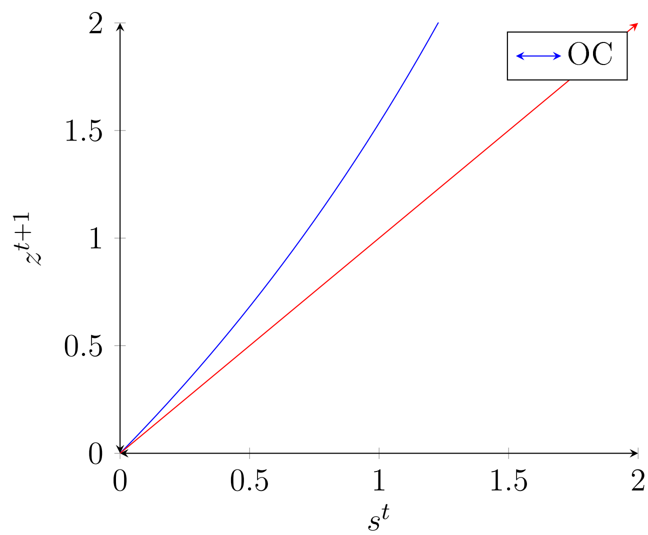
Plotting this in the phase space (z^t, z^{t+1}) yields



The reflected offer curve will be

$$z^{t+1} = \frac{12s^t}{9.8 - 1.98s^t}$$

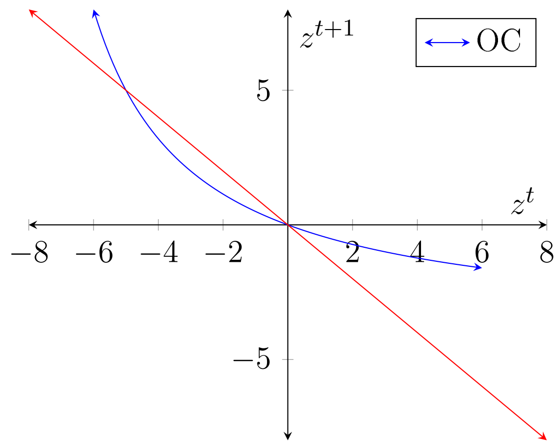
We may plot this globally, but the relevant picture here really focuses on the first quadrant:



Part b)

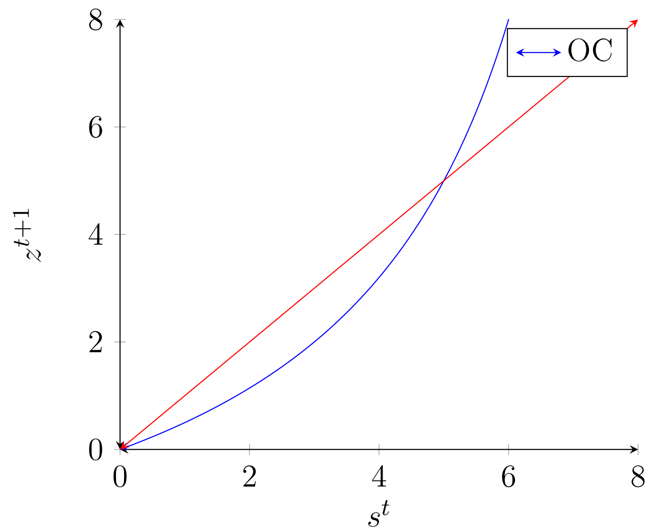
If $A = 15$, $B = 10$, $C = 2$, and $D = 3$, then the offer curve is

$$z^{t+1} = \frac{-20z^t}{5z^t + 45}$$



The reflected offer curve will be

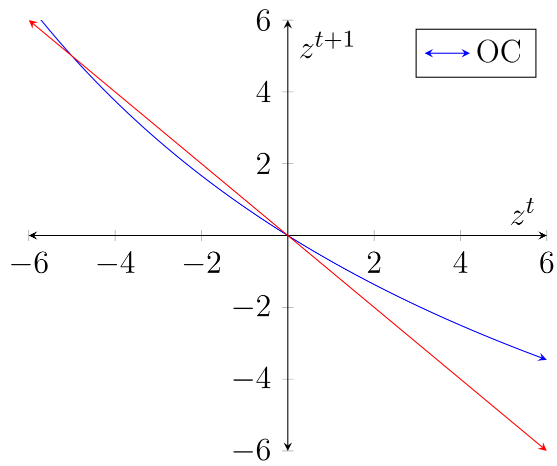
$$z^{t+1} = \frac{20s^t}{45 - 5s^t}$$



Part c)

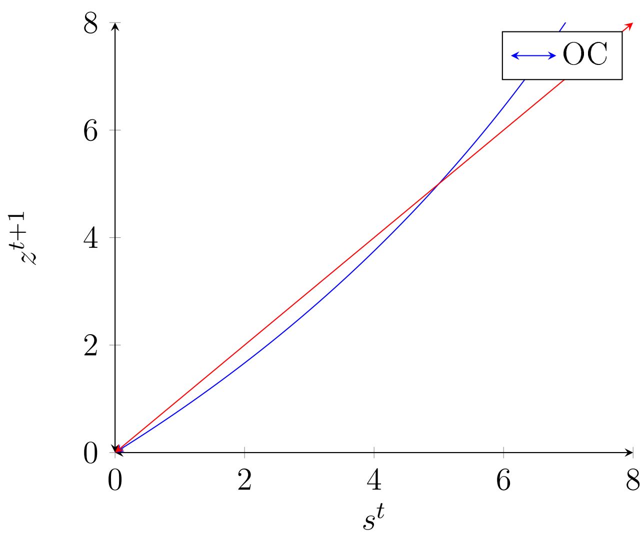
If $A = 40$, $B = 30$, $C = 0.5$, and $D = 0.5$, then the offer curve is

$$z^{t+1} = \frac{-15z^t}{z^t + 20}$$



The reflected offer curve will be

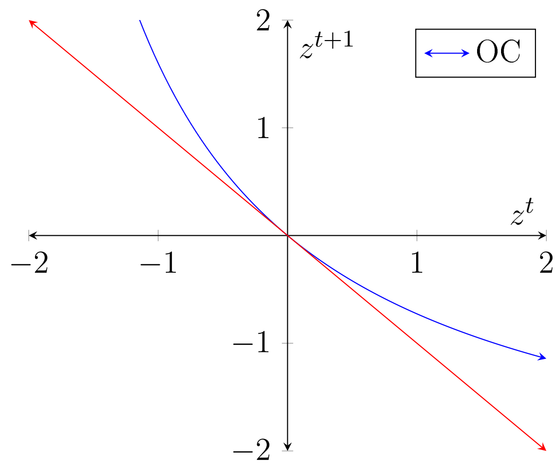
$$z^{t+1} = \frac{15s^t}{20 - s^t}$$



Part d)

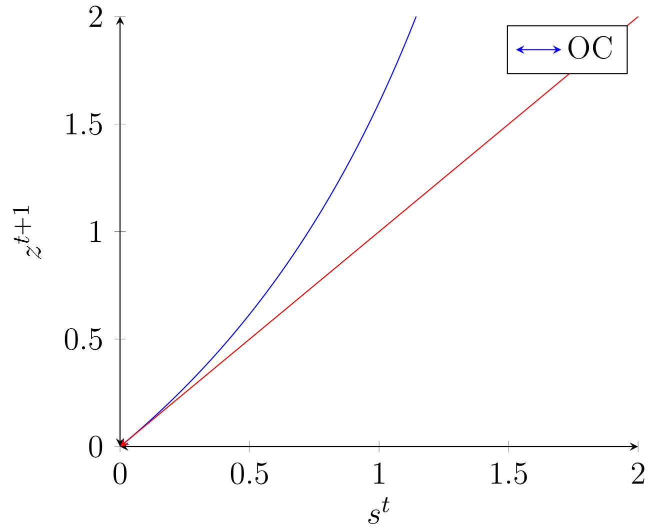
If $A = 8$, $B = 4$, $C = 1.9$, and $D = 0.95$, then the offer curve is

$$z^{t+1} = \frac{-7.6z^t}{2.85z^t + 7.6}$$



The reflected offer curve will be

$$z^{t+1} = \frac{7.6s^t}{7.6 - 2.85s^t}$$



For parts a) and d), the global dynamics are Ricardo; the only stationary point $z^{t+1} = z^t$ is at $z^t = 0$, the non-monetary (NM) equilibrium, or autarky.

In contrast, for parts b) and c), we are in the Samuelson case. There exist fixed points to the mapping, the non-monetary and monetary (PO) steady states. In the latter, where $z^{t+1} = z^t = \bar{p}^m$, the PO equilibrium is unstable. If $0 < p^m < \bar{p}^m$, the economy will be inflationary, as the current commodity price of money converges to 0. The money bubble thereby fades away, although it does not burst.

However, if $p^m > \bar{p}^m$, then the economy enters a hyperdeflationary spiral; the goods price of money grows so that the demand for goods exceeds the supply.

To be in the Ricardo case, it must be that if

$$MRS = \frac{\frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^t}}{\frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^{t+1}}} = 1 + r$$

Then $r \geq 0$.
The slope of the indifference curve of the agent at the endowment point will be

$$MRS = \left[\frac{dx_t^{t+1}}{dx_t^t}\right]_{(\omega^t, \omega^{t+1})} = \left[\frac{\frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^t}}{\frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^{t+1}}}\right]_{(\omega^t, \omega^{t+1})} = 1 + r$$

So

$$\left[\frac{\frac{C}{A}}{\frac{D}{B}}\right] = 1 + r \;\Rightarrow\; r = \frac{BC}{AD} - 1$$

So then if $r \geq 0$ for the Ricardo case,

$$\frac{BC}{AD} - 1 > 0 \;\Rightarrow\; BC \geq AD.$$

In the Samuelson case, it must then be that $BC < AD$.

2. Overlapping Generations, Part II

Consider the following OLG economy:

- Pure exchange, 2-period lives, one consumer per generation.

$$u_0(x_0^1) = x_0^1 \text{ and } \omega_0^1 = 1 \text{ for } t = 0$$

$$u_t(x_t^t, x_t^{t+1}) = x_t^t + x_t^{t+1} \text{ and } (\omega_t^t, \omega_t^{t+1}) = (1, 1) \text{ for } t = 1, 2, ...$$

- Money transfers:

$$m_0^1 = 2, \quad m_1^1 = -1$$

$$m_1^2 = 1, \quad m_t^s = 0 \text{ otherwise.}$$

a) What is the non-monetary equilibrium allocation? What are the prices? What are the interest rates?

b) Derive the reflected offer curve for consumer $t = 1, 2, ...$

c) Derive the set of equilibrium money prices.

d) Draw the phase diagram and show the full evolution of this economy (depending on the price of money).

e) What is the Pareto optimal allocation associated with the above (money) tax-transfer policy?

f) Find an alternative tax-transfer policy and associated allocation which is not Pareto optimal but in which everyone is strictly better off than they would be in autarky.

g) Find an alternative tax-transfer policy and associated allocation which is Pareto optimal and in which everyone is strictly better off than they would be in the non-monetary equilibrium.

Solution:

a) When there is no money, the initial old problem is

$$\max_{x_0^1} x_0^1$$

$$\text{s.t. } p^1 x_0^1 \leq p^1$$

And the problem of a person born in date t is

$$\max_{x_t^t + x_t^{t+1}} \{x_t^t + x_t^{t+1}\}$$

$$s.t. p^t x_t^t + p^{t+1} x_t^{t+1} \leq p^t + p^{t+1}$$

Normalizing the price of money in date 1 to be equal to 1, we get that the initial old choose $x_0^1 = 1$. Now looking at the date one budget constraint we must have that $x_1^1 = 1$, which from the budget constraint of the person born in date 1 implies that $x_1^2 = 1$. And we see that this argument goes on ad infinitum. So our equilibrium allocation is

$$(c_0^1, \{c_t^t, c_t^{t+1}\}) = (1, \{1, 1\}) \forall t$$

And from the first order conditions of the date t generation (since we are at an interior solution) we can get the prices, $p^t = 1 \forall t$. This implies the interest rate is $1 + r^t = 1$.

b) The problem of the initial old is now

$$\max_{x_0^1} x_0^1$$

$$\text{s.t. } x_0^1 + p^m x_0^{1,m} \leq 1 + 2p^m$$

And the problem of generation t is now

$$\max_{(x_t^t, x_t^{t+1})} x_t^t + x_t^{t+1}$$

$$\text{s.t. } p^t x_t^t + p^{t+1} x_t^{t+1} + p^m (x_t^{t,m} + x_t^{t+1,m}) \leq p^t + p^{t+1} + p^m (m_t^t + m_t^{t+1})$$

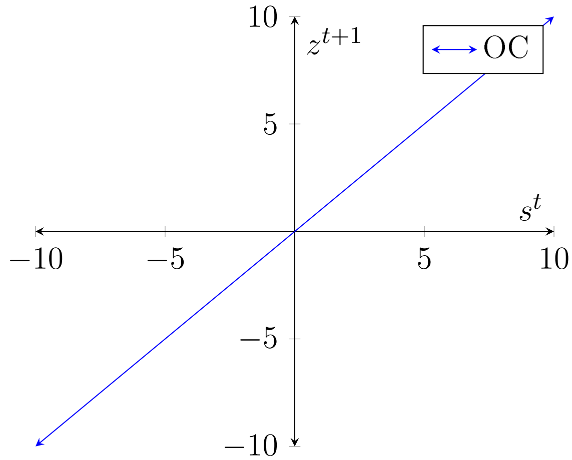
We know that $p^m (x_t^{t,m} + x_t^{t+1,m}) = x_0^{1,m} = 0$ in equilibrium. And for every date t we note also that $(m_t^t + m_t^{t+1}) = 0$.
First order conditions again imply that $p^t = 1 \forall t$. We note that the solution for the initial old is now $x_0^1 = 1 + 2p^m$, which implies by the resource constraint $x_1^1 = 1 - 2p^m$. We continue this ad infinitum as in part a) to drive at the solutions

$$(x_{t-1}^t, x_t^t) = (1 + 2p^m, 1 - 2p^m) \quad \forall t$$

We now get that the reflected offer curve from knowing $x_t^{t+1} = 2 - x_t^t$, which implies our reflected offer curve,

$$z^{t+1} = s^t$$

The graph is as follows:



c) The equilibrium set of money prices must be such that no individuals consumption is negative. This implies we must look at the consumption of the young. So we need that

$$x_t^t = 1 - 2p^m \geq 0$$

This implies our equilibrium set of money prices is $p^m \in [0, \frac{1}{2}]$.

d) The phase diagram is as above, noting that the offer curve in this case is exactly equal to the resource constraint (the $z^{t+1} = s^t$ line). Thus, for any price of money in the equilibrium set, this picks out a stationary point on the phase diagram.

e) The Pareto optimal allocation associated with the above policy is when $p^m = \frac{1}{2}$, corresponding to the allocation $x_0^1 = 2$ and $(x_t^t, x_t^{t+1}) = (0, 2)$. The utility of the initial old generation is maximized ($u_0 = 2$, up from $u_0 = 1$ in autarky), and every date-t generation is made no worse off.

f) Consider the tax and transfer policy such that the young in date t are taxed an amount of goods $\tau = \frac{1}{2}(1 - \left(\frac{1}{2}\right)^t)$ and the transfer is $-\tau$ to the old. This allocation will make everyone strictly better off, but will not be Pareto Optimal, as we will see in part g).

g) We're looking for a solution similar to f) but Pareto Optimal this time, so we're looking for a tax-transfer system that converges to $\tau = 1$.
Such a tax on the young in date t is $\tau = 1 - (\frac{1}{2})^t$ and the transfer to the old of $-\tau$. Every generation is strictly better off than they would be in the non-monetary equilibrium (i.e., autarky). The utility of the date-t generation is now

$$u_t = \left(1 - 1 + \left(\frac{1}{2}\right)^t\right) + \left(1 + 1 - \left(\frac{1}{2}\right)^{t+1}\right)$$

$$= 2 + \left(\frac{1}{2}\right)^t - \left(\frac{1}{2}\right)^{t+1}$$

This is an improvement to f), where

$$u_t = \left(1 - \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^t\right) + \left(1 + \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^{t+1}\right)$$

$$= 2 + \frac{1}{2}\left(\left(\frac{1}{2}\right)^t - \left(\frac{1}{2}\right)^{t+1}\right)$$

and an improvement to autarky, where $u_t = 2$. This also underscores that our solution to f) is strictly Pareto improving relative to autarky.