Economics 4905: Augmented Lecture 8
Overlapping Generations, Inter-temporal Demands, Offer Curves and Phase Diagrams

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Fall 2017

Overlapping Generations, Worked Example 1

2 period lives.

1 commodity per period, $\ell = 1$.

Stationary endowments:

$$\omega_0^1 = 2 > 0 \text{ for } t = 0$$
$$(\omega_t^t, \omega_t^{t+1}) = (2, 2) > 0 \text{ for } t = 1, 2, ...$$

Stationary preferences:

$$u_0(x_0^1) = 4 \log x_0^1 \text{ for } t = 0$$
$$u_t(x_t^t, x_t^{t+1}) = \log x_t^t + 4 \log x_t^{t+1} \text{ for } t = 1, 2, ...$$

$$m_0^1 = 10 \qquad m_t^s = 0 \text{ otherwise}$$

Goods price of money is $p^m \geq 0$.

Solving the Consumer Problem by Substitution Method

Consumer problem:

$$\max_{x_t^t, x_t^{t+1}} \quad \log x_t^t + 4 \log x_t^{t+1}$$

$$\text{subject to} \quad p^t x_t^t + p^{t+1} x_t^{t+1} = p^t \omega_t^t + p^{t+1} \omega_t^{t+1}$$

Solving the budget constraint for x_t^t gives

$$x_t^t = \omega_t^t + \frac{p^{t+1}}{p^t} \omega_t^{t+1} - \frac{p^{t+1}}{p^t} x_t^{t+1}$$

Plugging into the consumer's utility function

$$\max_{x_t^{t+1}} \quad \log \left(\omega_t^t + \frac{p^{t+1}}{p^t} \omega_t^{t+1} - \frac{p^{t+1}}{p^t} x_t^{t+1} \right) + 4 \log x_t^{t+1}$$

Solving the Consumer Problem by Substitution Method

Taking derivative with respect to x_t^{t+1}

$$\frac{\frac{p^{t+1}}{p^t}}{\omega_t^t + \frac{p^{t+1}}{p^t}\omega_t^{t+1} - \frac{p^{t+1}}{p^t}x_t^{t+1}} = \frac{4}{x_t^{t+1}}$$

Solving for x_t^{t+1}

$$x_t^{t+1} = \frac{4}{5}\frac{p^t\omega_t^t + p^{t+1}\omega_t^{t+1}}{p^{t+1}}$$

Plugging this back to the budget constraint and solve for x_t^t gives

$$x_t^t = \frac{1}{5}\frac{p^t\omega_t^t + p^{t+1}\omega_t^{t+1}}{p^t}$$

Solving the Consumer Problem by Lagrange Method

Consumer problem:

$$\begin{aligned} \max_{x_t^t, x_t^{t+1}} \quad & \log x_t^t + 4 \log x_t^{t+1} \\ \text{subject to} \quad & p^t x_t^t + p^{t+1} x_t^{t+1} = p^t \omega_t^t + p^{t+1} \omega_t^{t+1} \end{aligned}$$

Setting up the Lagrangian:

$$\mathcal{L} = \log x_t^t + 4 \log x_t^{t+1} + \lambda (p^t \omega_t^t + p^{t+1} \omega_t^{t+1} - p^t x_t^t - p^{t+1} x_t^{t+1})$$

Solving the Consumer Problem by Lagrange Method

Taking derivatives of Lagrangian with respect to x_t^t, x_t^{t+1} and λ gives

$$\frac{1}{x_t^t} = \lambda p^t$$
$$\frac{4}{x_t^{t+1}} = \lambda p^{t+1}$$
$$p^t x_t^t + p^{t+1} x_t^{t+1} = p^t \omega_t^t + p^{t+1} \omega_t^{t+1}$$

Solving the three equations for x_t^t, x_t^{t+1} and λ

$$x_t^t = \frac{1}{5} \frac{p^t \omega_t^t + p^{t+1} \omega_t^{t+1}}{p^t}$$
$$x_t^{t+1} = \frac{4}{5} \frac{p^t \omega_t^t + p^{t+1} \omega_t^{t+1}}{p^{t+1}}$$
$$\lambda = \frac{1}{5} (p^t \omega_t^t + p^{t+1} \omega_t^{t+1})$$

Solving for Offer Curve

Rearrange the solutions for x_t^t and x_t^{t+1}

$$\frac{p^{t+1}}{p^t} = \frac{5x_t^t - \omega_t^t}{\omega_t^{t+1}}$$

$$\frac{p^{t+1}}{p^t} = \frac{\omega_t^t}{\frac{5}{4}x_t^{t+1} - \omega_t^{t+1}}$$

Setting the two equations equal

$$\frac{5x_t^t - \omega_t^t}{\omega_t^{t+1}} = \frac{\omega_t^t}{\frac{5}{4}x_t^{t+1} - \omega_t^{t+1}}$$

Solving for Offer Curve

Define excess demands

$$y^t = (x_t^t - \omega_t^t)$$

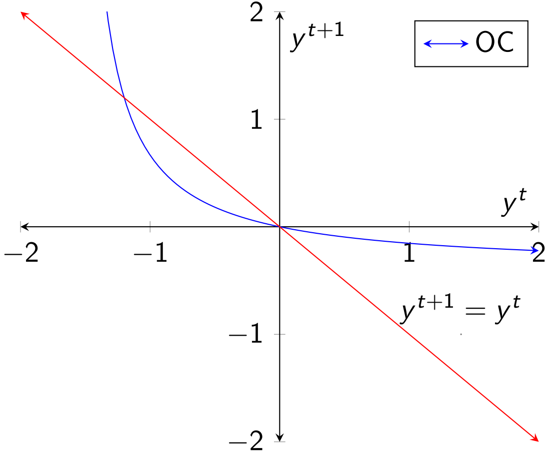
and

$$y^{t+1} = (x_t^{t+1} - \omega_t^{t+1})$$

Using the definitions for y^t and y^{t+1}, and $\omega_t^t = \omega_t^{t+1} = 2$ gives

$$y^{t+1} = \frac{-2y^t}{8 + 5y^t}$$

Offer Curve



The Reflected Offer Curve

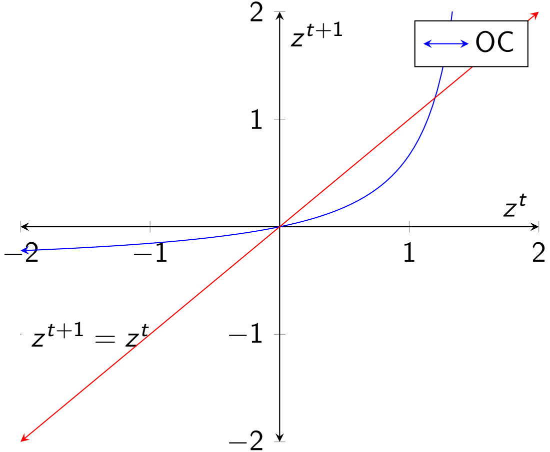
The reflected OC, which is more familiar in economic dynamics, lies in quadrants 1 and 3. We will focus on quadrant 1 when we do the dynamics. Let $z^t = \text{ excess demand }$ by Mr $(t-1)$ for goods in period $t = \text{ excess supply }$ of Mr. t for goods in period t.

$$z^t = (x^t_{t-1} - \omega^t_{t-1}) = (\omega^t_t - x^t_t)$$
$$\frac{z^{t+1}}{z^t} = \frac{p^t}{p^{t+1}} = R^t = (1 + r^t)$$

where p^t and p^{t+1} are present prices, R^t is the interest factor, and r^t is the interest rate.
The reflected OC is given by

$$z^{t+1} = \frac{2z^t}{8 - 5z^t}$$

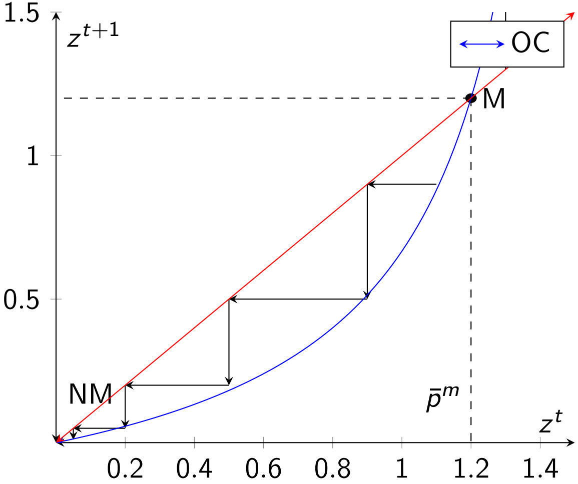
Reflected Offer Curve



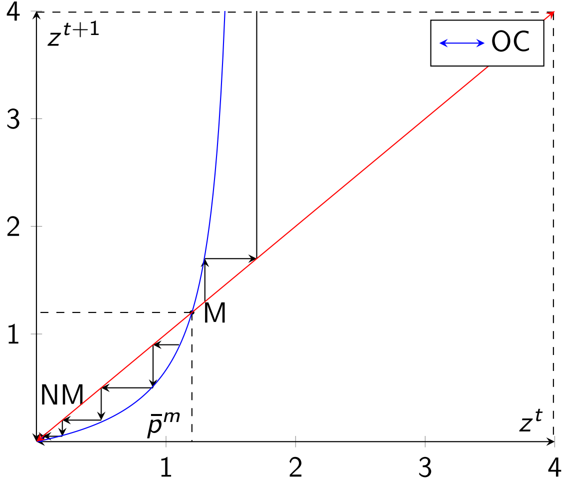
Samuelson Case

We redraw the reflected OC below solely in quadrant 1 as our phase diagram. We are in the Samuelson case, since

$$\frac{\frac{\partial u_t(\omega_t^t,\omega_t^{t+1})}{\partial x_t^t}}{\frac{\partial u_t(\omega_t^t,\omega_t^{t+1})}{\partial x_t^{t+1}}} = 1 + r = \frac{1}{4} < 1, \text{ i.e. } r = -\frac{3}{4} < 0$$



1.5
1
0.5
z^{t+1}
OC
M
NM
$\bar{p}^m$
z^t
0.2
0.4
0.6
0.8
1
1.2
1.4



4
3
2
1
z^{t+1}
OC
M
NM
$\bar{p}^m$
z^t
1
2
3
4

Steady States

There are 2 stationary states.

(a) The non-monetary (non-PO) steady state with $z^t = z^{t+1} = 0$ (autarky) labeled NM.

(b) The monetary (PO) steady state with $z^t = z^{t+1} = 10\bar{p}^m = \frac{6}{5}$ and $\bar{p}^m = \frac{6}{5} \times \frac{1}{10} = \frac{3}{25}$, labeled M.

The non-monetary steady state is locally stable. The monetary steady state is unstable. If $0 < p^m < \bar{p}^m$, the economy is *inflationary*. The current goods price of money tends asymptotically to zero. The money bubble fades away, but it does not burst.

Deflationary Trajectory

If $p^m > \bar{p}^m$, the economy is deflationary. The current goods price of money grows so that in finite time demand for goods exceeds supply, so the deflationary bubble must burst (in finite time) since $x_t^t + x_t^{t+1} > \omega_t^t + \omega_t^{t+1}$, i.e. the demand for goods excess supply, see the phase diagram. Outside the 4×4 box competitive equilibrium cannot be obtained. The hyper-deflationary path does not satisfy long-run perfect foresight. The bubble must burst in finite time.

Overlapping Generations, Worked Example 2

2 period lives.

1 commodity per period, $\ell = 1$.

Stationary endowments:

$$\omega_0^1 = 10 \text{ for } t = 0$$
$$(\omega_t^t, \omega_t^{t+1}) = (10, 10) \text{ for } t = 1, 2, ...$$

Stationary preferences:

$$u_0(x_0^1) = 10 \log x_0^1 \text{ for } t = 0$$
$$u_t(x_t^t, x_t^{t+1}) = 2 \log x_t^t + 10 \log x_t^{t+1} \text{ for } t = 1, 2, ...$$

$$m_0^1 = 1 \qquad m_t^s = 0 \text{ otherwise}$$

Present goods price of money is $p^m \geq 0$.

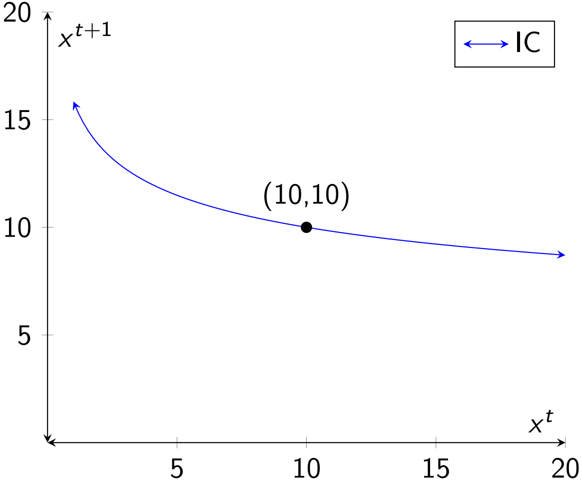
The indifference curve (IC) running through the endowment is

$$\{(x_t^t, x_t^{t+1}) : 2\log x_t^t + 10\log x_t^{t+1} = 2\log 10 + 10\log 10 = 12\log 10\}$$

Exponentiating both sides to derive an analytical expression for graphing,

$$\begin{aligned}
&(x_t^t)^2(x_t^{t+1})^{10} = 10^{12} \\
\Leftrightarrow &(x_t^{t+1})^{10} = \frac{10^{12}}{(x_t^t)^2} \\
\Leftrightarrow &x_t^{t+1} = \frac{10^{6/5}}{(x_t^t)^{2/5}}.
\end{aligned}$$

Graphically:



The slope of the IC at the endowment in (x^t, x^{t+1}) space is

$$\begin{aligned}
\frac{dx_t^{t+1}}{dx_t^t} &= -\frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^t} \bigg/ \frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^{t+1}} \\
&= -\frac{2}{x_t^t} \bigg/ \frac{10}{x_t^{t+1}}
\end{aligned}$$

Evaluated at the endowment,

$$\begin{aligned}
\left.\frac{dx_t^{t+1}}{dx_t^t}\right|_{(10,10)} &= -\frac{2}{10} \bigg/ \frac{10}{10} \\
&= -1/5
\end{aligned}$$

Samuelson Case

The corresponding interest rate indicates we are in the Samuelson case:

$$\frac{\frac{\partial u_t(\omega_t^t, \omega_t^{t+1})}{\partial x_t^t}}{\frac{\partial u_t(\omega_t^t, \omega_t^{t+1})}{\partial x_t^{t+1}}} = 1 + r$$

$$\Leftrightarrow 1/5 = 1 + r$$

$$\Leftrightarrow r = -4/5 < 0$$

(Modified) Excess Demand

We next derive the reflected offer curve (ROC) in terms of Mr. t's excess demand. Define

$$z^t = (\omega_t^t - x_t^t)$$

and

$$z^{t+1} = (x_t^{t+1} - \omega_t^{t+1}).$$

Utility Maximization

Consumer problem:

$$\begin{array}{ll} \max\limits_{x_t^t, x_t^{t+1}} & 2\log x_t^t + 10\log x_t^{t+1} \\ \text{subject to} & p^t x_t^t + p^{t+1} x_t^{t+1} = p^t \omega_t^t + p^{t+1}\omega_t^{t+1} \end{array}$$

Solving the budget constraint for x_t^t gives

$$x_t^t = \omega_t^t + \frac{p^{t+1}}{p^t}\omega_t^{t+1} - \frac{p^{t+1}}{p^t}x_t^{t+1}$$

Plugging into the consumer's utility function

$$\max_{x_t^{t+1}} \quad 2\log\left(\omega_t^t + \frac{p^{t+1}}{p^t}\omega_t^{t+1} - \frac{p^{t+1}}{p^t}x_t^{t+1}\right) + 10\log x_t^{t+1}$$

Solving the Consumer Problem by Substitution Method

Taking derivative with respect to x_t^{t+1}

$$\frac{2\frac{p^{t+1}}{p^t}}{\omega_t^t + \frac{p^{t+1}}{p^t}\omega_t^{t+1} - \frac{p^{t+1}}{p^t}x_t^{t+1}} = \frac{10}{x_t^{t+1}}$$

Solving for x_t^{t+1}

$$x_t^{t+1} = \frac{5}{6}\frac{p^t\omega_t^t + p^{t+1}\omega_t^{t+1}}{p^{t+1}}$$

Plugging this back to the budget constraint and solve for x_t^t gives

$$x_t^t = \frac{1}{6}\frac{p^t\omega_t^t + p^{t+1}\omega_t^{t+1}}{p^t}$$

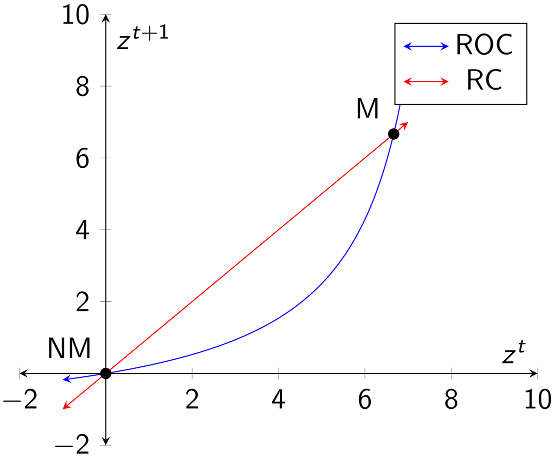
Offer Curve

Plugging in the excess demands and endowment values gives the following ROC:

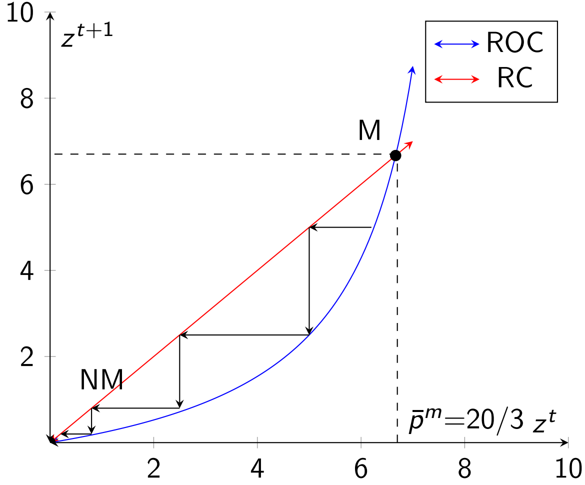
$$\begin{aligned} z^{t+1} &= \frac{(2)(10)z^t}{(10)(10) - (10+2)z^t} \\ &= \frac{20z^t}{100 - 12z^t} \end{aligned}$$

Equating $z^{t+1} = z^t$ yields $100 - 12z^t = 20 \Leftrightarrow z^t = 20/3$.

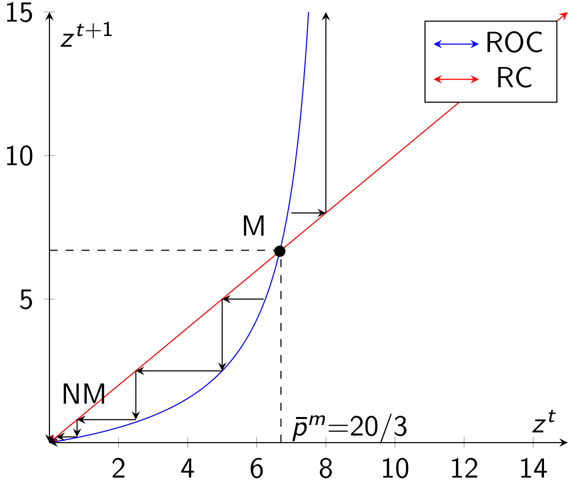
Reflected Offer Curve



Dynamics for $p^m < \bar{p}^m$



Global Dynamics



There are two stationary states:

(a) The non-monetary (non-PO) steady state with $z^t = z^{t+1} = 0$ (i.e., autarky), labeled NM.

(b) The monetary (PO) steady state with $z^t = z^{t+1} = \bar{p}^m = \frac{20}{3}$, labeled M.

The non-monetary steady state is locally stable. The monetary steady state is unstable. If $0 < p^m < \bar{p}^m$, the economy is inflationary. The current goods price of money tends asymptotically to zero. The money bubble fades away, but it does not burst.

If $p^m > \bar{p}^m$, the economy is deflationary. The current goods price of money grows so that in finite time demand for goods exceeds supply, so the deflationary bubble must burst (in finite time) since $x_{t-1}^t + x_t^t > \omega_{t-1}^t + \omega_t^t$, i.e. the demand for goods excess supply, see the phase diagram. The hyper-deflationary path does not satisfy long-run perfect foresight. The bubble must burst in finite time.

Overlapping Generations, Worked Example 3

2 period lives.

1 commodity per period, $\ell = 1$.

Stationary endowments:

$$\omega_0^1 = 3 \text{ for } t = 0$$
$$(\omega_t^t, \omega_t^{t+1}) = (3, 3) > 0 \text{ for } t = 1, 2, ...$$

Stationary preferences:

$$u_0(x_0^1) = \log x_0^1 \text{ for } t = 0$$
$$u_t(x_t^t, x_t^{t+1}) = 10 \log x_t^t + \log x_t^{t+1} \text{ for } t = 1, 2, ...$$

Taxation:

Two monies, R and B

$$m_0^{R,1} = 5,\ m_0^{B,1} = -2 \qquad m_t^{R,s} = m_t^{B,s} = 0 \text{ otherwise}$$

Present goods prices of money are $p^{R,t} \geq 0$, $p^{B,t} \geq 0$.

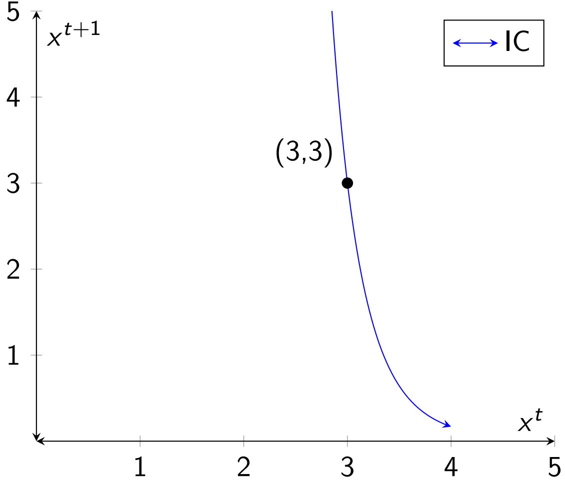
The indifference curve (IC) running through the endowment is characterized by

$$\{(x_t^t, x_t^{t+1}) : 10 \log x_t^t + \log x_t^{t+1} = 10 \log 3 + \log 3 = 11 \log 3\}$$

Exponentiating both sides to derive an analytical expression for graphing,

$$\begin{aligned} &(x_t^t)^{10}(x_t^{t+1}) = 3^{11} \\ \Leftrightarrow &(x_t^{t+1}) = \frac{3^{11}}{(x_t^t)^{10}} \end{aligned}$$

Graphically:



The slope of the IC at the endowment in (x^t, x^{t+1}) space is calculated as follows:

$$\frac{dx_t^{t+1}}{dx_t^t} = -\frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^t} \Big/ \frac{\partial u_t(x_t^t, x_t^{t+1})}{\partial x_t^{t+1}}$$
$$= -\frac{10}{x_t^t} \Big/ \frac{1}{x_t^{t+1}}$$

Evaluated at the endowment,

$$\left.\frac{dx_t^{t+1}}{dx_t^t}\right|_{(3,3)} = -\frac{10}{3} \Big/ \frac{1}{3}$$
$$= -10$$

Ricardo Case

The corresponding interest rate indicates we are in the Ricardo case:

$$\frac{\frac{\partial u_t(\omega_t^t,\omega_t^{t+1})}{\partial x_t^t}}{\frac{\partial u_t(\omega_t^t,\omega_t^{t+1})}{\partial x_t^{t+1}}} = 1 + r$$

$$\Leftrightarrow 10 = 1 + r$$

$$\Leftrightarrow r = 9 > 0$$

ROC

We next derive the reflected offer curve (ROC) in terms of Mr. t's excess demand. Again define

$$z^t = (\omega_t^t - x^t)$$

and

$$z^{t+1} = (x_t^{t+1} - \omega_t^{t+1}).$$

The Lagrangian from the utility maximization problem is:

$$\mathcal{L} = 10 \log x_t^t + \log x_t^{t+1} + \lambda(p^t \omega_t^t + p^{t+1} \omega_t^{t+1} - p^t x_t^t - p^{t+1} x_t^{t+1})$$

Taking the first order derivatives of the Lagrangian gives:

$$\frac{10}{x_t^t} = \lambda p^t$$
$$\frac{1}{x_t^{t+1}} = \lambda p^{t+1}$$

ROC

Dividing through yields the following optimality condition:

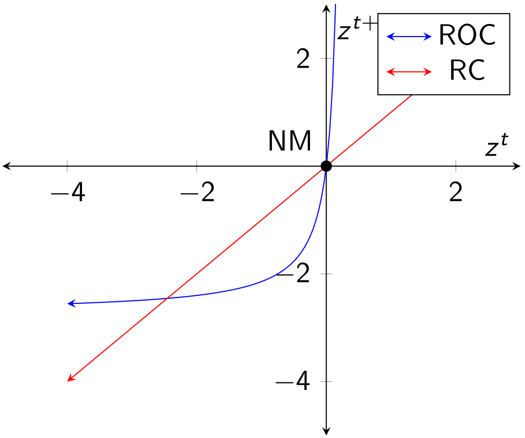
$$\frac{p^t}{p^{t+1}} = \frac{10}{1}\frac{x_t^{t+1}}{x_t^t}$$

Plugging into Mr. t's the budget constraint, we derive the following ROC:

$$\begin{aligned} z^{t+1} &= \frac{(10)(3)z^t}{(1)(3)-(1+10)z^t} \\ &= \frac{30z^t}{3-11z^t} \end{aligned}$$

ROC in the Ricardo Case

Graphically:



Non-Monetary Steady State

There is only one stationary equilibrium for this Ricardo economy:

(a) The non-monetary (PO) steady state with $z^t = z^{t+1} = 0$ (i.e., autarky), labeled NM. From the budget constraint of the initial old generation, we have

$$\begin{aligned} z_0^1 &= m_0^{R,1} P^R + m_0^{B,1} P^B \\ 0 &= 5P^R - 2P^B \\ P^B &= \frac{5}{2} P^R \end{aligned}$$

The non-monetary steady state is locally stable.

Similarly, there is no non-stationary equilibrium.

The End

Overlapping Generations is the last lecture subject. From now on, we will focus on the student presentations. Please see Professor Shell in his office hours or by appointment.

Prelim 2

1. Prelim 2 is scheduled for the class period on Monday, November 6. All material up to and including OG will be fair game.
2. Professor Shell is due back in Ithaca on late night on Sunday, November 5. His flights have several close connections. Do not count on his Monday, November 5 office hour. He will be there if the flights work out.