Bank Runs: The Pre-Deposit Game*

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Abstract

We analyze in some detail the full *pre-deposit* game in a simple, tractable, yet very rich, banking environment. How does run-risk affect the optimal deposit contract? If there is a run equilibrium in the *post-deposit* game, then the optimal contract in the *pre-deposit* game tolerates small-probability runs. However, this does not mean that small changes in run-risk are ignored. In some cases, the optimal contract becomes – as one would expect – strictly more conservative as the run-probability increases (until it switches to the best run-proof contract), and the equilibrium allocation is not a mere randomization over the equilibrium allocations from the post-deposit game. In other cases, the allocation *is* a mere randomization over the equilibria from the post-deposit game. In the first cases (the more intuitive cases), the incentive constraint does not bind. In the second cases, the incentive constraint does bind.

Keywords: bank runs, constrained efficient allocation, deposit contract, impulse demand, pre-deposit game, post-deposit game, run probability, strategic complementarity, sunspots, unconstrained efficient allocation

JEL classification numbers: G21; E44

1 Introduction

Bryant (1980) and Diamond and Dybvig (1983) – hereafter DD – introduced the modern literature on panic-based bank runs. The bank deposit contract is a mechanism designed to improve the welfare of depositors facing an uncertain impulse demand (i.e., when they become impatient). Since the impulse demand itself is not directly observable, it is uninsurable in the market. The deposit contract facilitates the needed "insurance" through early and late withdrawals that are Bayesian incentive compatible (hereafter BIC), i.e., depositors with different liquidity needs correctly self-select their types.[1] But the "good" equilibrium in which the depositors correctly self-select is not the *unique* Bayes-Nash equilibrium (hereafter BNE). DD show that there is also a "bad" equilibrium in which there is self-fulfilling bank run. When a bank run occurs, depositors attempt to withdraw early no matter what their liquidity needs are.

But given the two equilibria of the *post-deposit* game, consumers will never deposit in the bank if they anticipate the run: a probability-one bank-run will not be an equilibrium for the *pre-deposit* game. DD recognized this problem and offered sunspots as an answer.[2] The question is how safe a bank or banking system should be given the risk of runs.

Peck and Shell (2003) – hereafter PS – have shown that the optimal banking contract allows for runs if the run probability is small.[3] PS showed

[1] That is, an impatient depositor chooses early withdrawal and a patient depositor chooses late withdrawal if he expects that the other patient depositors will also choose late withdrawal.

[2] DD write on Page 410 "This could happen if the selection between the bank run equilibrium and the good equilibrium depended on some commonly observed random variable in the economy. This could be a bad earnings report, a commonly observed run at some other bank, a negative government forecast, or *even sunspots* [emphasis ours]."

Postlewaite and Vives (1987) show how bank runs can be seen as a prisoner's dilemma-type situation in which there is a unique equilibrium that involves a positive probability of a run.

[3] See also Cooper and Ross (1998) and Ennis and Keister (2006). These two papers analyze how banks respond to the possibility of runs in their design of deposit contracts and in their investment decisions. Gu (2011) analyzes noisy sunspots and bank runs. These three papers focus on simple deposit contracts, while PS allows for partial or full suspension of convertibility. The PS model is more general, but we adopt their 2-depositor example.

that a sunspot-driven run can be an equilibrium in the pre-deposit game as long as the run probability is below a threshold level. PS employed a 2-depositor numerical example in calculating the optimal deposit contract as a function of the run probability.[4] The banking contract is completely characterized by the scalar c, the withdrawal of the first depositor in line in period 1. The smaller the value of c, the more conservative is the deposit contract. The optimal c is denoted by $c^*(s)$, which is a function of the exogenous, sunspot run-probability s.[5] In the calculated example in PS, $c^*(s)$ is a step function: If the probability s is greater than some fraction s_0, the optimal contract is the best run-proof contract. If the probability s is less than the threshold probability s_0, the optimal contract $c^*(s)$ tolerates runs and is a constant.

In the present paper, we ask: Why doesn't the optimal contract become strictly more conservative as the run probability increases (until runs are no longer tolerated)? In other words, shouldn't we expect $c^*(s)$ to be strictly decreasing in s until the contract switches to the best run-proof contract? If yes, which economies exhibit this property and in which economies is $c^*(s)$ a step function? Are there other (perhaps mixed) cases? To answer these questions and to keep the analysis tractable, we employ the general 2-depositor banking example from PS,[6] but instead of relying solely on a single numerical

In contrast to DD, there is also intrinsic uncertainty (as opposed to extrinsic uncertainty in the form of sunspots) in the PS model: (1) The aggregate number of impatient consumers is uncertain. It could be $0, 1$ or 2. This is important because the bank will never know with certainty that a run is underway. (2) Sequential service is taken seriously (Wallace (1988)). When 2 depositors withdraw early, their positions in the queue are random. (3) Partial suspension of convertibility is feasible for the bank (Wallace (1990)). Typically the first in line in period 1 will be allowed a larger withdrawal than the second in line.

[4]The numerical example is in their proof for Proposition 3 (Page 114).

[5]Like other papers in the literature, PS assume that in equilibrium the bank chooses the contract that maximizes the *ex-ante* expected utility of depositors. This assumption can be justified when banks compete for deposits, there is no entry cost for banks, and the depositors are *ex-ante* identical.

[6]Green and Lin (2000), Andolfatto, Nosal and Wallace (2007) and Nosal and Wallace (2009) analyze a model similar to PS. The main differences among the models are on the amount of information that a depositor has at the time of making his withdrawal decision. Ennis and Keister (2009) show that the PS assumptions on marginal utilities are not necessary for the qualitative results in PS. Ennis and Keister (2009) also study the Green-Lin model under a more general specification of the distribution of types across agents. See Ennis and Keister (2010) for a good survey on this part of the literature.

example, we provide the global comparative statics of the optimal contract.

We analyze the game backward and start the analysis from the *post-deposit* game for an arbitrary contract c. We characterize the conditions on c which allow for a run equilibrium and a non-run equilibrium in the post-deposit game. The optimal contract c^* is the BIC contract which maximizes the expected utility of the depositor in the *pre-deposit* game.

A contract is BIC if, for this contract, there exists a non-run Bayes-Nash equilibrium in the *post-deposit* game. Therefore, the set of contracts with non-run as the *unique* BNE is a subset of the set of BIC contracts. For "unusual" values of the parameters, the two sets are the same. With "unusual parameters", bank runs are not relevant for the optimal contract and hence the analysis of the *pre-deposit* game is similar to that of the corresponding *post-deposit* game. Analysis of c^* for the "unusual" parameters is in the Appendix.

For the "usual" values of the parameters – which are our focus – the set of contracts with non-run as the *unique* BNE is a *strict* subset of the set of BIC contracts. For the "usual" parameters, c^* is affected by s unless non-run is the unique BNE. The shape of the function $c^*(s)$ depends on further specification of the remaining parameters.

To further examine $c^*(s)$, we divide the "usual" parameter space into three cases: (1) the *unconstrained efficient allocation* is the unique BNE;[7] (2) it is not the unique BNE but it is BIC; or (3) it is not even BIC. In the first case, c^* is the contract supporting the unconstrained efficient allocation and it is not affected by s since the "good equilibrium" is the unique BNE.[8] In the second case, c^* balances the trade-off between the non-run and run equilibria. As s increases, the trade-off changes continuously and the expected utility of

[7] The *unconstrained efficient allocation* is the best allocation that can be attained when agent types (patient or impatient) are observable. In other words, the allocation maximizes the ex ante expected utility of agents without imposing incentive compatibility, but it is still subject to the sequential service and resource feasibility constraints. See Ennis and Keister (2010). The associated contract is sometimes called the "first-best contract".

[8] Note the difference between the parameters with "unusual" values and Case 1. With "unusual" parameter values, the non-run equilibrium and run equilibrium cannot co-exist for *any* contract c. But for Case 1, the non-run equilibrium and run equilibrium can co-exist for some contracts, but these contracts are not optimal.

the depositor is more dependant on the run equilibrium. Hence the optimal contract $c^*(s)$ is continuous and strictly decreasing until it switches to the best run-proof contract. In the third case, the trade-off is constrained by the incentive-compatibility constraint (hereafter ICC). When s is small, the ICC binds and the optimal contract does not respond to changes in s since the binding ICC is independent of s. In this case, a larger run probability does not induce a more conservative contract since the binding ICC has already forced $c^*(s)$ to be more conservative than it would have been absent the ICC.

In Section 2, we introduce the notation and the formal set-up. In Section 3, we analyze the post-deposit game and characterize the non-run equilibrium and the run equilibrium in the post-deposit game. We identify the "usual" and "unusual" values of the parameters.

In Section 4, we analyze the pre-deposit game and provide comparative statics for the optimal contract for the "usual" parameter values. We focus in Section 4 on comparative statics with respect to the impulse demand parameter. Holding the other parameters constant, the impulse demand parameter determines whether we are in Case 1 or Case 2 or Case 3.

The proofs of our results are in the Appendix. In the Online Appendix, we complete the full comparative statics with respect to the parameters.

2 The Environment

There are two consumers and three periods: 0, 1 and 2.[9] In period 0, each consumer is endowed with y units of the consumption good.[10] Let c^1 and c^2 denote the withdrawals of the depositor in period 1 and 2 respectively. The impatient consumers derive utility only from period-1 consumption, while the patient consumers derive utility only from period-2 consumption. The patient consumers can store consumption goods costlessly across the two periods. The impatient and patient consumers, respectively, have utility

[9] The notation is from PS.

[10] There are no endowments in periods 1 and 2.

functions $u(c^1)$ and $v(c^1 + c^2)$, where

$$u(x) = A\frac{(x)^{1-b}}{1-b} \text{ where } A > 0, \tag{1}$$

$$v(x) = \frac{(x)^{1-b}}{1-b}. \tag{2}$$

A is the intensity of the impulse demand of an impatient consumer. PS introduced the parameter A in the bank runs literature. In DD, we have $A = 1$. We analyze in section 3 the effects of A on the optimal deposit contract. The parameter $b > 1$ is the coefficient of relative risk aversion. Consumers are identical in period 0. In period 1, each consumer becomes either impatient with probability p or patient with probability $1 - p$. Types are uncorrelated and private information. Since the number of consumers is finite, the aggregate number of patient depositors is stochastic. In period 1, each depositor also observes a sunspot variable δ distributed uniformly on $[0, 1]$. Besides the storage technology, there is another investment technology. Investing one unit of period 0 consumption yields $R > 1$ units if harvested in period 2 and yields 1 unit if harvested in period 1.

The sequential service constraint is part of the physical environment. A depositor visits the bank only when he makes a withdrawal. When a depositor learns his type and makes his withdrawal decision, he does not know his position in the bank queue. If more than one depositor chooses to withdraw, a depositor's position in the queue is random; in particular, positions in the queue are equally probable.

The pre-deposit game is a sequential game. In the first-stage (period 0), the bank chooses the contract c and the consumers choose whether to deposit or not. If the consumers choose to deposit,[11] they move on to the second stage (period 1) which is the post-deposit game and decide when to make withdrawals. As usual in sequential games, we solve it by backward induction.

[11] Consumers always weakly choose to deposit in equilibrium since the contract chosen by the bank can at least mimic the autarky allocation.

3 The Post-Deposit Game

The post-deposit game is a static game between two depositors with incomplete information. The incomplete information is due to the fact that one's "type" is private information. The action set of each depositor is $\{E, L\}$, where E (L) stands for early (late) withdrawal in period 1 (period 2). An impatient depositor always chooses E. Hence the relevant strategy of a depositor is (E, E) or (L, E), where the first (second) element in the parenthesis stands for the action chosen by the depositor when he is patient (impatient). The pay-off matrix is below. The first symbol in each cell represents the payoff to the row player and the second symbol represents the payoff to the column player.

Table 1

	(E,E)	(L,E)
(E,E)	T_1, T_1	T_3, T_2
(L,E)	T_2, T_3	T_4, T_4

The entries in the payoff matrix are given by:

$$T_1 = (1-p)\frac{v(c)+v(2y-c)}{2} + p\frac{u(c)+u(2y-c)}{2},$$

$$T_2 = (1-p)v[(2y-c)R] + p\frac{u(c)+u(2y-c)}{2},$$

$$\begin{aligned} T_3 = {} & (1-p)[(1-p)v(c) + p\tfrac{v(c)+v(2y-c)}{2}] \\ & +p[(1-p)u(c) + p\tfrac{u(c)+u(2y-c)}{2}], \end{aligned}$$

$$\begin{aligned} T_4 = {} & (1-p)[(1-p)v(yR) + pv[(2y-c)R]] \\ & +p[(1-p)u(c) + p\tfrac{u(c)+u(2y-c)}{2}]. \end{aligned}$$

3.1 Run Equilibrium in the Post-Deposit Game

A run equilibrium in the post-deposit game is defined as a Bayes-Nash equilibrium in which both depositors choose (E, E), which requires that $T_1 > T_2$, or equivalently, that $c \in [0, 2y]$ satisfies[12]

$$[v(c) + v(2y - c)]/2 > v[(2y - c)R]. \tag{3}$$

Whether there is $c \in [0, 2y]$ satisfying inequality (3) depends on the parameters b and R. This is because, from inequality (3), the patient depositor's comparison between the early and late withdrawals depends on: (1) his attitude toward the risk of being the second in line when participating in runs (i.e., b), and (2) the productivity of the investment technology (i.e., R). Given R, the more risk-averse the patient depositor, the less preferable is it for him to run on the bank. Given his attitude toward risk, the more productive the investment, the less preferable is it for him to run because more resources will be left in the last period. Hence the existence of $c \in [0, 2y]$ satisfying inequality (3) restricts the parameters b and R. We have the following lemma.[13]

Lemma 1 *If $b < 1 + \ln 2/\ln R$ holds, then there is a run equilibrium in the post-deposit game if and only if c satisfies the inequality*

$$c > c^{early} = 2y/[(2/R^{b-1} - 1)^{1/(b-1)} + 1]. \tag{4}$$

If $b \geq 1 + \ln 2/\ln R$ holds, then there is no run equilibrium in the post-deposit game for any $c \in [0, 2y]$.

If $b < 1 + \ln 2/\ln R$ holds, then c^{early} is the level of c beyond which a patient depositor chooses early withdrawal if he expects that the other depositor – if patient – will also choose early withdrawal.

3.2 Non-Run Equilibrium in the Post-Deposit Game

A non-run equilibrium in the post-deposit game is defined as a Bayes-Nash equilibrium in which both depositors choose (L, E), which requires $T_4 \geq T_3$,

[12]As in other papers in the literature, we assume that a patient depositor chooses early withdrawal if he *strictly* prefers the period-1 withdrawal to the period-2 withdrawal. And he chooses period-2 withdrawal if he *weakly* prefers to do so.

[13]y only changes the scale of the economy.

or equivalently, that $c \in [0, 2y]$ satisfies

$$(1-p)v(yR) + pv[(2y-c)R] \geq (1-p)v(c) + p[v(c) + v(2y-c)]/2. \quad (5)$$

Inequality (5) is also the Incentive Compatibility Constraint (ICC).

Lemma 2 *If $b < 1 + \ln 2/\ln R$ holds, there is a non-run equilibrium in the post-deposit game if and only if c satisfies the inequality*

$$c \leq c^{wait}, \quad (6)$$

where c^{wait} is the level of c such that (5) holds as an equality.

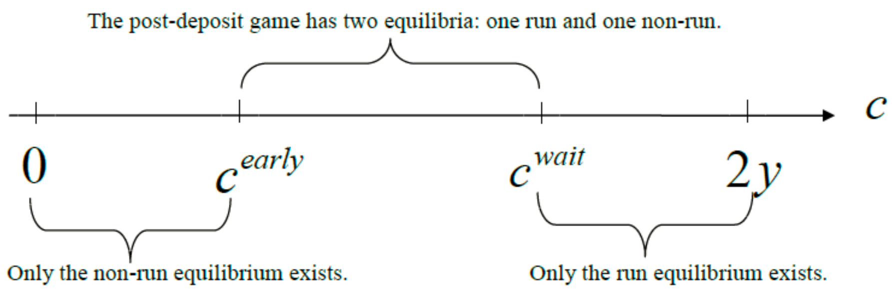
Thus c^{wait} is the level of c *below* which a patient depositor chooses late withdrawal if he expects that the other patient depositor will also choose late withdrawal. Thus, if we have $b < 1+\ln 2/\ln R$, then the set of BIC contracts is $[0, c^{wait}]$.

3.3 Equilibria in the Post-Deposit Game for an Arbitrary Contract

From the analysis above, we know that if we have $b < 1+\ln 2/\ln R$, then c^{early} and c^{wait} are well-defined and they are two critical thresholds in the contract space. Furthermore, if we have $c^{early} < c^{wait}$, then the set of contracts with the non-run outcome as the unique BNE is the interval $[0, c^{early}]$, which is a strict subset of the set of BIC contracts, the interval $[0, c^{wait}]$. Therefore, the post-deposit game has a unique non-run equilibrium for $c \in [0, c^{early}]$, two equilibria (one non-run equilibrium and one run equilibrium) for $c \in (c^{early}, c^{wait}]$, and a unique run equilibrium for $c \in (c^{wait}, 2y]$. (See Figure 1.) The interval $(c^{early}, c^{wait}]$ is the region of c for which the patient depositors' withdrawal decisions exhibit strategic complementarity.

In other words, for $c \in [0, c^{early}]$, we have $T_2 \geq T_1$ and $T_4 > T_3$. (L, E) is the dominant strategy for each depositor. The post-deposit game has a unique BNE which is also a dominant strategy equilibrium with Pareto efficiency. For $c \in (c^{early}, c^{wait}]$, we have $T_2 < T_1$ and $T_4 \geq T_3$. The decisions of the two depositors are strategically complementary, leading to both one run and one non-run equilibrium. The post-deposit game is "stag hunt". For $c \in (c^{wait}, 2y]$, we have $T_2 < T_1$ and $T_4 < T_3$. (E, E) is the dominant strategy for each depositor. The post-deposit game is of "prisoner's dilemma" type; running is the unique equilibrium strategy.

Figure 1. Equilibrium in the Post-Deposit Game



In the following lemma, we characterize the set of parameters b and R for which we have $c^{early} < c^{wait}$.

Lemma 3 *We have $c^{early} < c^{wait}$ if and only if we have*

$$b < \min\{2, 1 + \ln 2/\ln R\}. \tag{7}$$

If (7) is satisfied we say that the parameters are "usual" since then the set of contracts with non-run as the unique BNE is a strict subset of the set of BIC contracts. From PS, we know that the *pre-deposit* game has a run equilibrium only if the post-deposit game has both a non-run equilibrium and a run equilibrium.[14] Given the "usual" values of b and R, we know that a run-equilibrium exists in the *pre-deposit* game only if the optimal deposit contract belongs to $(c^{early}, c^{wait}]$. In the next section, we solve for the optimal deposit contract. Before we describe the optimal contract, we give a numerical example in which b and R are "usual".

[14]Proposition 2 in PS.

Example 1 *The parameters are*

$$b = 1.01; p = 0.5; y = 3; R = 1.5.$$

These parameters will be fixed throughout the examples. We see that b *and* R *satisfy (7). Hence* c^{early} *must be strictly smaller than* c^{wait}*. Indeed, we have* $c^{early} = 4.155955$ *and* $c^{wait} = 4.280878$*. Therefore, whenever the contract* c *is larger than* 4.155955 *and smaller than or equal to* 4.280878*, both a run equilibrium and a non-run equilibrium exist in the post-deposit game.*

For completeness, we take a look at the *post-deposit* game when b and R are "unusual" (i.e., they do not satisfy (7)). They are "unusual" since, under these values of parameters, the set of BIC contracts is the same as the set of contracts with non-run as the unique BNE. According to the Revelation Principle,[15] when we search for the optimal contract we only have to focus on c which is BIC. Therefore, for "unusual" parameters, bank runs are not relevant. We discuss the optimal contract under "unusual" parameters in the Appendix. For the remainder of the paper, we focus on the economy with "usual" values of b and R.

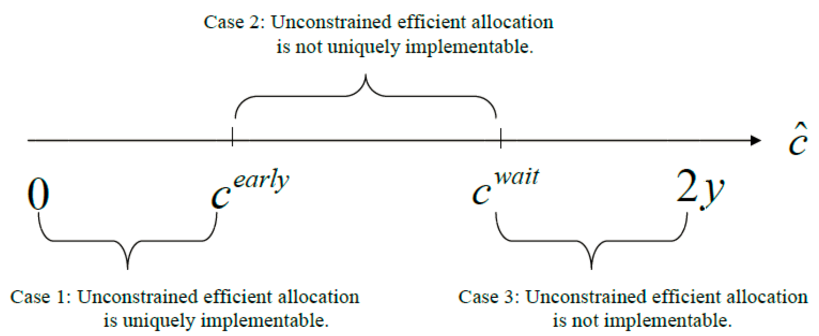
4 The Optimal Contract for the Pre-Deposit Game

We have shown that, when the values of b and R are "usual", for any $c \in (c^{early}, c^{wait}]$ we have multiple equilibria in the *post-deposit* game. We assume – as in PS – that bank runs are sunspot-driven. Hence whether a run equilibrium exists in the *pre-deposit* game depends on whether in the first stage the contract chosen by the bank (i.e., the optimal contract) belongs to the set $(c^{early}, c^{wait}]$. To characterize the optimal contract, we divide the "usual parameters" into three cases depending on $\widehat{c}$, the contract supporting the *unconstrained efficient allocation*. The three cases are: $\widehat{c} \leq c^{early}$ (Case 1), $\widehat{c} \in (c^{early}, c^{wait}]$ (Case 2), and $\widehat{c} > c^{wait}$ (Case 3). We next characterize

[15] Myerson (1979)

the parameters for each case. In particular, when b and R are "usual", we show in the following section that each case defines a range of values of the parameter A, the impulse intensity of the impatient consumer.

Figure 2. Three Cases



4.1 The Impulse Parameter A and the 3 Cases

The contract $\widehat{c}$ supports the *unconstrained efficient allocation*, where $\widehat{c}$ is defined by

$$\widehat{c} = \arg \max_{c \in [0,2y]} \widehat{W}(c),$$

where

$$\widehat{W}(c) = p^2[u(c)+u(2y-c)]+2p(1-p)[u(c)+v[(2y-c)R]]+2(1-p)^2 v(yR). \quad (8)$$

$\widehat{W}(c)$ is the depositor's expected utility if the types of the depositors are publicly observable *ex-post*.[16] Given the CRRA utility function, we have

$$\widehat{c} = \frac{2y}{\{p/(2-p) + 2(1-p)/[(2-p)AR^{b-1}]\}^{1/b} + 1}. \quad (9)$$

From (9), we can write $\widehat{c}$ as an increasing function of A, $\widehat{c}(A)$. As the "impulse demand" is more intense, the unconstrained efficient allocation provides

[16] $\widehat{W}(c)$ is also the depositor's expected utility in the non-run equilibrium of the *pre-deposit* game.

for a larger first-period withdrawal. From equation (9), we have

$$\lim_{A\to 0}\widehat{c}(A) = 0$$

and

$$\lim_{A\to\infty}\widehat{c}(A) = \frac{2y}{[p/(2-p)]^{1/b}+1}.$$

From equations (4) and (5), we know that neither c^{early} nor c^{wait} depends on A. This is intuitive: c^{early} and c^{wait} are determined solely by the patient depositor's utility, which is independent of A.

Hence if A is sufficiently small, we have Case 1. Furthermore, if

$$\frac{2y}{[p/(2-p)]^{1/b}+1} \leq c^{early},$$

only Case 1 obtains.

If

$$c^{early} < \frac{2y}{[p/(2-p)]^{1/b}+1} \leq c^{wait},$$

there is a unique level of A, denoted by A^{early}, such that

$$\widehat{c}(A) = c^{early}. \tag{10}$$

Hence if $A \leq A^{early}$, we have Case 1. If $A > A^{early}$, we have Case 2. The unconstrained efficient allocation never leads to Case 3.

If

$$c^{wait} < \frac{2y}{[p/(2-p)]^{1/b}+1},$$

there is a unique level of A, denoted by A^{wait}, such that

$$\widehat{c}(A) = c^{wait}. \tag{11}$$

We now have all three cases: if $A \leq A^{early}$, we are in Case 1; if $A^{early} < A \leq A^{wait}$, we are in Case 2; if $A > A^{wait}$, we are in Case 3.[17]

[17] It is easy to see that c^{early} does not depend on p and that we have $y < c^{early} < c^{wait} < Ry$. $\lim_{p\to 1}\frac{2y}{[p/(2-p)]^{1/b}+1} = y$ and $\lim_{p\to 0}\frac{2y}{[p/(2-p)]^{1/b}+1} = 2y$. Hence we know that for

Example 2 *In Example 1, we know that* $c^{early} = 4.155955$ *and* $c^{wait} = 4.280878$*. Now we calculate the thresholds of* A *for each case. It is easy to see that*

$$c^{wait} < \frac{2y}{[p/(2-p)]^{1/b} + 1}$$

for the specified parameter values. Therefore, all three cases exist. We have $A^{early} = 6.217686$ *and* $A^{wait} = 10.27799$*. Hence if* $A \leq 6.217686$*, we are in Case 1; if* $6.217686 < A \leq 10.27799$*, we are in Case 2; if* $A > 10.27799$*, we are in Case 3.*

In the Online Appendix, we show how the other parameters – namely p, R and b – affect $\widehat{c}$. Unlike A, these parameters also affect the values of c^{early} and/or c^{wait}, and $\widehat{c}$ is not monotonic in the parameters. The analysis is slightly more complicated in these cases, but once the parameter values are given, we can readily compute the values of $\widehat{c}$, c^{early} and c^{wait} to determine which case is applicable.

4.2 The Optimal Contract

In this subsection, we focus on the parameter values of b and R satisfying condition (7). We will describe the optimal contract c^* for the three cases specified above.

For an arbitrary contract $c \in [0, 2y]$, we have one or two equilibria in the post-deposit game depending on whether c belongs to $(c^{early}, c^{wait}]$ or not. As equation (8) shows, the depositor's expected utility in the non-run equilibrium is $\widehat{W}(c)$. Let $W^{run}(c)$ denote the depositor's expected utility in the run equilibrium (if it exists). It is given by

$$\begin{aligned} W^{run}(c) &= p^2[u(c) + u(2y-c)] + p(1-p)[u(c) + v(2y-c) + v(c) + u(2y-c)] \\ &\quad + (1-p)^2[v(c) + v(2y-c)]. \end{aligned} \tag{12}$$

When $c \leq c^{early}$, only the non-run-equilibrium exists and the depositor's

sufficiently large p, we have $\frac{2y}{[p/(2-p)]^{1/b}+1} \leq c^{early}$. If $R < 2$, we have $c^{wait} < \frac{2y}{[p/(2-p)]^{1/b}+1}$ for sufficiently small p. For intermediate values of p, we have $c^{early} < \frac{2y}{[p/(2-p)]^{1/b}+1} \leq c^{wait}$.

ex-ante expected utility is simply $\widehat{W}(c)$. When $c^{early} < c \leq c^{wait}$, both the run-equilibrium and the non-run-equilibrium exist in the *post-deposit* game. If the run probability is s, the depositor's ex-ante expected utility is

$$(1-s)\widehat{W}(c) + sW^{run}(c).$$

When we have $c^{wait} < c \leq 2y$, only the run-equilibrium exists and therefore, no consumer would want to deposit in this bank. Hence when we consider the optimal contract, $[0, c^{wait}]$ is the relevant choice interval for c.

Let $c^*(s)$ denote the optimal contract which maximizes the depositor's ex-ante expected utility in the *pre-deposit* game given the run probability s.[18] We have

$$c^*(s) = \arg\max_{c\in[0,c^{wait}]} W(c;s),$$

where

$$W(c;s) = \begin{cases} \widehat{W}(c) & \text{if } c \leq c^{early}. \\ (1-s)\widehat{W}(c) + sW^{run}(c) & \text{if } c^{early} < c \leq c^{wait}. \end{cases} \tag{13}$$

Case 1 *The unconstrained efficient allocation is the unique BNE, i.e., we have* $\widehat{c} \leq c^{early}$.

Since the unconstrained efficient allocation is uniquely implementable, it is straightforward to see that the optimal contract for the pre-deposit game supports the unconstrained efficient allocation

$$c^*(s) = \widehat{c} \text{ for } s \text{ belonging to } [0,1]$$

and that a bank run does not occur in equilibrium. Other contracts cannot be optimal because they either deliver lower welfare in the non-run equilibrium or, to make things worse, they might also support a run equilibrium. The next is a numerical example for Case 1.

[18] At $c^*(s;A)$, consumers must weakly prefer depositing to autarky. This is because the deposit contract can always mimic the autarky allocation by setting $c^*(s;A)$ equal to y. Hence the participation constraint is not an issue for $c \in [0, c^{wait}]$

Example 3 *In Example 2, we have seen that as long as* $A \leq A^{early} = 6.217686$*, we have Case 1 for this economy. Let* A *be equal to* 1*. We have* $c^*(s) = \widehat{c} = 3.004012$ *for any* $s \in [0,1]$*. Since* $c^*(s) < c^{wait} = 4.280878$*, a bank run is not an equilibrium.*

As we have discussed in section 4.1, when A is larger than A^{wait}, we have Case 2:

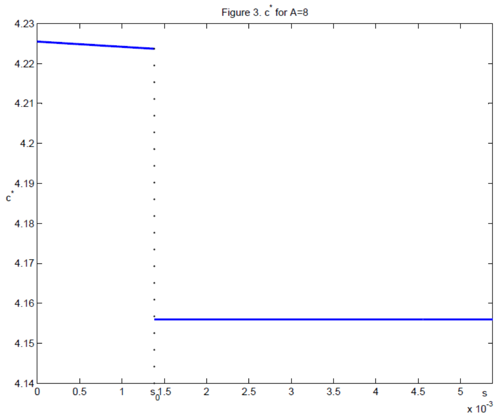
Case 2 *The unconstrained efficient allocation is BIC but not the unique BNE, i.e., we have* $c^{early} < \widehat{c} \leq c^{wait}$.

The optimal contract for the *pre-deposit* game cannot be $\widehat{c}$ except for the degenerate case when $s = 0$. This is because a run equilibrium exists at $\widehat{c}$. How much the optimal contract deviates from $\widehat{c}$ depends on s, which changes the trade-off between the expected utilities over the 2 *post-deposit* game equilibria. We show in Proposition 1 how the optimal contract to the *pre-deposit* game changes with the probability s.

Proposition 1 *In Case 2, the optimal contract* $c^*(s)$ *satisfies: (1) if* s *is larger than the threshold probability* s_0 *(specified in equation (21) in the proof), the optimal contract is run-proof,* $c^*(s) = c^{early}$ *and (2) if* s *is smaller than* s_0*, the optimal contract* $c^*(s)$ *tolerates runs and it is a strictly decreasing function of* s*. We have that* $c^*(s) \leq \widehat{c}$ *(with equality if and only if* $s = 0$*).*

From Proposition 1, we see that in Case 2 the contract supporting the unconstrained efficient allocation also supports the run equilibrium from the post-deposit game. Except for the degenerate case of the run probability $s = 0$, this contract cannot be optimal since it delivers very low welfare in the run equilibrium. The optimal contract should optimize the ex-ante trade-off between the depositors' welfares in the run and the non-run equilibria. Intuitively: The trade-off depends on the run probability s. For positive s, a more conservative contract, still tolerating runs or eliminating runs completely, is desirable. If s is larger than the threshold probability s_0, eliminating runs is less costly (in terms of ex-ante welfare) than tolerating runs; hence the optimal contract is the best run-proof contract, c^{early}, in which the non-run equilibrium is implemented uniquely. On the other hand, if s is smaller than s_0, tolerating runs is less costly. Furthermore, $c^*(s)$ will be strictly decreasing in s since, as s increases, the ex-ante welfare leans more towards the welfare in the run equilibrium. The sunspot equilibrium allocation in this case is *not* a mere randomization over the *unconstrained efficient allocation* and the corresponding *run allocation.*

Example 4 *In Example 2, we have seen that as long as* $6.217686 < A \leq 10.277988$*, we are in Case 2. Set* $A = 8$*. We have* $s_0 = 1.382358 \times 10^{-3}$*. If* $s > s_0$*, the optimal contract is run-proof and* $c^*(s) = c^{early} = 4.155955$*. If* $s < s_0$*, the optimal contract tolerates runs and* $c^*(s)$ *is strictly decreasing in* s *with* $c^*(0) = \widehat{c} = 4.225479$*. If* $s = s_0$*, both the run-proof contract* (c^{early}) *and the run-tolerating contract are optimal. Figure 3 is a plot of* $c^*(s)$ *for this case.*



As we have discussed in section 4.1, we shift from Case 2 to Case 3 for even larger values of A:

Case 3 *The unconstrained efficient allocation is not BIC, i.e., we have* $c^{wait} < \widehat{c}$.

In this case, the unconstrained efficient allocation is not implementable since $\widehat{c}$ cannot satisfy the ICC. Like Case 2, the optimal contract for the *pre-deposit* game also involves the trade-off between the two *post-deposit* game equilibria, but the trade-off is constrained by the ICC.[19] This changes how $c^*(s)$ tolerates runs. To provide sufficient incentives for the patient depositors to choose late withdrawal, the ICC requires that c is not too large. Hence the binding ICC forces $c^*(s)$ to be more conservative than it would be without the ICC. If the ICC binds, then for small s when runs are still tolerated, c^* is independent of s.

Proposition 2 *In Case 3: (1) If s is larger than the threshold probability s_1 (specified in equation (26) in the proof in the Appendix), we have $c^*(s) = c^{early}$ and the optimal contract is run-proof, and (2) if s is smaller than s_1, the optimal contract $c^*(s)$ tolerates runs and it is a weakly decreasing function of s. Furthermore, we have $c^*(s) = c^{wait}$ (i.e., ICC binds) for at least part of the run tolerating range of s.*

At least when s is sufficiently small, the ICC binds. So for Case 3, when s is sufficiently small, $c^*(s)$ does not change with s since the ICC does not depend on s. The ICC might bind for merely part of the run-tolerating range of s (the first sub-case) or the whole run-tolerating range of s (the second sub-case). The PS numerical example belongs to the second sub-case. When the ICC binds, the allocation supported by the optimal contract *is* a simple randomization over the *constrained efficient allocation*[20] and the corresponding run allocation. The *unconstrained efficient allocation* is never supported by $c^*(s)$ since it is not implementable.

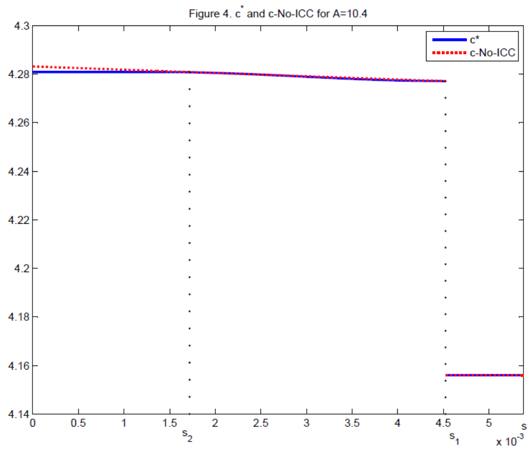
We next provide two examples of the optimal contract for Case 3, which correspond to the two sub-cases.

Example 5 *In Example 2, we have seen that as long as* $A > 10.277988$*, we are in Case 3. Let A be equal to* 10.4 *and we are in the first sub-case of Case*

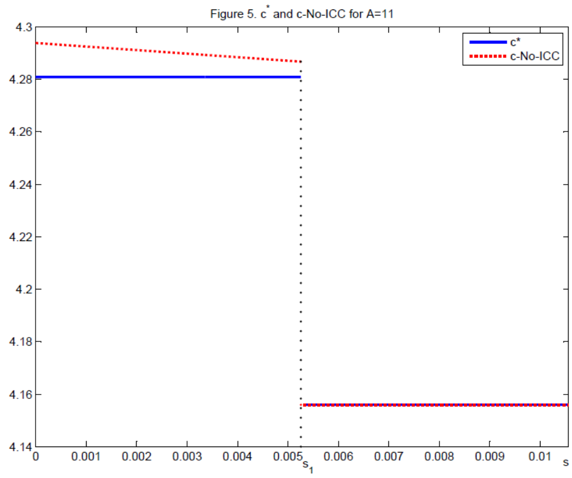
[19] In Case 2, the ICC cannot bind. To see this, (1) we know that a contract c which is larger than the contract supporting the *unconstrained efficient allocation* makes the run equilibrium more devastating, and (2) it does not improve welfare at the non-run equilibrium. Hence to find the optimal contract, we need only to focus on contracts which are more conservative than the contract supporting the *unconstrained efficient allocation.* So binding incentive compatibility cannot occur in Case 2.

[20] Ennis and Keister (2010) define the *constrained efficient allocation* as "a (contingent) consumption allocation to maximize the ex ante expected utility of agents subject to incentive compatibility, sequential service, and resource feasibility constraints". We use the same definition.

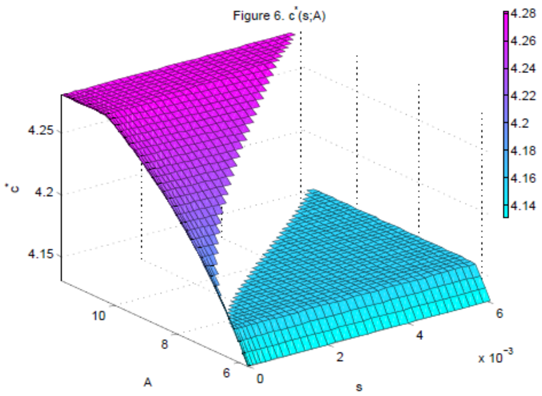
3. If s is larger than $s_1 = 4.524181 \times 10^{-3}$, the optimal contract is run-proof and $c^*(s) = c^{early} = 4.155955$. If s is smaller than s_1, the optimal contract $c^*(s)$ tolerates runs and it is a weakly decreasing function of s. To be more specific, if s is smaller than $s_2 = 1.719643 \times 10^{-3}$, the ICC binds and $c^*(s) = c^{wait} = 4.280878$; if $s_2 \leq s < s_1$, the ICC does not bind and $c^*(s)$ is strictly decreasing in s. To show the effect of the binding ICC, we plot in Figure 4 the hypothetical optimal contract in which the ICC is removed (labelled as c-No-ICC) along with $c^*(s)$. The hypothetical contract is infeasible if – as we assume – types are private information.



Example 6 *Let A be equal to 11. We are in the second sub-case of Case 3. If s is larger than $s_1 = 5.281242 \times 10^{-3}$, the optimal contract is run-proof and $c^*(s) = c^{early} = 4.155955$. If $s < s_1$, the optimal contract tolerates runs and $c^*(s) = c^{wait} = 4.280878$ since ICC binds. When $s = s_1$, both the run-proof contract (c^{early}) and the run-tolerating contract (c^{wait}) are optimal. Again we plot both c^* and the hypothetical optimal contract c-No-ICC in Figure 5.*



We plot c^* versus both s and A in Figure 6. Given the parameters we have used in previous examples, we have: if $A \leq A^{early} = 6.217686$, we are in Case 1 and the optimal contract doesn't tolerate runs and it is equal to $\widehat{c}$ the value of which depends on A. If $6.217686 < A \leq 10.277988$, we are in Case 2 and the optimal contract is strictly decreasing in s until it levels off at the best run-proof contract $c^{early} = 4.155955$. If $10.277988 < A$, we are in Case 3 and the ICC binds when s is small. The ICC may bind either in part of the run-tolerating range of s (the first sub-case) or the whole run-tolerating range of s (the second sub-case).



5 Summary

This paper is about bank runs in the pre-deposit banking model. Understanding the pre-deposit model is essential for understanding bank runs: “Runs” in the DD post-deposit model are not necessarily runs in the full sequential analysis, in which the bank’s design of the optimal contract anticipates the withdrawal decisions of the depositors. We show that the optimal contract can depend in a non-trivial way on the probability of sunspot-driven runs.

Peck and Shell (2003) have shown that the probability of runs determine *whether or not* runs are tolerated: The optimal banking contract tolerates runs if and only if the run probability is small. Our contribution in the present paper is the analysis of *how* the optimal banking contract responds to changes in run risk. Intuitively, changes in the run risk should not be ignored and the contract should be more conservative as run risk increases. In the numerical example in PS, however, the optimal contract does not change with the run probability for as long as the run is tolerated. We show that this (seemingly counter-intuitive) result in the PS example follows from the fact that the ICC constraint binds in the PS example. The binding ICC forces the contract to be more conservative than it would have been absent the ICC.

A lesson from daily life might be instructive. It can be risky for a walker to cross the street, but if the risks are sufficiently small, crossing might be his best option. However, this does not mean that the small risk should be ignored. We would expect that his response to a small increase in risk due to heavier traffic might lead the walker to be more cautious by looking more attentively to the left and the right. This will be the case unless the pedestrian has been constrained by a traffic law that requires him to cross the street more cautiously than he would absent the binding law. The binding law is analogous to the banking ICC constraint.

We characterize the parameters (Case 2 and 3) in which runs matter. In each of these cases, the optimal contract switches to being run-proof when the run probability is sufficiently large. But *how* the optimal contract tolerates runs differs between the two cases: In Case 2, the ICC doesn’t bind since the unconstrained efficient allocation is (weakly) implementable. As a result of balancing the trade-off between the run equilibrium and non-run equilibrium in the post-deposit game, the optimal contract adjusts continuously and becomes more conservative as the run probability increases. However, in Case 3 - to which the PS numerical example belongs - the ICC binds for small run-probabilities. Hence, for Case 3, the optimal contract does not change with s until the ICC no longer binds.

Understanding *how* the optimal contract tolerates runs is important to banks and regulators. Optimal contracts and regulations could well be different based on the financial sector's level of fragility or pessimism as measured by the run-risk parameter s.[21] For instance, as the financial system becomes more fragile but not to the level of activating a completely run-proof mode, what should the optimal response of the financial sector be? Our result shows that if the problem of incomplete information is so severe that the ICC binds (i.e., Case 2 in our analysis), then no change should be made (as in the PS example). Otherwise, the optimal contract should become tighter and/or the regulation should be tightened.

Our paper also makes a contribution to the wider literature on sunspot equilibrium. These are yet other examples in which (1) not all sunspot equilibria are mere randomizations over certainty equilibria and (2) not all randomizations over certainty equilibria are sunspot equilibria. See Shell (2008) and Shell and Smith (1992).

[21]The parameter s might be small for an economy with a mature financial system that has recently experienced few or no recent financial breakdowns. For an emerging country with an immature financial sector or a country with some recent, large financial breakdowns, the parameter s might be large.

6 Appendix

6.1 Proof of Lemma 1

Proof. Inequality (3) holds if and only if

$$\frac{-(c^{1-b})/2 + (2y-c)^{1-b}(R^{1-b} - 1/2)}{(b-1)} > 0.$$

For $c \in [0, 2y]$ to satisfy the above inequality, it is necessary that $(R^{1-b} - 1/2) > 0$, which can be re-written as

$$b < 1 + \ln 2/\ln R. \tag{14}$$

When b and R satisfy condition (14), define c^{early} to be the value of c such that inequality (3) holds as an equality. We have

$$c^{early} = 2y/[(2/R^{b-1} - 1)^{1/(b-1)} + 1].$$

Inequality (3) is equivalent to

$$c \in (c^{early}, 2y]. \tag{15}$$

■

6.2 Proof of Lemma 2

Proof. The difference between the left-hand side and the right hand side of inequality (5) is a continuous function of c. If $b < 1+\ln 2/\ln R$, the difference is decreasing in c. It changes from $+\infty$ when $c = 0$ to $-\infty$ when $c = 2y$. Hence there is a unique $c = c^{wait} \in (0, 2y)$ that solves the equation

$$pv[(2y-c)R] + (1-p)v(yR) = p[v(c) + v(2y-c)]/2 + (1-p)v(c).$$

So when b and R satisfy the condition $b < 1 + \ln 2/\ln R$, inequality (5) is equivalent to

$$c \in [0, c^{wait}]. \tag{16}$$

■

6.3 Proof of Lemma 3

Proof. If condition (14) holds, c^{wait} and c^{early} are well defined. To get the condition on b and R that implies the inequality

$$c^{wait} > c^{early}, \tag{17}$$

we merely replace c in inequality (5) by c^{early}. This results in

$$\frac{2/R}{(2/R^{b-1} - 1)^{1/(b-1)} + 1} < 1. \tag{18}$$

When b and R satisfy condition (14), $(2/R^{b-1} - 1)^{1/(b-1)}$ is decreasing in b. Hence inequality (18) is equivalent to

$$b < 2. \tag{19}$$

To summarize: the set of c satisfying both conditions (3) and (5) is non-empty if and only if b and R satisfy both inequality (14) and inequality (19), which results in condition (7). ■

6.4 Proof of Proposition 1

Proof. Since we have $\widehat{W}(c) > W^{run}(c)$, $W(c; s)$ is not continuous at c^{early} if $s > 0$. We study the two regions, $[0, c^{early}]$ and $(c^{early}, c^{wait}]$, separately, and compare the maximum values of $W(c; s)$ in these two regions.

For $c \in [0, c^{early}]$, $W(c; s)$ is strictly increasing in c since $c^{early} < \widehat{c}$. Hence the maximum value of $W(c; s)$ over $[0, c^{early}]$ is achieved at c^{early}. Therefore the best run-proof contract is $c = c^{early}$.

For $c \in (c^{early}, c^{wait}]$, the maximum value of $W(c; s)$ may not be achievable because $(c^{early}, c^{wait}]$ is not closed. To fix this problem, we define the function $\widetilde{W}(c; s)$ on $[c^{early}, c^{wait}]$ by

$$\widetilde{W}(c; s) = (1 - s)\widehat{W}(c) + sW^{run}(c).$$

When $c \in (c^{early}, c^{wait}]$, $\widetilde{W}(c; s) = W(c; s)$. When $c = c^{early}$, $\widetilde{W}(c; s) < W(c; s)$. Let $\widetilde{c}(s)$ be defined by

$$\widetilde{c}(s) = \arg \max_{c \in [c^{early}, c^{wait}]} \widetilde{W}(c; s).$$

We have

$$\widetilde{c}(s) = \max\{\frac{2y}{\gamma^{1/b} + 1}, c^{early}\}, \tag{20}$$

where

$$\gamma = \frac{s(1-p)(pA + 1 - p\frac{2}{R^{b-1}}) + (p^2A + (1-p)p\frac{2}{R^{b-1}})}{s(1-p)(1-pA) + p(2-p)A}.$$

It can be shown that $\widetilde{c}(s)$ is continuous in s. Furthermore, $\widetilde{c}(s)$ is strictly decreasing in s when s is small such that $\widetilde{c}(s) > c^{early}$.[22] We also have $c^{early} = \widetilde{c}(1) < \widetilde{c}(0) = \widehat{c}$. $\widetilde{W}(\widetilde{c}(s); s)$ is continuous in s and it is also strictly decreasing in s since $\widehat{W}(c) > W^{run}(c)$. Furthermore, we have

$$\widetilde{W}(\widetilde{c}(0); 0) = \widehat{W}(\widehat{c}) > \widehat{W}(c^{early})$$

and

$$\widetilde{W}(\widetilde{c}(1); 1) = W^{run}(c^{early}) < \widehat{W}(c^{early}).$$

Hence there is a unique $s_0 \in (0, 1)$ such that

$$\widetilde{W}(\widetilde{c}(s_0); s_0) = \widehat{W}(c^{early}). \tag{21}$$

Obviously, we have $\widetilde{c}(s_0) > c^{early}$.

Hence if $s < s_0$, we have $c^*(s) = \widetilde{c}(s)$. The optimal contract $c^*(s)$ tolerates runs and it is a strictly decreasing function of s. We have $c^{early} < c^*(s) \leq \widehat{c}$

[22] It is easy to check that if $\widetilde{c}(s; A) > c^{early}$, $\widetilde{c}(s; A)$ is strictly decreasing in s because $AR^{b-1} > 1$. $AR^{b-1} > 1$ must hold in Case 2. To see why: it is trivial to establish that c^{early} must be larger than y. Hence in Case 2, we have $\widehat{c}(A) > y$, which leads to $AR^{b-1} > 1$.

(with equality if and only if $s = 0$).

If $s > s_0$, $c^*(s) = c^{early}$. The optimal contract is run-proof.

If $s = s_0$, $\widetilde{W}(\widetilde{c}(s); s) = \widehat{W}(c^{early})$. So both the run-proof contract (c^{early}) and the run-tolerating contract ($\widetilde{c}(s_0)$) are optimal at $s = s_0$. ■

6.5 Proof of Proposition 2

Proof. The proof is similar to that for Proposition 1. The only difference is that the ICC may bind. As before, we analyze separately the two regions $[0, c^{early}]$ and $(c^{early}, c^{wait}]$ separately, and compare the maximum values of $W(c; s)$ in these two regions.

For $c \in [0, c^{early}]$, it is easy to see that $W(c; s)$ is strictly increasing. Hence, as in Case 2, the best run-proof contract is $c = c^{early}$.

For $c \in (c^{early}, c^{wait}]$, the maximum value of $W(c; s)$ may not be achievable because $(c^{early}, c^{wait}]$ is not closed. To fix this problem and characterize the possibly binding ICC, we define the function $\overline{W}(c; s)$ on $[c^{early}, 2y]$ by:

$$\overline{W}(c; s) = (1 - s)\widehat{W}(c) + sW^{run}(c).$$

When $c \in (c^{early}, c^{wait}]$, we have $\overline{W}(c; s) = W(c; s)$. When $c = c^{early}$, we have $\overline{W}(c; s) < W(c; s)$. Let $\overline{c}(s)$ be defined by

$$\overline{c}(s) = \arg \max_{c \in [c^{early}, 2y]} \overline{W}(c; s).$$

We have

$$\overline{c}(s) = \frac{2y}{\eta^{1/b} + 1}, \tag{22}$$

where

$$\eta = \frac{s(1-p)(pA + 1 - p\frac{2}{R^{b-1}}) + (p^2 A + (1-p)p\frac{2}{R^{b-1}})}{s(1-p)(1-pA) + p(2-p)A}.$$

By using the same argument as that in Proposition 2, we can show that $\overline{c}(s)$ is continuous in s. Furthermore, $\overline{c}(s)$ is strictly decreasing in s when s is small such that $\overline{c}(s) > c^{early}$. We also have $c^{early} = \overline{c}(1) < \overline{c}(0) = \widehat{c}$. Note that in Case 3, we have $c^{wait} < \widehat{c}$. Hence there is a unique level of $s \in (0, 1)$, denoted by s_2, such that

$$\overline{c}(s_2) = c^{wait}. \tag{23}$$

That is, s_2 is the threshold run probability below which the ICC binds. Next, we need to check, when $s = s_2$, whether the optimal contract still tolerates runs. To do that, we define s_4 by

$$s_4 = \frac{\widehat{W}(c^{wait}) - \widehat{W}(c^{early})}{\widehat{W}(c^{wait}) - W^{run}(c^{early})}. \tag{24}$$

Obviously, we have $s_4 \in (0,1)$. There will be two sub-cases depending on whether the optimal contract still tolerates runs when when $s = s_2$.

In the first sub-case of Case 3, we have $s_4 > s_2$, that is, at the threshold run probability which makes the ICC just become non-binding, the optimal contract still tolerates runs. Now we need to determine the threshold run probability beyond which the optimal contract switches to being run-proof. That threshold level is s_3 which is defined by

$$\overline{W}(\overline{c}(s_3); s_3) = \widehat{W}(c^{early}). \tag{25}$$

Using the same argument as in Proposition 2, we know that $\overline{W}(\overline{c}(s); s)$ is continuous and strictly decreasing in s. Therefore, s_3 is unique. Since $s_4 > s_2$, we know that $s_3 > s_2$. The contract $c^*(s)$ satisfies the following: When $s < s_2$, the ICC binds and $c^*(s) = c^{wait}$ since we have

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s) > \widehat{W}(c^{early}).$$

When $s_2 \le s < s_3$, the ICC no longer binds and $c^*(s) = \overline{c}(s)$ since we have

$$W(\overline{c}(s); s) = \overline{W}(\overline{c}(s); s) > \widehat{W}(c^{early}).$$

When $s = s_3$, both $\overline{c}(s)$ and c^{early} are optimal since

$$W(\overline{c}(s); s) = \overline{W}(\overline{c}(s); s) = \widehat{W}(c^{early}).$$

When $s > s_3$, $c^*(s) = c^{early}$ since

$$W(\overline{c}(s); s) = \overline{W}(\overline{c}(s); s) < \widehat{W}(c^{early}).$$

To summarize, if $s_4 > s_2$ we have

$$c^*(s) = \begin{cases} c^{wait} \text{ if } s < s_2 \\ \overline{c}(s) \text{ if } s_2 \leq s \leq s_3 \\ c^{early} \text{ if } s_3 \leq s. \end{cases}$$

In the second sub-case of Case 3, we have $s_4 \leq s_2$, that is, at the run probability which makes the ICC just become non-binding, the optimal contract does not tolerate runs. Hence the optimal contract will switch to the best run-proof contract (c^{early}) when the ICC still binds. $c^*(s)$ satisfies the following property: When $s < s_4$, the ICC binds and $c^*(s) = c^{wait}$ since we have

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s) > \widehat{W}(c^{early}).$$

When $s = s_4$, both c^{wait} or c^{early} are optimal since we have

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s_4) = \widehat{W}(c^{early}).$$

When $s_4 < s < s_2$, we have $c^*(s) = c^{early}$. This is because the ICC binds and

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s) < \widehat{W}(c^{early}).$$

When $s_2 \leq s$, $c^*(s; A)$ is still equal to c^{early}. This is because the ICC no longer binds and

$$W(\overline{c}(s); s) = \overline{W}(\overline{c}(s); s) < \overline{W}(\overline{c}(s_2); s_2) = \overline{W}(c^{wait}; s_2) < \widehat{W}(c^{early}).$$

To summarize, if $s_4 \leq s_2$, we have

$$c^*(s) = \begin{cases} c^{wait} \text{ if } s \leq s_4 \\ c^{early} \text{ if } s \geq s_4. \end{cases}$$

We can see, in both of the two sub-cases, $c^*(s)$ switches to run-proof if the run probability is larger than the threshold. Let s_1 denote that threshold run probability and we can have

$$s_1 = \begin{cases} s_3 \text{ if } s_4 > s_2 \\ s_4 \text{ if } s_4 \leq s_2. \end{cases} \tag{26}$$

■

6.6 The Optimal Contract for b and R with "unusual" values.

6.6.1 The Post-Deposit Game

For b and R with "unusual" values (i.e., where condition (7) is not satisfied), we have either

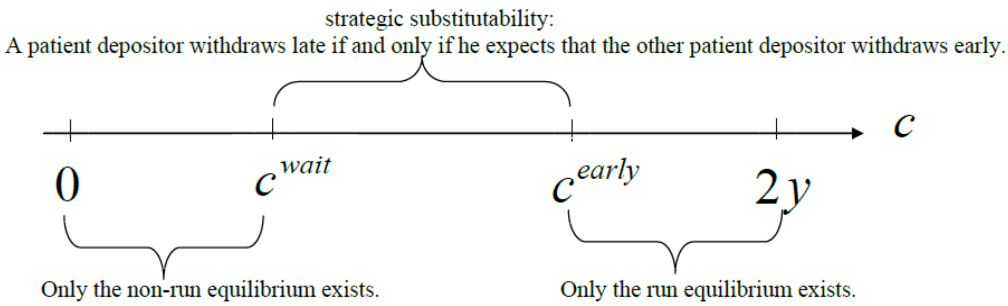
$$2 \leq b < 1 + \ln 2/\ln R \tag{27}$$

or

$$b \geq 1 + \ln 2/\ln R. \tag{28}$$

For b and R satisfying inequality (27), we have $c^{wait} \leq c^{early}$. (This can be seen directly from the proof of Lemma 3). In contrast to the "usual" parameter values, now the order of c^{early} and c^{wait} is reversed. Thus, compared to "usual" parameter values, the post-deposit game possesses different game forms. (See Figure 8.): From the pay-off matrix of the post-deposit game, we see that for $c \in [0, c^{wait}]$, we have $T_2 > T_1$ and $T_4 \geq T_3$. (L, E) is the dominant strategy for each depositor. The post-deposit game has a dominant strategy equilibrium with Pareto efficiency (i.e., non-run equilibrium). For $c \in (c^{early}, 2y]$, we have $T_2 < T_1$ and $T_4 < T_3$. (E, E) is the dominant strategy for each depositor. The post-deposit game is "prisoner's dilemma" and run is the unique equilibrium. For $c \in (c^{wait}, c^{early}]$, we have $T_2 \geq T_1$ and $T_4 < T_3$. The interval $(c^{wait}, c^{early}]$ is the region of c for which the post-deposit game is "chicken" type and the patient depositors' withdrawal decisions exhibit strategic substitutability (rather than strategic complementarity): A patient depositor withdraws late if and only if he expects that the other depositor – if patient – to withdraw early.[23] The chicken behavior might seem a bit exotic in banking, but nonetheless the equilibrium is like a partial run. Thus, in contrast to the "usual" parameter values for which the set of contracts with non-run as the unique BNE is a strict subset of the set of BIC contracts, now the two sets are the same and both of them are $[0, c^{wait}]$.

Figure 8. Equilibrium in the Post-Deposit Game



For b and R satisfying inequality (28), from the proof of Lemma 1, we can see that there is no run equilibrium for any contract $c \in [0, 2y]$ in the post-deposit game. Therefore any BIC contract is also a contract with non-run as the unique BNE.

6.6.2 The Optimal Contract for the Pre-Deposit Game

According to the Revelation Principle, to find $c^*(s)$ for the pre-deposit game, we need only focus on the BIC contracts. As we have seen, for the "unusual" parameter values, a BIC contract is also a contract with non-run as the unique BNE. Hence, bank runs are not relevant for the optimal contract c^* and c^*

[23] It is well-known that a mixed strategy equilibrium also exists in the "chicken game". We do not put much emphasis on the analysis of the *post-deposit* game here since *not* all of the *post-deposit* games are relevant for the optimal contract for the *pre-deposit* game. According to the Revelation Principle, the optimal contract must be BIC. Thus, we only need to consider contracts $c \in [0, c^{wait}]$. In other words, the bank will never choose a contract which leads to a "chicken game" in the *post-deposit* game.

maximizes the expected welfare of the depositors at the non-run equilibrium:

$$c^*(s) = \arg\max_c \widehat{W}(c) \text{ for } s \in [0,1] \tag{29}$$
$$s.t.\ c \text{ satisfies ICC (i.e. condition (5))}$$

For b and R satisfying inequality (27), we know that c satisfies (5) if and only if $c \leq c^{wait}$. Hence the solution to problem (29) is

$$c^* = \min\{\widehat{c}(A), c^{wait})\}.$$

For b and R satisfying inequality (28), c^{wait} is not well-defined. From the proof of Lemma 2, we know that the difference between the left-hand side and the right hand side of inequality (5) is no longer decreasing in c. Let us denote that difference by $Diff(c)$. $Diff(c)$ is strictly decreasing in c for $c \in [0, \overline{c^{wait}}]$ and strictly increasing in c when $c \in [\overline{c^{wait}}, 2y]$, where

$$\overline{c^{wait}} = \frac{2y}{[\frac{1-p/2}{-p(R^{1-b}-1/2)}]^{-1/b} + 1}.$$

Furthermore, $Diff(0) = +\infty$ and $Diff(2y) = +\infty$. Therefore, if $Diff(\overline{c^{wait}}) \geq 0$, (5) holds for any $c \in [0, 2y]$. If $Diff(\overline{c^{wait}}) < 0$, (5) holds for

$$c \in [0, c^{wait1}] \cup [c^{wait2}, 2y], \tag{30}$$

where $c^{wait1} < c^{wait2}$ and they are the two solutions for $Diff(c) = 0$. Hence if $Diff(\overline{c^{wait}}) \geq 0$, or $Diff(\overline{c^{wait}}) < 0$ but at the same time $\widehat{c}(A)$ satisfies condition (30), the ICC doesn't bind and the solution to the problem (29) is

$$c^* = \widehat{c}(A).$$

If $Diff(\overline{c^{wait}}) < 0$ and at the same time $\widehat{c}(A)$ doesn't satisfy condition (30), the ICC binds and c^* is equal to c^{wait1} or c^{wait2} depending on which one delivers higher expected welfare at the non-run equilibrium $\widehat{W}(c)$.

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