1 Connections between Futures Market Economy and Money Market Economy

One good per period, $\ell = 1$, two periods, $t = 1,2$.

**Futures Market:**

$$ \begin{align*} 
\max & \quad u_h(x^1_h, x^2_h) \\
\text{s.t.} & \quad p^1 x^1_h + p^2 x^2_h = p^1 \omega^1_h + p^2 \omega^2_h 
\end{align*} $$

Equilibrium is a price vector $(p^1, p^2)$ such that:

$$ \sum_h x^t_h = \sum_h \omega^t_h \text{ for } t = 1,2 $$

Define the interest factor $R$ and the interest rate $r$ in terms of the equilibrium commodity prices $(p^1, p^2)$.

**Money Market:**

$$ \begin{align*} 
\max & \quad u_h(x^1_h, x^2_h) \\
\text{s.t.} & \quad p^1 x^1_h + p^{m1} m^1_h = p^1 \omega^1_h \\
& \quad p^2 x^2_h + p^{m2} m^2_h = p^2 \omega^2_h 
\end{align*} $$

Equilibrium $(p^1, p^2, p^{m1}, p^{m2})$ such that:

$$ \sum_h x^t_h = \sum_h \omega^t_h \text{ and } \sum_h m^t_h = 0 \text{ for } t = 1,2 $$

1) Prove that in equilibrium $p^{m1} = p^{m2} = p^m \geq 0$. This is a no-arbitrage-property result.

**Solution:**

$$ \begin{align*} 
m^1_h + m^2_h &= 0 \\
m^2_h &= -m^1_h \\
p^1 (x^1_h - \omega^1_h) &= -p^{m1} m^1_h \\
p^1 (x^1_h - \omega^1_h) + p^2 (x^2_h - \omega^2_h) &= (p^{m2} - p^{m1}) m^1_h 
\end{align*} $$
Suppose $p^{m_2} > p^{m_1}$. The optimal strategy is to choose $m_h^1$ to be arbitrarily large, which contradicts the market clearing condition. Similarly, suppose $p^{m_2} < p^{m_1}$. The optimal strategy is to choose $m_h^1$ to be negative and arbitrarily large in absolute value, which also contradicts the market clearing condition. Therefore, $p^{m_1} = p^{m_2} = p^m ≥ 0$.

2) Show that if $(x_{1h}, x_{2h}), h = 1, \ldots, n$ solves the futures market problem, it also solves the money market problem.

**Solution:**

Rearranging the budget constraint in the futures market:

\[ p^1(x_{1h} - \omega_{1h}^1) + p^2(x_{2h} - \omega_{2h}^2) = 0 \]
\[ p^1(x_{1h} - \omega_{1h}^1) = -p^2(x_{2h} - \omega_{2h}^2) \]

Using $-p^m m_h^1 = p^1(x_{1h} - \omega_{1h}^1)$ and $p^m m_h^2 = -p^2(x_{2h} - \omega_{2h}^2)$ from the money market, we can get:

\[ -p^m m_h^1 = p^m m_h^2 \]
\[ p^m(m_h^1 + m_h^2) = 0 \]

For any $p^m > 0$, we know for sure that

\[ m_h^1 + m_h^2 = 0 \]

3) Show that if $(x_{1h}, x_{2h}), h = 1, \ldots, n$ solves the money market problem with $p^m > 0$, then it also solves the futures market problem.

**Solution:**

Combining the two budget constraints in the money market:

\[ p^1 x_{1h} + p^2 x_{2h} + p^m m_h^1 + p^m m_h^2 = p^1 \omega_{1h}^1 + p^2 \omega_{2h}^2 \]

Since $p^{m_1} = p^{m_2} = p^m$ and $m_h^1 + m_h^2 = 0$, the equation becomes

\[ p^2 x_{1h} + p^2 x_{2h} + p^m(m_h^1 + m_h^2) = p^1 \omega_{1h}^1 + p^2 \omega_{2h}^2 \]
\[ p^2 x_{1h} + p^2 x_{2h} = p^1 \omega_{1h}^1 + p^2 \omega_{2h}^2 \]

which is exactly the same as the constraint in the futures market.

4) **Example A:** 1 good, 2 individuals $h = 1, 2$, 2 periods $t = 1, 2$. Futures markets.

\[ u_h = \log(x_{1h}^t) + \beta \log(x_{2h}^t) \text{ for } h = 1, 2 \]

Mr 1: $\omega_1 = (100, 50) = (\omega_{11}, \omega_{12})$

Mr 2: $\omega_2 = (50, 100) = (\omega_{21}, \omega_{22})$

Set up the CP and CE for when there is (only) perfect futures markets.

Solve for the CE allocations, the CE prices, the interest factors $R$, and the CE interest rate $r$ for the following cases:
a) $\beta = 1$

b) $\beta = 5$

c) $\beta = 1/5$

Discuss the economics of your answers to parts (a), (b) and (c).

**Solution:**

Consumer Problem (CP):

\[
\max_{x_1^h, x_2^h} \log x_1^h + \beta \log x_2^h
\]

subject to \( p^1 x_1^h + p^2 x_2^h = p^1 \omega^1_h + p^2 \omega^2_h \)

A competitive equilibrium (CE) is a set of prices \((p^1, p^2) \geq 0\) such that CP is satisfied for all consumers \(h = 1, 2\) and goods market clear in both periods.

Using Lagrangian to solve the CP, the first order conditions can be summarized as

\[
\beta = \frac{p^2 \omega^2_h}{p^1 \omega^1_h}
\]

Plugging this into the budget constraint in the CP gives

\[
x_1^h = \frac{1}{p^1(1+\beta)}(p^1 \omega^1_h + p^2 \omega^2_h)
\]

\[
x_2^h = \frac{\beta}{p^2(1+\beta)}(p^1 \omega^1_h + p^2 \omega^2_h)
\]

When \(\beta = 1\),

\[
x_1^1 = \frac{1}{p^1(1+\beta)}(p^1 \omega^1_1 + p^2 \omega^2_1) = \frac{1}{p^1(1+1)}(100p^1 + 50p^2) = 50 + 25 \frac{p^2}{p^1}
\]

\[
x_2^1 = \frac{\beta}{p^2(1+\beta)}(p^1 \omega^1_1 + p^2 \omega^2_1) = \frac{1}{p^2(1+1)}(100p^1 + 50p^2) = 50 \frac{p^1}{p^2} + 25
\]

\[
x_2^2 = \frac{\beta}{p^2(1+\beta)}(p^1 \omega^1_2 + p^2 \omega^2_2) = \frac{1}{p^2(1+1)}(50p^1 + 100p^2) = 25 + 50 \frac{p^1}{p^2}
\]

\[
x_2^2 = \frac{\beta}{p^2(1+\beta)}(p^1 \omega^1_2 + p^2 \omega^2_2) = \frac{1}{p^2(1+1)}(50p^1 + 100p^2) = 25 \frac{p^1}{p^2} + 50
\]

Set \(p^1 = 1\), from market clearing condition for good 1:

\[
\sum_h x_1^h = \sum_h \omega^1_h
\]

\[
50 + 25p^2 + 25 + 50p^2 = 100 + 50
\]

\[
p^2 = 1
\]
Since \( p^1 = p^2 = 1 \), we have \( R = 1 \) and \( r = 0 \). The CE allocations are

\[ x^1_h = x^2_h = 75 \quad \text{for } h = 1, 2 \]

When \( \beta = 5 \),

\[
(x^1_1, x^2_1) = \left( \frac{50}{3} + \frac{25}{3} p^2, \frac{250}{3} p^1 + \frac{125}{3} \right)
\]

\[
(x^1_2, x^2_2) = \left( \frac{25}{3} + \frac{50}{3} p^2, \frac{125}{3} p^1 + \frac{250}{3} \right)
\]

Set \( p^1 = 1 \), from market clearing condition for good 1:

\[
\frac{50}{3} + \frac{25}{3} p^2 + \frac{25}{3} + \frac{50}{3} p^2 = 100 + 50
\]

\[ p^2 = 5 \]

We have \( R = 1/5 \) and \( r = -4/5 \). The CE allocations are

\[
(x^1_1, x^1_2) = \left( \frac{175}{3}, \frac{175}{3} \right)
\]

\[
(x^2_1, x^2_2) = \left( \frac{275}{3}, \frac{275}{3} \right)
\]

When \( \beta = 1/5 \):

\[
(x^1_1, x^2_1) = \left( \frac{250}{3} + \frac{125}{3} p^2, \frac{50}{3} p^1 + \frac{25}{3} \right)
\]

\[
(x^1_2, x^1_2) = \left( \frac{125}{3} + \frac{250}{3} p^2, \frac{25}{3} p^1 + \frac{50}{3} \right)
\]

Set \( p^1 = 1 \), from market clearing condition for good 1:

\[
\frac{250}{3} + \frac{125}{3} p^2 + \frac{125}{3} + \frac{250}{3} p^2 = 100 + 50
\]

\[ p^2 = \frac{1}{5} \]

We have \( R = 5 \) and \( r = 4 \). The CE allocations are

\[
(x^1_1, x^1_2) = \left( \frac{275}{3}, \frac{275}{3} \right)
\]

\[
(x^2_1, x^2_2) = \left( \frac{175}{3}, \frac{175}{3} \right)
\]

Notice that in all three cases, we have \( R = 1/\beta \). There is also perfect consumption smoothing because the aggregate endowment are constant across time. When \( \beta \) is large, demand for period 2 consumption goes up. Consumption in period 2 gets more expensive. Consumer with higher endowment in period 2 ends up consuming more in both periods in equilibrium.
Example B: 1 good, 2 individuals, 1 inside money. Money markets.

Replace futures markets in Example A with (inside) money markets.

Set up the CP and the CE in this problem. Show that the CE allocations in Example A are also a CE allocations in Example B. Identify in (a), (b) and (c) which individual is a borrower and which one is a lender. Discuss the economics of your answers.

Show that there is a CE allocation in Example B that is not a CE allocation in Example A. Discuss the economics.

Solution:

Consumer Problem (CP):

\[
\max_{x_1^h, x_2^h} \log x_1^h + \beta \log x_2^h
\]
subject to
\[
p_1^1 x_1^h + p_1^m m_1^h = p_1^1 \omega_1^h
\]
\[
p_2^2 x_2^h + p_2^m m_2^h = p_2^2 \omega_2^h
\]

A competitive equilibrium (CE) is a set of prices \((p_1^1, p_2^2, p_1^m, p_2^m) \geq 0\) such that CP is satisfied for all consumers \(h = 1, 2\) and goods market clear in both periods.

We know from part (a) that \(p_1^m = p_2^m\) and \(m_1^h + m_2^h = 0\). The two constraints in the CP can be combined into

\[
p_1^1 x_1^h + p_1^m m_1^h + p_2^2 x_2^h + p_2^m (-m_1^h) = p_1^1 \omega_1^h + p_2^2 \omega_2^h
\]
\[
p_1^1 x_1^h + p_2^2 x_2^h = p_1^1 \omega_1^h + p_2^2 \omega_2^h
\]
which is exactly the constraint in the futures market economy. Therefore, the two problems are equivalent. To find out whether a consumer is a borrower or lender, we simply compute \(\omega_h^1 - x_h^1\). If this number is negative, the consumer is a borrower. If this number is positive, the consumer is a lender.

Following these steps, we should get that Mr 1 is always a lender and Mr 2 is always a borrower. This is because Mr 1 has high endowment in period 1 and low endowment in period 2. Therefore he tends to lend in period 1 to trade some consumption in period 1 for consumption in period 2.

The CE that exists in Example B but not in Example A is the special case when \(p^m = 0\). In this case, the constraints in Example B become

\[
p_1^1 x_1^h = p_1^1 \omega_1^h
\]
\[
p_2^2 x_2^h = p_2^2 \omega_2^h
\]
Good prices $p^1$ and $p^2$ can be any positive number. The only choice the consumer has is to consume his endowment in each period. This constitutes a CE in the money market economy but not the futures market economy.