Economics 4905
Financial Fragility and the Macroeconomy
Fall 2017
Problem Set 2
Due Thursday, October 5, 2017

1 Connections between Futures Market Economy and Money Market Economy

One good per period, $\ell = 1$, two periods, $t = 1, 2$.

Futures Market:

$$\begin{aligned} \max \quad & u_h(x_h^1, x_h^2) \\ \text{s.t.} \quad & p^1 x_h^1 + p^2 x_h^2 = p^1 \omega_h^1 + p^2 \omega_h^2 \end{aligned}$$

Equilibrium is a price vector (p^1, p^2) such that:

$$\sum_h x_h^t = \sum_h \omega_h^t \text{ for } t = 1, 2$$

Define the interest factor R and the interest rate r in terms of the equilibrium commodity prices (p^1, p^2).

Money Market:

$$\begin{aligned} \max \quad & u_h(x_h^1, x_h^2) \\ \text{s.t.} \quad & p^1 x_h^1 + p^{m^1} m_h^1 = p^1 \omega_h^1 \\ & p^2 x_h^2 + p^{m^2} m_h^2 = p^2 \omega_h^2 \end{aligned}$$

Equilibrium $(p^1, p^2, p^{m^1}, p^{m^2})$ such that:

$$\sum_h x_h^t = \sum_h \omega_h^t \text{ and } \sum_h m_h^t = 0 \text{ for } t = 1, 2$$

1) Prove that in equilibrium $p^{m^1} = p^{m^2} = p^m \geq 0$. This is a no-arbitrage-property result.

Solution:

$$m_h^1 + m_h^2 = 0$$

$$m_h^2 = -m_h^1$$

$$p^1(x_h^1 - \omega_h^1) = -p^{m^1} m_h^1$$

$$p^1(x_h^1 - \omega_h^1) + p^2(x_h^2 - \omega_h^2) = (p^{m^2} - p^{m^1}) m_h^1$$

Suppose $p^{m^2} > p^{m^1}$. The optimal strategy is to choose m_h^1 to be arbitrarily large, which contradicts the market clearing condition. Similarly, suppose $p^{m^2} < p^{m^1}$. The optimal strategy is to choose m_h^1 to be negative and arbitrarily large in absolute value, which also contradicts the market clearing condition. Therefore, $p^{m^1} = p^{m^2} = p^m \geq 0$.

2) Show that if (x_h^1, x_h^2), $h = 1, \ldots, n$ solves the futures market problem, it also solves the money market problem.

Solution:

Rearranging the budget constraint in the futures market:

$$p^1(x_h^1 - \omega_h^1) + p^2(x_h^2 - \omega_h^2) = 0$$
$$p^1(x_h^1 - \omega_h^1) = -p^2(x_h^2 - \omega_h^2)$$

Using $-p^m m_h^1 = p^1(x_h^1 - \omega_h^1)$ and $p^m m_h^2 = -p^2(x_h^2 - \omega_h^2)$ from the money market, we can get:

$$-p^m m_h^1 = p^m m_h^2$$
$$p^m(m_h^1 + m_h^2) = 0$$

For any $p^m > 0$, we know for sure that

$$m_h^1 + m_h^2 = 0$$

3) Show that if $(x_h^1, x_h^2), h = 1, \ldots, n$ solves the money market problem with $p^m > 0$, then it also solves the futures market problem.

Solution:

Combining the two budget constraints in the money market:

$$p^1 x_h^1 + p^2 x_h^2 + p^{m^1} m_h^1 + p^{m^2} m_h^2 = p^1 \omega_h^1 + p^2 \omega_h^2$$

Since $p^{m^1} = p^{m^2} = p^m$ and $m_h^1 + m_h^2 = 0$, the equation becomes

$$p^2 x_h^1 + p^2 x_h^2 + p^m(m_h^1 + m_h^2) = p^1 \omega_h^1 + p^2 \omega_h^2$$
$$p^2 x_h^1 + p^2 x_h^2 = p^1 \omega_h^1 + p^2 \omega_h^2$$

which is exactly the same as the constraint in the futures market.

4) **Example A:** 1 good, 2 individuals $h = 1, 2$, 2 periods $t = 1, 2$. Futures markets.

$$u_h = \log(x_h^1) + \beta \log(x_h^2) \text{ for } h = 1, 2$$
$$\text{Mr 1:} \quad \omega_1 = (100, 50) = (\omega_{11}, \omega_{12})$$
$$\text{Mr 2:} \quad \omega_2 = (50, 100) = (\omega_{21}, \omega_{22})$$

Set up the CP and CE for when there is (only) perfect futures markets.

Solve for the CE allocations, the CE prices, the interest factors R, and the CE interest rate r for the following cases:

a) $\beta = 1$
b) $\beta = 5$
c) $\beta = 1/5$

Discuss the economics of your answers to parts (a), (b) and (c).

Solution:

Consumer Problem (CP):

$$\max_{x_h^1, x_h^2} \quad \log x_h^1 + \beta \log x_h^2$$
$$\text{subject to} \quad p^1 x_h^1 + p^2 x_h^2 = p^1 \omega_h^1 + p^2 \omega_h^2$$

A competitive equilibrium (CE) is a set of prices $(p^1, p^2) \geq 0$ such that CP is satisfied for all consumers $h = 1, 2$ and goods market clear in both periods.

Using Lagrangian to solve the CP, the first order conditions can be summarized as

$$\beta = \frac{p^2 \omega_h^2}{p^1 \omega_h^1}$$

Plugging this into the budget constraint in the CP gives

$$x_h^1 = \frac{1}{p^1(1+\beta)}(p^1 \omega_h^1 + p^2 \omega_h^2)$$
$$x_h^2 = \frac{\beta}{p^2(1+\beta)}(p^1 \omega_h^1 + p^2 \omega_h^2)$$

When $\beta = 1$,

$$x_1^1 = \frac{1}{p^1(1+\beta)}(p^1 \omega_1^1 + p^2 \omega_1^2) = \frac{1}{p^1(1+1)}(100p^1 + 50p^2) = 50 + 25\frac{p^2}{p^1}$$
$$x_1^2 = \frac{\beta}{p^2(1+\beta)}(p^1 \omega_1^1 + p^2 \omega_1^2) = \frac{1}{p^2(1+1)}(100p^1 + 50p^2) = 50\frac{p^1}{p^2} + 25$$
$$x_2^1 = \frac{1}{p^1(1+\beta)}(p^1 \omega_2^1 + p^2 \omega_2^2) = \frac{1}{p^1(1+1)}(50p^1 + 100p^2) = 25 + 50\frac{p^2}{p^1}$$
$$x_2^2 = \frac{\beta}{p^2(1+\beta)}(p^1 \omega_2^1 + p^2 \omega_2^2) = \frac{1}{p^2(1+1)}(50p^1 + 100p^2) = 25\frac{p^1}{p^2} + 50$$

Set $p^1 = 1$, from market clearing condition for good 1:

$$\sum_h x_h^1 = \sum_h \omega_h^1$$

$$50 + 25p^2 + 25 + 50p^2 = 100 + 50$$
$$p^2 = 1$$

Since $p^1 = p^2 = 1$, we have $R = 1$ and $r = 0$. The CE allocations are

$$x_h^1 = x_h^2 = 75 \quad \text{for } h = 1, 2$$

When $\beta = 5$,

$$(x_1^1, x_1^2) = (\frac{50}{3} + \frac{25}{3}\frac{p^2}{p^1}, \frac{250}{3}\frac{p^1}{p^2} + \frac{125}{3})$$

$$(x_1^1, x_1^2) = (\frac{25}{3} + \frac{50}{3}\frac{p^2}{p^1}, \frac{125}{3}\frac{p^1}{p^2} + \frac{250}{3})$$

Set $p^1 = 1$, from market clearing condition for good 1:

$$\frac{50}{3} + \frac{25}{3}p^2 + \frac{25}{3} + \frac{50}{3}p^2 = 100 + 50$$

$$p^2 = 5$$

We have $R = 1/5$ and $r = -4/5$. The CE allocations are

$$(x_1^1, x_1^2) = (\frac{175}{3}, \frac{175}{3})$$

$$(x_2^1, x_2^2) = (\frac{275}{3}, \frac{275}{3})$$

When $\beta = 1/5$:

$$(x_1^1, x_1^2) = (\frac{250}{3} + \frac{125}{3}\frac{p^2}{p^1}, \frac{50}{3}\frac{p^1}{p^2} + \frac{25}{3})$$

$$(x_1^1, x_1^2) = (\frac{125}{3} + \frac{250}{3}\frac{p^2}{p^1}, \frac{25}{3}\frac{p^1}{p^2} + \frac{50}{3})$$

Set $p^1 = 1$, from market clearing condition for good 1:

$$\frac{250}{3} + \frac{125}{3}p^2 + \frac{125}{3} + \frac{250}{3}p^2 = 100 + 50$$

$$p^2 = \frac{1}{5}$$

We have $R = 5$ and $r = 4$. The CE allocations are

$$(x_1^1, x_1^2) = (\frac{275}{3}, \frac{275}{3})$$

$$(x_2^1, x_2^2) = (\frac{175}{3}, \frac{175}{3})$$

Notice that in all three cases, we have $R = 1/\beta$. There is also perfect consumption smoothing because the aggregate endowment are constant across time. When β is large, demand for period 2 consumption goes up. Consumption in period 2 gets more expensive. Consumer with higher endowment in period 2 ends up consuming more in both periods in equilibrium.

Example B: 1 good, 2 individuals, 1 inside money. Money markets.

Replace futures markets in Example A with (inside) money markets.

Set up the CP and the CE in this problem. Show that the CE allocations in Example A are also a CE allocations in Example B. Identify in (a), (b) and (c) which individual is a borrower and which one is a lender. Discuss the economics of your answers.

Show that there is a CE allocation in Example B that is not a CE allocation in Example A. Discuss the economics.

Solution:

Consumer Problem (CP):

$$\begin{aligned} \max_{x_h^1, x_h^2} \quad & \log x_h^1 + \beta \log x_h^2 \\ \text{subject to} \quad & p^1 x_h^1 + p^{m^1} m_h^1 = p^1 \omega_h^1 \\ & p^2 x_h^2 + p^{m^2} m_h^2 = p^2 \omega_h^2 \end{aligned}$$

A competitive equilibrium (CE) is a set of prices $(p^1, p^2, p^{m^1}, p^{m^2}) \geq 0$ such that CP is satisfied for all consumers $h = 1, 2$ and goods market clear in both periods.

We know from part (a) that $p^{m^1} = p^{m^2}$ and $m_h^1 + m_h^2 = 0$. The two constraints in the CP can be combined into

$$p^1 x_h^1 + p^m m_h^1 + p^2 x_h^2 + p^m(-m_h^1) = p^1 \omega_h^1 + p^2 \omega_h^2$$

$$p^1 x_h^1 + p^2 x_h^2 = p^1 \omega_h^1 + p^2 \omega_h^2$$

which is exactly the constraint in the futures market economy. Therefore, the two problems are equivalent. To find out whether a consumer is a borrower or lender, we simply compute $\omega_h^1 - x_h^1$. If this number is negative, the consumer is a borrower. If this number is positive, the consumer is a lender.

Following these steps, we should get that Mr 1 is always a lender and Mr 2 is always a borrower. This is because Mr 1 has high endowment in period 1 and low endowment in period 2. Therefore he tends to lend in period 1 to trade some consumption in period 1 for consumption in period 2.

The CE that exists in Example B but not in Example A is the special case when $p^m = 0$. In this case, the constraints in Example B become

$$p^1 x_h^1 = p^1 \omega_h^1$$

$$p^2 x_h^2 = p^2 \omega_h^2$$

Good prices p^1 and p^2 can be any positve number. The only choice the consumer has is to consume his endowment in each period. This constitutes a CE in the money market economy but not the futures market economy.