Economics 4905
Financial Fragility and the Macroeconomy
Fall 2017
Problem Set 4
Due before class on Wednesday, November 1, 2017

1. The Overlapping Generations Model

The model is set up as follow:

- 2 period lives
- 1 commodity per period, $\ell = 1$
- Stationary environment
- 1 person per generation

The utility functions are given as:

$$u_0(x_0^1) = \beta \log x_0^1$$
$$u_t(x_t^t, x_t^{t+1}) = \alpha \log x_t^t + \beta \log x_t^{t+1} \text{ for } t = 1, 2, \ldots$$

The endowments are 1 unit for each period each person is alive:

$$\omega_0^1 = \omega_t^t = \omega_t^{t+1} = 1 \text{ for } t = 1, 2, \ldots$$

Define the excess demands:

$$z^t = x_t^t - \omega_t^t$$
$$z^{t+1} = \omega_t^{t+1} - x_t^{t+1}$$

Case 1: $\alpha = 1, \beta = 10, m_0^1 = 1, m_s^t = 0$ otherwise
Case 2: $\alpha = 5, \beta = 1, m_0^1 = 1, m_s^t = 0$ otherwise

For both of the above cases, solve for the following:

a) The equilibrium demand (x_t^t, x_t^{t+1})

b) The offer curve (OC)

c) The steady states

d) The set of equilibrium money prices, $\mathcal{P}^m$

e) The full dynamic analysis, including the stability of steady states

2. Solutions

2.1 Case 1

a) The consumer utility maximization problem is:

$$\max_{x_t^t, x_t^{t+1}} \quad \alpha \log x_t^t + \beta \log x_t^{t+1}$$
$$\text{subject to} \quad p^t x_t^t + p^{t+1} x_t^{t+1} = p^t \omega_t^t + p^{t+1} \omega_t^{t+1}$$

Setting up the Lagrangian:

$$\mathcal{L} = \alpha \log x_t^t + \beta \log x_t^{t+1} + \lambda(p^t \omega_t^t + p^{t+1} \omega_t^{t+1} - p^t x_t^t - p^{t+1} x_t^{t+1})$$

Taking derivatives with respect to x_t^t, x_t^{t+1} and λ yields:

$$\frac{\alpha}{x_t^t} = \lambda p^t \tag{1}$$
$$\frac{\beta}{x_t^{t+1}} = \lambda p^{t+1} \tag{2}$$
$$p^t x_t^t + p^{t+1} x_t^{t+1} = p^t \omega_t^t + p^{t+1} \omega_t^{t+1} \tag{3}$$

Dividing (2) by (1) gives:

$$\frac{p^{t+1}}{p^t} = \frac{\beta}{\alpha} \frac{x_t^t}{x_t^{t+1}} \tag{4}$$

Using (3) and (4) to solve for the demands:

$$x_t^t = \frac{\alpha}{\alpha + \beta} \frac{p^t \omega_t^t + p^{t+1} \omega_t^{t+1}}{p^t} \tag{5}$$
$$x_t^{t+1} = \frac{\beta}{\alpha + \beta} \frac{p^t \omega_t^t + p^{t+1} \omega_t^{t+1}}{p^{t+1}} \tag{6}$$

Plugging in the values of $\alpha, \beta, \omega_t^t$ and ω_t^{t+1} gives

$$x_t^t = \frac{1}{11}\left(1 + \frac{p^{t+1}}{p^t}\right)$$
$$x_t^{t+1} = \frac{10}{11}\left(\frac{p^t}{p^{t+1}} + 1\right)$$

b) Dividing both sides of the budget constraint of the consumer problem by p^t gives:

$$x_t^t + \frac{p^{t+1}}{p^t} x_t^{t+1} = \omega_t^t + \frac{p^{t+1}}{p^t} \omega_t^{t+1} \tag{7}$$

Plugging (4) into (7) gives:

$$x_t^t + \frac{\beta}{\alpha} \frac{x_t^t}{x_t^{t+1}} x_t^{t+1} = \omega_t^{t+1} + \frac{\beta}{\alpha} \frac{x_t^t}{x_t^{t+1}} \omega_t^{t+1} \tag{8}$$

Plugging in the equations from the definition of excess demand:

$$(z^t + \omega_t^t) + \frac{\beta}{\alpha}\frac{(z^t + \omega_t^t)}{(\omega_t^{t+1} - z^{t+1})}(\omega_t^{t+1} - z^{t+1}) = \omega_t^{t+1} + \frac{\beta}{\alpha}\frac{(z^t + \omega_t^t)}{(\omega_t^{t+1} - z^{t+1})}\omega_t^{t+1}$$

Solving for z^{t+1} yields:

$$z^{t+1} = \frac{\alpha\omega_t^{t+1} z^t}{\beta\omega_t^t - (\alpha+\beta)z^t} \tag{9}$$

Plugging the values of endowments, α and β the offer curve for case 1:

$$z^{t+1} = \frac{z^t}{10 - 11z^t} \tag{10}$$

c) Setting $\bar{z} = z^t = z^{t+1}$ in equation (10):

$$\bar{z} = \frac{\bar{z}}{10 - 11\bar{z}}$$

Solving for $\bar{z}$ gives:

$$\bar{z} = \frac{9}{11} \text{ or } 0$$

These are the two steady states.

d) Since $m_0^1 = 1$, the set of equilibrium money prices must be

$$\mathcal{P}^m = \left[0, \frac{9}{11}\right]$$

e) If $0 < P^m < \frac{9}{11}$, then z^t is declining, and the bubble fades away through inflation. $z = 0$ is a stable steady state, in which money is worthless $P^m = 0$. $z = \frac{9}{11}$ is an unstable steady state. If $z > \frac{9}{11}$, hyperinflation ensues and the bubble bursts in finite time. We may note that this is the Samuelson case.

2.2 Case 2

a) Plugging the new parameter values into equations (5) and (6) gives:

$$x_t^t = \frac{5}{6}\left(1 + \frac{p^{t+1}}{p^t}\right)$$
$$x_t^{t+1} = \frac{1}{6}\left(\frac{p^t}{p^{t+1}} + 1\right)$$

b) Plugging the new parameter values into equation (9) gives:

$$z^{t+1} = \frac{5z^t}{1 - 6z^t} \tag{11}$$

c) Setting $\bar{z} = z^t = z^{t+1}$ in equation (11):

$$\bar{z} = \frac{5\bar{z}}{1 - 6\bar{z}}$$

Solving for $\bar{z}$ gives:

$$\bar{z} = -\frac{2}{3} \text{ or } 0$$

Since one of the solutions is negative, there is only one steady state $z = 0$.

d) The set of equilibrium money prices is

$$\mathcal{P}^m = \{0\}$$

e) The non-monetary steady state where $P^m = 0$ is unstable, unique and Pareto optimal. Trajectories originating away from it will be deflationary. This is the Ricardo case.