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Indivisibilities, lotteries and sunspot equilibria

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Overview

- Why are indivisible goods important?
- Pure exchange economies
  - One indivisible good
  - 2 states of the world
  - N agents
  - Heterogeneous endowments
  - Continuous sunspots
  - 2 goods
- Application: employment lotteries

Pure exchange – Setup I

- Consumption set $R^J_+ \times \{0,1\}^{K-J}$
- Measure space (I, Ω, α) of consumers
- vNM utility function $U^i: \bar{X} \to R$
- Endowments $e^i \in R^K_+$
- $\sum_k p_k = 1$
- Walrasian equilibrium

$$max_{x \in X}\ U^i(x)$$
$$px\ \leq pe^i$$
$$\int x^i \alpha(di) \leq \int e^i \alpha(di)$$

Pure exchange – One indivisible good

- X = {0, 1}, I = 2, $e^1 = e^2 = \frac{1}{2}$
- WE: $x^1 = x^2 = 0$, $U^1 = U^2 = 0$
  - Not Pareto optimal: (1, 0) or (0, 1) strictly dominates
  - First Welfare Theorem doesn't hold
- Randomization:

$$\left(x^1, x^2\right) = \begin{cases} (1,0)\ with\ prob. = \pi_1 \\ (0,1) with\ prob. = \pi_2 \end{cases}$$

- $EU^1 = \pi_1 U^1(1) > 0, EU^2 = \pi_2 U^2(1) > 0$
  - LE Pareto dominates WE

Pure exchange – Setup II

- Probability space (S, Σ, π)
- Commodity space: $x: S \rightarrow R^J$
- $\int \sum_k \widetilde{p_k}(\mathrm{s})\pi(ds) = 1$
- Sunspot equilibrium

$$\text{maximize } EU^i = \int_S U^i[x^i(s)]\pi(ds)$$

$$subject\ to \int_S \tilde{p}(s)x^i(s)\pi(ds) \leq \int_S \tilde{p}(s)e^i\pi(ds) \equiv W_i$$

$$\int x^i(s)\alpha(di) \leq \int e^i\alpha(di)$$

Pure exchange – two states

- X = {0, 1}, I = 2, $e^1 = e^2 = \frac{1}{2}$
- S = $\{s_1, s_2\}$
- $maximize\ EU^i = \pi_1 U^i\left[x^i(s_1)\right] + \pi_2 U^i\left[x^i(s_2)\right]$

$$subject\ to\ p(s_1)x^i(s_1) + p(s_2)x^i(s_2) \leq \frac{1}{2}$$

Proposition 1. *In the economy with* $X = \{0, 1\}$, $N = 2$, $e^1 = e^2 = 1/2$, *and* $S = \{s_1, s_2\}$, *we have:* (a) If $\pi_1 \neq \pi_2$ *then* SE *do not exist.* (b) *If* $\pi_1 = \pi_2$ *then there are exactly two* SE, *with prices* $p(s_1) = p(s_2) = 1/2$ *and one of the following two allocations*

$$[x^1(s_1), x^1(s_2)] = (1, 0) \quad and \quad [x^2(s_1), x^2(s_2)] = (0, 1)$$

$$[x^1(s_1), x^1(s_2)] = (0, 1) \quad and \quad [x^2(s_1), x^2(s_2)] = (1, 0), \qquad (2.4)$$

which are simply relabelings of the same outcome. (c) *All* SE *are nondegenerate, and in particular, the* WE *allocation cannot be supported as a* SE. (d) *The* SE *are Pareto optimal with respect to* Z *and dominate the* WE *allocation.*

Pure exchange – two states

- $[x(s_1), x(s_2)]$ = (1, 0) or $[x(s_1), x(s_2)]$ = (0, 1) is affordable $\forall p$
- Feasibility constraint: $\sum_i x^i(s) \leq 1 \rightarrow$ (c) holds
- Let $[x^1(s_1), x^1(s_2) = (1, 0)]$ and $[x^2(s_1), x^2(s_2) = (0, 1)]$ be a SE.
  $\pi_1 U^1(1) + (1 - \pi_1)U^1(0) \geq \pi_1 U^1(0) + (1 - \pi_1)U^1(1)$ or $\pi_1 \geq \frac{1}{2}$

  $\pi_1 U^2(0) + (1 - \pi_1)U^2(1) \geq \pi_1 U^2(1) + (1 - \pi_1)U^2(0)$ or $\pi_1 \leq \frac{1}{2}$

  Hence (a) holds
- Allocations solve the maximization problem if and only if $p(s_1) = p(s_2) \rightarrow$ (b) holds

Pure exchange – N agents

Proposition 2. *The economy with* $X = \{0, 1\}$ *and* N *consumers with* $e^i = e < 1$ *for all i has a unique* WE *with* $x^i = 0$ *for all i. Let* $n = \text{int}(Ne)$ *and let* n^*/N^* reduce n/N *to its lowest terms. Then this economy has a* SE *with* N^* *states,* $\pi(s_j) = p(s_j) = 1/N^*$, *and an allocation where* $x^i(s_j) = 1$ *in* n^* *states and* $x^i(s_j) = 0$ *in* $N^* - n^*$ *states for all i. If* $n^* \geqq 1$, *then the* SE *is optimal with respect to* Z *and dominates the* WE, *and the* WE *does not reappear as a* SE.

- Each agent can afford at most n* units
- U is strictly concave, hence a SE has the following properties:
  - $x^i = 1$ in n* states and $x^i = 0$ in N* - n* states for all i
  - $x^i(s) = 1$ for $\frac{n^*}{N^*}$ agents and $x^i(s) = 0$ for others for all s (market clearing)

Pure exchange – N agents

	s_1	s_2	s_3
1	1	0	0
2	0	1	0
3	0	0	1

a

	s_1	s_2	s_3
1	1	0	1
2	1	1	0
3	0	1	1

b

Fig. 1a. $N^* = 3$ and $n^* = 1$. **b** $N^* = 3$ and $n^* = 2$.

- $x^i(s_j) = a_{ij}$ for $i = 1, \ldots, N^*$, for others reproduce the allocation

Pure exchange – heterogeneous endowments

- X = {0, 1}, N = 2, $0 < e^1 < e^2$, $e^1 + e^2 = 1$, S = $\{s_1, s_2\}$
- Either A or B is affordable
- $\frac{p(s_2)}{p(s_1)} = \frac{e^2}{e^1} \rightarrow$
- $[x^1(s_1), x^1(s_2)]$ = (1, 0) and

  $[x^2(s_1), x^2(s_2)]$ = (0, 1)

  if and only if $\pi_1 \leq \frac{1}{2}$
- Different probabilities imply

different SE and EU

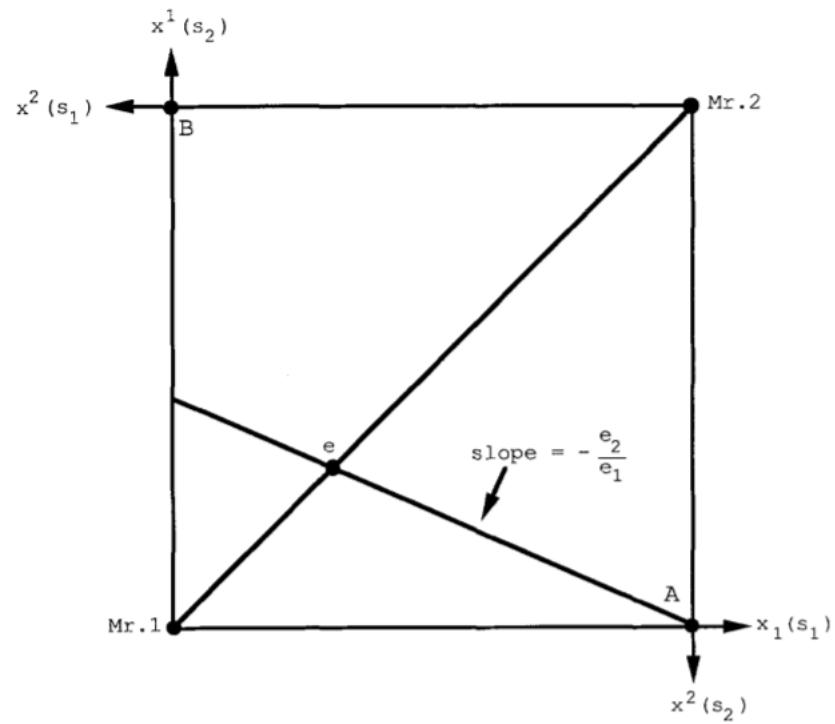


Fig. 2. The Edgeworth Box.

Pure exchange – heterogeneous endowments

- if $\pi_1 = \pi_2 = \frac{1}{2}$ then $EU^1 = EU^2 = \frac{1}{2}U^i(1)$ regardless of e_1 and e_2
- if N = 2, SE is in the core for any π_1
- in case of replication, agents can form coalitions to influence $\pi(s)$
- Continuum of agents, types t = 1,…,T, fraction α_t, $\sum_t \alpha_t e^t = 1$
- Any coalition of type t agents can have

$$x^i = \begin{cases} 1 \; with \; prob. = e^t \\ 0 \; with \; prob. = 1 - e^t \end{cases}$$

- Any core allocation must have prob.$(x^i = 1) \geq e^t$, but feasibility ensures prob.$(x^i = 1) > e^t$ is impossible

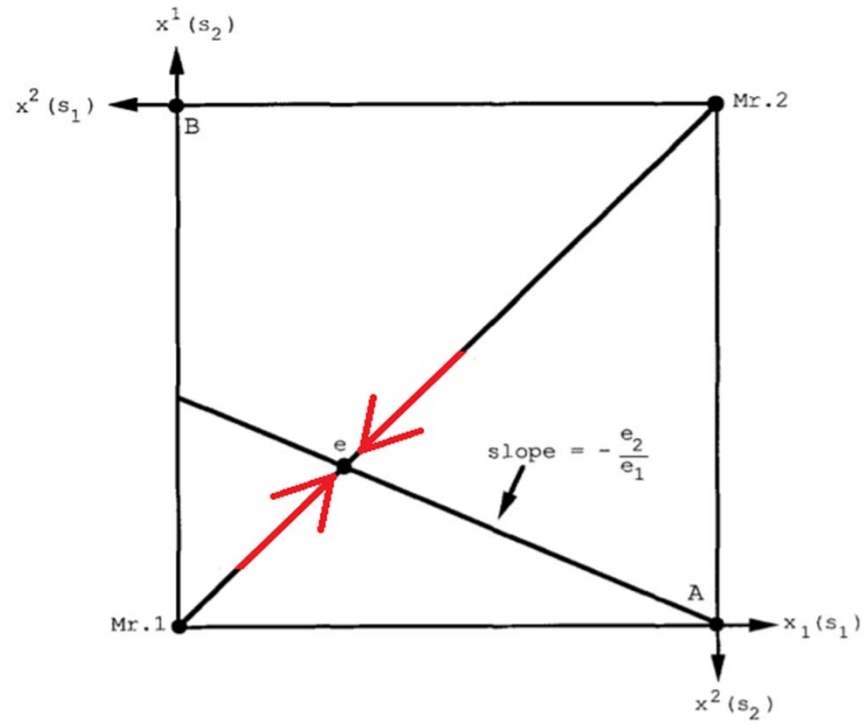


Fig. 2. The Edgeworth Box.

Pure exchange – Continuous sunspots

- $e^i \in [0,1], \int e^i \alpha(di) = 1$, s has density φ(s)
- $$maximize \int_S U^i[x^i(s)]\varphi(s)ds$$

  $$\text{subject to } \int_S p(s)x^i(s)ds \leq e^i$$
- In equilibrium $p(s) = \varphi(s)$ - otherwise arbitrage is possible
- $x^i(s) = \begin{cases} 1 \; if \; s \in S_i \\ 0 \; else \end{cases}$ where $prob(S_i) = e^i$ for all i

Proposition 3. *Suppose* $\int e^i \alpha(di) = 1$. *Let s be continuous with density* $\varphi(s)$. *Then, up to a relabeling, there is a unique* SE, *and it has the following properties:* (a) $p(s) = \varphi(s)$ *for all s, and* (b) $\text{prob}(x^i) = e^i$ *for almost all i. The corresponding allocation is the unique core allocation that necessarily survives replication. With a finite distribution for s, there can be other* SE, *with allocations such that* $\text{prob}(x^i) \neq e^i$.

Pure exchange – 2 goods

- X = $R_+ \times \{0,1\}$, continuum of agents, $e^i = (\frac{1}{2}, \frac{1}{2})$, $U^i(x_1, x_2) = u(x_1) + u(x_2)$
- WE: fraction $\frac{1}{2}$ gets $(x^i{}_1, x^i{}_2) = (1,0)$ and fraction $\frac{1}{2}$ gets $(x^i{}_1, x^i{}_2) = (0,1)$
- $U^i = u(1)$ and WE is **Pareto optimal**
- However $(x^i{}_1, x^i{}_2) = \begin{cases} \left(\frac{1}{2}, 0\right) with\ \pi_1 = \frac{1}{2} \\ \left(\frac{1}{2}, 1\right) with\ \pi_2 = \frac{1}{2} \end{cases}$ **dominates** WE by strict concavity of u
- SE: $p_1(s_j) = p_2(s_j)$ for all s_j

Employment lotteries – Setup

- X = $R_+ \times \{0,1\}$
- Continuum of agents with unit mass
- U(x_1, x_2) = U(c, l), $e = (0, 1)$
- Production set Y = $\{y \in R^2_+ : y_1 \leq f(y_2)\}$, (y_1, y_2) = (q, h)
- Profit is distributed equally as dividends

$$maximize\ U(c, l)$$

$$\text{subject to } pc + wl \leq w + \Pi$$

$$\text{maximize } \Pi = pq - wh$$

$$\int l^i di + h = 1\ ,\ \int c^i di = q$$

- WE: $(c^i, l^i) = \begin{cases} (\Pi, 1)\ for\ fraction\ 1 - \mu\ (unemployed) \\ (w + \Pi, 0)\ for\ fraction\ \mu\ (employed) \end{cases}$
- WE is unique and Pareto optimal, $U(\Pi, 1) = U(w + \Pi, 0)$

Employment lotteries – Social planner

- Randomization: $(c^i, l^i) = \begin{cases} (c_0, 0)\ with\ \pi_1 = \mu \\ (c_1, 1)\ with\ \pi_2 = 1 - \mu \end{cases}$

$$\text{Maximize } V = \mu U(c_0, 0) + (1 - \mu) U(c_1, 1)$$

$$\text{Subject to } \mu c_0 + (1 - \mu) c_1 \leq f(\mu),\ \mu \leq 1$$

- FOC:

$$U(c_0, 0) - U(c_1, 1) + \lambda[f'(\mu) - c_0 + c_1] = \beta$$

$$\mu U_1(c_0, 0) - \mu\lambda = 0$$

$$(1 - \mu) U_1(c_1, 1) - (1 - \mu)\lambda = 0$$

$$f(\mu) - \mu c_0 - (1 - \mu) c_1 = 0$$

$$\mu \leq 1, \qquad \beta(1 - \mu) = 0$$

- If $\mu^* < 1$ welfare is improved

Employment lotteries - Decentralization

$$\text{Maximize } EU = \int_S U[c(s), l(s)]\pi(ds)$$

$$\text{Subject to } \int_S [p(s)c(s) + w(s)l(s)]\pi(ds) \leq \Pi + \int_S w(s)\pi(ds)$$

$$\text{Maximize } \Pi = \int_S [p(s)q(s) - w(s)h(s)]\pi(ds)$$

$$\int l^i(s)di + h(s) = 1, \int c^i(s)di = q(s)$$

Proposition 4. *In the indivisible labor economy, the planner's randomized allocation can be supported as nondegenerate* SE.

Employment lotteries - Decentralization

- $s \sim U[0,1], p(s) = 1, w(s) = f'(\mu^*)$
- Maximize profit $\rightarrow f'[h(s)] = w(s) \rightarrow h(s) = \mu^*$
- Denote $S_0 = \{s \in S: l(s) = 0\}$ and $S_1 = \{s \in S: l(s) = 1\}$, $prob.(S_0) = \hat{\mu}$

Maximize $EU = \hat{\mu} \int_{S_0} U[c(s), 0]ds + (1 - \hat{\mu}) \int_{S_1} U[c(s), 1]ds$

Subject to $\int_{S_0} c(s)ds + \int_{S_1} c(s)ds + (1 - \hat{\mu})f'(\mu^*) = \Pi + f'(\mu^*)$

- By strict concavity $c(s) = \begin{cases} \hat{c}_0 \; for \; s \in S_0 \\ \hat{c}_1 \; for \; s \in S_1 \end{cases}$

Employment lotteries - Decentralization

$$\text{Maximize } EU = \hat{\mu}U(\hat{c}_0, 0) + (1-\hat{\mu})U_1(\hat{c}_1, 1)$$

$$\text{Subject to } \hat{\mu}\hat{c}_0 + (1-\hat{\mu})\,\hat{c}_1 - \hat{\mu}f'(\mu^*) = f(\mu^*) - \mu^* f'(\mu^*)$$

- FOC:

$$U(\hat{c}_0, 0) - U(\hat{c}_1, 1) + \lambda[f'(\mu^*) - \hat{c}_0 + \hat{c}_1] = \beta$$

$$\hat{\mu}U_1(\hat{c}_0, 0) - \hat{\mu}\lambda = 0$$

$$(1-\hat{\mu})U_1(\hat{c}_1, 1) - (1-\hat{\mu})\lambda = 0$$

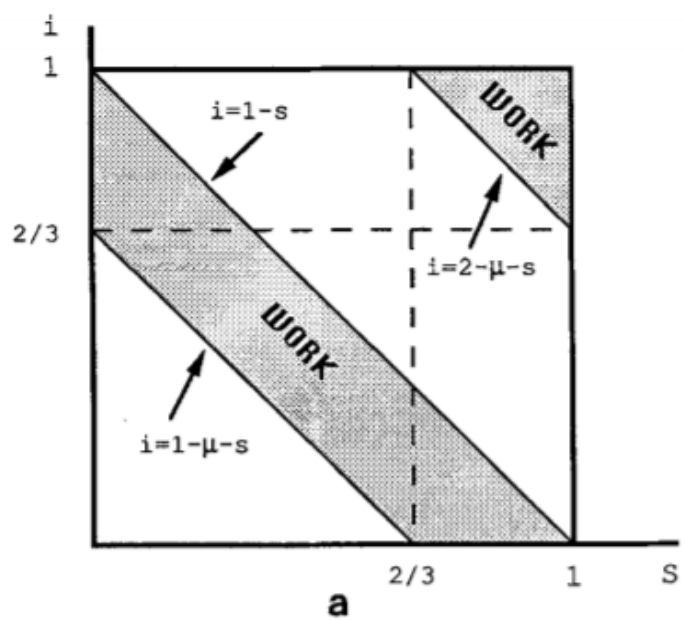
$$f(\mu^*) + (\hat{\mu} - \mu^*)f'(\mu^*) - \hat{\mu}\hat{c}_0 - (1-\hat{\mu})\hat{c}_1 = 0$$

$$\hat{\mu} \leq 1, \qquad \beta(1-\hat{\mu}) = 0$$

- Social planner allocation is a solution

Employment lotteries - Decentralization

- If $s \leq 1 - \mu^*$ then $l^i(s) = \begin{cases} 0 \; if \; i \in [1 - \mu^* - s, 1 - s] \\ 1 \; otherwise \end{cases}$
- If $s > 1 - \mu^*$ then $l^i(s) = \begin{cases} 1 \; if \; i \in [1 - s, 2 - \mu^* - s] \\ 0 \; otherwise \end{cases}$
- Then $\int l^i(s)ds = 1 - \mu^*$ and $\int l^i(s)di = 1 - \mu^*$
- SE is Pareto optimal and dominates WE
- $\mu^* = \frac{1}{3}$



Employment lotteries - Decentralization

- If $s \leq 1 - \mu^*$ then $l^i(s) = \begin{cases} 0 \; if \; i \in [1 - \mu^* - s, 1 - s] \\ 1 \; otherwise \end{cases}$
- If $s > 1 - \mu^*$ then $l^i(s) = \begin{cases} 1 \; if \; i \in [1 - s, 2 - \mu^* - s] \\ 0 \; otherwise \end{cases}$
- Then $\int l^i(s)ds = 1 - \mu^*$ and $\int l^i(s)di = 1 - \mu^*$
- SE is Pareto optimal and dominates WE
- $\mu^* = \frac{1}{3}$
- If $\mu^* = \frac{n^*}{N^*}$, then there is a SE

with N^*equiprobable states, where each individual works in n^*