Appendix A: Proofs

Proof of Lemma 1

Proof. Inequality (3) holds if and only if

$$\frac{-(c^{1-b})/2 + (2y - c)^{1-b}(R^{1-b} - 1/2)}{(b-1)} > 0.$$

For $c \in [0, 2y]$ to satisfy the above inequality, it is necessary that $(R^{1-b} - 1/2) > 0$, which can be re-written as

$$b < 1 + \ln 2 / \ln R. \tag{14}$$

When b and R satisfy condition (14), define c^{early} to be the value of c such that inequality (3) holds as an equality. We have

$$c^{early} = 2y / [(2/R^{b-1} - 1)^{1/(b-1)} + 1].$$

Inequality (3) is equivalent to

$$c \in (c^{early}, 2y]. \tag{15}$$

■

Proof of Lemma 2

Proof. It is easy to see that if $b < 1 + \ln 2/\ln R$, $d(c)$ is decreasing in c. It changes from $+\infty$ when $c = 0$ to $-\infty$ when $c = 2y$. Hence there is a unique

$c = c^{wait} \in (0, 2y)$ that solves the equation

$$pv[(2y - c)R] + (1 - p)v(yR) = p[v(c) + v(2y - c)]/2 + (1 - p)v(c).$$

So when b and R satisfy the condition $b < 1 + \ln 2/\ln R$, inequality (5) is equivalent to

$$c \in [0, c^{wait}]. \tag{16}$$

■

Proof of Lemma 3

Proof. If condition (14) holds, c^{wait} and c^{early} are well defined. To get the condition on b and R that implies the inequality

$$c^{wait} > c^{early}, \tag{17}$$

we merely replace c in inequality (5) by c^{early}. This results in

$$\frac{2/R}{(2/R^{b-1} - 1)^{1/(b-1)} + 1} < 1. \tag{18}$$

When b and R satisfy condition (14), $(2/R^{b-1} - 1)^{1/(b-1)}$ is decreasing in b. Hence inequality (18) is equivalent to

$$b < 2. \tag{19}$$

To summarize: the set of c satisfying both conditions (3) and (5) is non-empty if and only if b and R satisfy both inequality (14) and inequality (19), which results in condition (7). ■

Proof of Proposition 1

Proof. Since we have $\widehat{W}(c) > W^{run}(c)$, $W(c;s)$ is not continuous at c^{early} if $s > 0$. We study the two regions, $[0, c^{early}]$ and $(c^{early}, c^{wait}]$, separately, and compare the maximum values of $W(c;s)$ in these two regions.

For $c \in [0, c^{early}]$, $W(c;s)$ is strictly increasing in c since $c^{early} < \widehat{c}$. Hence the maximum value of $W(c;s)$ over $[0, c^{early}]$ is achieved at c^{early}. Therefore the best run-proof contract is $c = c^{early}$.

For $c \in (c^{early}, c^{wait}]$, the maximum value of $W(c;s)$ may not be achievable because $(c^{early}, c^{wait}]$ is not closed. To fix this problem, we define the function $\widetilde{W}(c;s)$ on $[c^{early}, c^{wait}]$ by

$$\widetilde{W}(c;s) = (1-s)\widehat{W}(c) + sW^{run}(c).$$

When $c \in (c^{early}, c^{wait}]$, $\widetilde{W}(c;s) = W(c;s)$. When $c = c^{early}$, $\widetilde{W}(c;s) < W(c;s)$. Let $\widetilde{c}(s)$ be defined by

$$\widetilde{c}(s) = \arg \max_{c \in [c^{early}, c^{wait}]} \widetilde{W}(c;s).$$

We have

$$\widetilde{c}(s) = \max\{\frac{2y}{\gamma^{1/b} + 1}, c^{early}\}, \tag{20}$$

where

$$\gamma = \frac{s(1-p)(pA + 1 - p\frac{2}{R^{b-1}}) + (p^2 A + (1-p)p\frac{2}{R^{b-1}})}{s(1-p)(1-pA) + p(2-p)A}.$$

It can be shown that $\widetilde{c}(s)$ is continuous in s. Furthermore, $\widetilde{c}(s)$ is strictly decreasing in s when s is small such that $\widetilde{c}(s) > c^{early}$. We also have $c^{early} = \widetilde{c}(1) < \widetilde{c}(0) = \widehat{c}$. $\widetilde{W}(\widetilde{c}(s); s)$ is continuous in s and it is also strictly decreasing in s since $\widehat{W}(c) > W^{run}(c)$. Furthermore, we have

$$\widetilde{W}(\widetilde{c}(0); 0) = \widehat{W}(\widehat{c}) > \widehat{W}(c^{early})$$

and

$$\widetilde{W}(\widetilde{c}(1); 1) = W^{run}(c^{early}) < \widehat{W}(c^{early}).$$

Hence there is a unique $s_0 \in (0, 1)$ such that

$$\widetilde{W}(\widetilde{c}(s_0); s_0) = \widehat{W}(c^{early}). \tag{21}$$

Obviously, we have $\widetilde{c}(s_0) > c^{early}$.

Hence if $s < s_0$, we have $c^*(s) = \widetilde{c}(s)$. The optimal contract $c^*(s)$ tolerates runs and it is a strictly decreasing function of s. We have $c^{early} < c^*(s) \leq \widehat{c}$ (with equality if and only if $s = 0$).

If $s > s_0$, $c^*(s) = c^{early}$. The optimal contract is run-proof.

If $s = s_0$, $\widetilde{W}(\widetilde{c}(s); s) = \widehat{W}(c^{early})$. So both the run-proof contract (c^{early}) and the run-tolerating contract $(\widetilde{c}(s_0))$ are optimal at $s = s_0$. ■

Proof of Proposition 2

Proof. The proof is similar to that for Proposition 1. The only difference is that the ICC may bind. As before, we analyze separately the two regions $[0, c^{early}]$ and $(c^{early}, c^{wait}]$ separately, and compare the maximum values of $W(c; s)$ in these two regions.

For $c \in [0, c^{early}]$, it is easy to see that $W(c;s)$ is strictly increasing. Hence, as in Case 2, the best run-proof contract is $c = c^{early}$.

For $c \in (c^{early}, c^{wait}]$, the maximum value of $W(c;s)$ may not be achievable because $(c^{early}, c^{wait}]$ is not closed. To fix this problem and characterize the possibly binding ICC, we define the function $\overline{W}(c;s)$ on $[c^{early}, 2y]$ by:

$$\overline{W}(c;s) = (1-s)\widehat{W}(c) + sW^{run}(c).$$

When $c \in (c^{early}, c^{wait}]$, we have $\overline{W}(c;s) = W(c;s)$. When $c = c^{early}$, we have $\overline{W}(c;s) < W(c;s)$. Let $\overline{c}(s)$ be defined by

$$\overline{c}(s) = \arg \max_{c \in [c^{early}, 2y]} \overline{W}(c;s).$$

We have

$$\overline{c}(s) = \frac{2y}{\eta^{1/b} + 1}, \tag{22}$$

where

$$\eta = \frac{s(1-p)(pA + 1 - p\frac{2}{R^{b-1}}) + (p^2A + (1-p)p\frac{2}{R^{b-1}})}{s(1-p)(1-pA) + p(2-p)A}.$$

By using the same argument as that in Proposition 2, we can show that $\overline{c}(s)$ is continuous in s. Furthermore, $\overline{c}(s)$ is strictly decreasing in s when s is small such that $\overline{c}(s) > c^{early}$. We also have $c^{early} = \overline{c}(1) < \overline{c}(0) = \widehat{c}$. Note that in Case 3, we have $c^{wait} < \widehat{c}$. Hence there is a unique level of $s \in (0,1)$, denoted by s_2, such that

$$\overline{c}(s_2) = c^{wait}. \tag{23}$$

That is, s_2 is the threshold run probability below which the ICC binds. Next, we need to check, when $s = s_2$, whether the optimal contract still tolerates runs. To do that, we define s_4 by

$$s_4 = \frac{\widehat{W}(c^{wait}) - \widehat{W}(c^{early})}{\widehat{W}(c^{wait}) - W^{run}(c^{early})}. \tag{24}$$

Obviously, we have $s_4 \in (0, 1)$. There will be two sub-cases depending on whether the optimal contract still tolerates runs when when $s = s_2$.

In the first sub-case of Case 3, we have $s_4 > s_2$, that is, at the threshold run probability which makes the ICC just become non-binding, the optimal contract still tolerates runs. Now we need to determine the threshold run probability beyond which the optimal contract switches to being run-proof. That threshold level is s_3 which is defined by

$$\overline{W}(\overline{c}(s_3); s_3) = \widehat{W}(c^{early}). \tag{25}$$

Using the same argument as in Proposition 1, we know that $\overline{W}(\overline{c}(s); s)$ is continuous and strictly decreasing in s. Therefore, s_3 is unique. Since $s_4 > s_2$, we know that $s_3 > s_2$. The contract $c^*(s)$ satisfies the following: When $s < s_2$, the ICC binds and $c^*(s) = c^{wait}$ since we have

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s) > \widehat{W}(c^{early}).$$

When $s_2 \leq s < s_3$, the ICC no longer binds and $c^*(s) = \overline{c}(s)$ since we have

$$W(\overline{c}(s); s) = \overline{W}(\overline{c}(s); s) > \widehat{W}(c^{early}).$$

When $s = s_3$, both $\overline{c}(s)$ and c^{early} are optimal since

$$W(\overline{c}(s); s) = \overline{W}(\overline{c}(s); s) = \widehat{W}(c^{early}).$$

When $s > s_3$, $c^*(s) = c^{early}$ since

$$W(\overline{c}(s); s) = \overline{W}(\overline{c}(s); s) < \widehat{W}(c^{early}).$$

To summarize, if $s_4 > s_2$ we have

$$c^*(s) = \begin{cases} c^{wait} \text{ if } s < s_2 \\ \overline{c}(s) \text{ if } s_2 \leq s \leq s_3 \\ c^{early} \text{ if } s_3 \leq s. \end{cases}$$

In the second sub-case of Case 3, we have $s_4 \leq s_2$, that is, at the run probability which makes the ICC just become non-binding, the optimal contract does not tolerate runs. Hence the optimal contract will switch to the best run-proof contract (c^{early}) when the ICC still binds. $c^*(s)$ satisfies the following property: When $s < s_4$, the ICC binds and $c^*(s) = c^{wait}$ since we have

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s) > \widehat{W}(c^{early}).$$

When $s = s_4$, both c^{wait} or c^{early} are optimal since we have

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s_4) = \widehat{W}(c^{early}).$$

When $s_4 < s < s_2$, we have $c^*(s) = c^{early}$. This is because the ICC binds and

$$W(c^{wait}; s) = \overline{W}(c^{wait}; s) < \widehat{W}(c^{early}).$$

When $s_2 \leq s$, $c^*(s; A)$ is still equal to c^{early}. This is because the ICC no longer binds and

$$W(\bar{c}(s); s) = \overline{W}(\bar{c}(s); s) < \overline{W}(\bar{c}(s_2); s_2) = \overline{W}(c^{wait}; s_2) < \widehat{W}(c^{early}).$$

To summarize, if $s_4 \leq s_2$, we have

$$c^*(s) = \begin{cases} c^{wait} \text{ if } s \leq s_4 \\ c^{early} \text{ if } s \geq s_4. \end{cases}$$

We can see, in each of the two sub-cases, $c^*(s)$ switches to run-proof if the run probability is larger than the threshold value. Let s_1 denote that threshold run probability and we have

$$s_1 = \begin{cases} s_3 \text{ if } s_4 > s_2 \\ s_4 \text{ if } s_4 \leq s_2. \end{cases} \tag{26}$$

■

Appendix B: Comparative Statics with Respect to the Parameters p, R and b

In section 4.1 in the published paper, we analyzed the effects of the impulse demand parameter A on $\widehat{c}$, the contract supporting the *unconstrained efficient allocation*. Next, we analyze the effects of varying the remaining parameters, namely p, R and b, on $\widehat{c}$. We limit our discussion to the set of parameters permitting strategic complementarity, i.e., b and R satisfying inequality (5).

Appendix B.1: Probability of Impatience p

From equation (9), it is easy to see that $\widehat{c}$ is increasing in p if $AR^{b-1} < 1$, $\widehat{c}$ is equal to y if $AR^{b-1} = 1$, and $\widehat{c}$ is decreasing in p if $AR^{b-1} > 1$. Hence how p affects $\widehat{c}$ depends solely on the values of A and R. The intuition is the following: Because there is aggregate uncertainty, the economy might have 2 impatient consumers, 1 impatient consumer and 1 patient consumer, or 2 patient consumers. The parameter p affects the likelihood of the first scenario relative to the second scenario. The first scenario requires no cross-subsidy between the consumers. The second scenario requires a cross-subsidy, but how it is conducted depends on A and R. If $AR^{b-1} < 1$, the subsidy is from the impatient to the patient (i.e., $\widehat{c} < y$). While if $AR^{b-1} > 1$, the subsidy is from the patient to the impatient (i.e., $\widehat{c} > y$). As p increases, the second scenario becomes less likely compared to the first one and less subsidy is required (i.e., $\widehat{c}$ should be closer to y). Hence if $AR^{b-1} < 1$, $\widehat{c}$ increases as p increases. And if $AR^{b-1} > 1$, the opposite is true.

To see how different values of p correspond to the three cases of the optimal contract, note that c^{early} doesn't depend on p and

$$\lim_{p\to 1} \widehat{c} = y < c^{early}.$$

Hence we are in Case 1 whenever p is sufficiently large. Furthermore, we have

$$\lim_{p\to 0} \widehat{c} = \frac{2y}{(1/AR^{b-1})^{1/b} + 1}.$$

Hence if we have $\frac{2y}{(1/AR^{b-1})^{1/b}+1} \leq c^{early}$, then only Case 1 obtains.

If

$$\frac{2y}{(1/AR^{b-1})^{1/b} + 1} > c^{early},$$

which implies $AR^{b-1} > 1$, there is a unique level of p, denoted by p^{early}, such that

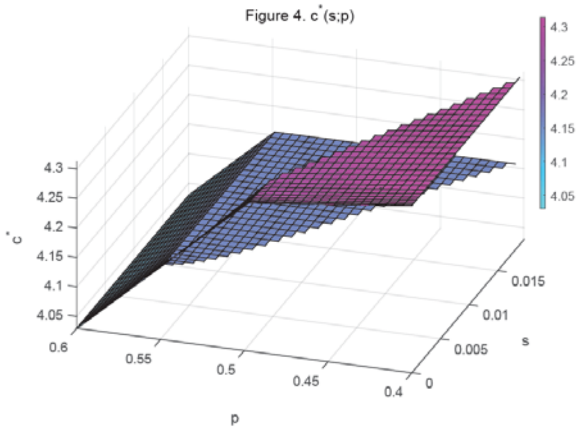
$$\widehat{c}(p^{early}) = c^{early}.$$

If $p \geq p^{early}$, we are in Case 1. If $p < p^{early}$, we are in Case 2 or Case 3 depending on whether $\widehat{c}(p)$ is smaller than c^{wait} or not. Note that c^{wait} does change with p.

Example 7 *Let*

$$b = 1.01, A = 10, y = 3, R = 1.5.$$

We have $c^{early} = 4.155955$. *It is easy to see that if* $p \geq 0.548823$, *we are in Case 1. If* $0.497423 \leq p < 0.548823$, *we are in Case 2. If* $p < 0.497423$, *we are in Case 3. We plot* c^* *versus* s *and* p *in Figure 4.*



Appendix B.2: Return on Bank Investment R

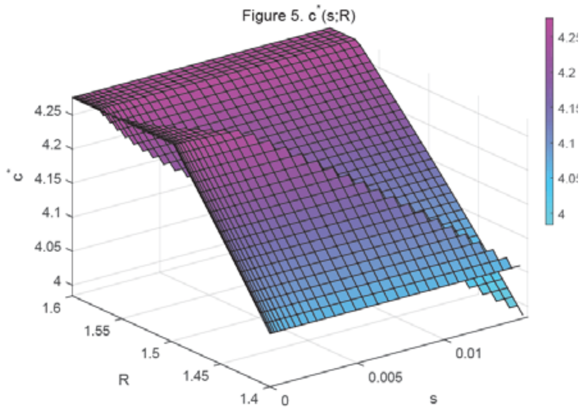
From equation (9), it is easy to see that $\widehat{c}$ is increasing in R. R affects $\widehat{c}$ by changing the optimal allocation when the economy has one impatient depositor and one patient depositor. For larger R, on the one hand, the marginal rate of transformation between the first period consumption and the second period consumption is increasing in R. On the other hand, the marginal rate of substitution between the first period consumption by the impatient depositor and the second period consumption by the patient depositor is also increasing in R. Since $b > 1$, the second effect is stronger and, therefore, the optimal allocation allows more first-period withdrawal, i.e., $\widehat{c}$ increases as R increases. It is easy to see that both c^{early} and c^{wait} increase in R. If $\widehat{c} \leq c^{early}$, we are in Case 1. If $c^{early} < \widehat{c} \leq c^{wait}$, we are in Case 2. If $\widehat{c} > c^{wait}$, we are in Case 3.

Example 8 *Let*

$$b = 1.01, A = 10, y = 3, p = 0.5.$$

It is easy to see that if $R \geq 1.572948$*, we are in Case 1. If* $1.497374 \leq R <$

1.572948*, we are in Case 2. If* $R < 1.497374$*, we are in Case 3. We plot* c^* *versus* s *and* R *in Figure 5.*



Appendix B.3: Risk Aversion Parameter b

The sign of $\frac{\partial \widehat{c}}{\partial b}$ is the same as the sign of

$$\ln\left(\frac{p}{2-p} + \frac{2(1-p)}{(2-p)AR^{b-1}}\right) + \frac{2(1-p)b\ln(R)}{2(1-p) + pAR^{b-1}}.$$

Hence if A is smaller than a threshold level, we have $\frac{\partial \widehat{c}}{\partial b} > 0$. Otherwise, we have $\frac{\partial \widehat{c}}{\partial b} < 0$. The intuition is the following: As b increases, consumption smoothing across the two depositors is more desirable. When A is small, $\widehat{c}$ is small and more consumption smoothing entails larger $\widehat{c}$. When A is large, $\widehat{c}$ is large and more consumption smoothing entails smaller $\widehat{c}$.

Example 9 *Let*

$$A = 10, y = 3, p = 0.5, R = 1.5.$$

It is easy to check that if $b \geq 1.112528$*, we are in Case 1. If* $1.00524 \leq b < 1.112528$*, we are in Case 2. If* $b < 1.00524$*, we are in Case 3. We plot* c^* *versus* s *and* b *in Figure 6.*

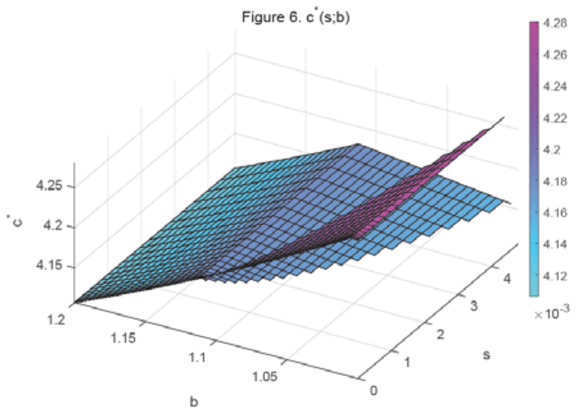


Figure 6. c*(s;b)
c*
4.25
4.2
4.15
1.2
1.15
1.1
1.05
b
0
1
2
3
4
s
4.28
4.26
4.24
4.22
4.2
4.18
4.16
4.14
4.12
×10^-3

Appendix C: The Optimal Contract for non-SC Parameter Values

Appendix C.1: The Post-Deposit Game

For the non-SC parameters (i.e., where condition (7) is not satisfied), we have either

$$2 \leq b < 1 + \ln 2/\ln R \tag{27}$$

or

$$b \geq 1 + \ln 2/\ln R. \tag{28}$$

For b and R satisfying inequality (27), we have $c^{wait} \leq c^{early}$. (This can be seen directly from the proof of Lemma 3). In contrast to the SC parameters, the order of c^{early} and c^{wait} is reversed. Thus, compared to SC parameters, the post-deposit game has different game forms. From the pay-off matrix of the post-deposit game, we see that for $c \in [0, c^{wait}]$, we have $T_2 > T_1$ and $T_4 \geq T_3$. (L, E) is the dominant strategy for each depositor. The post-deposit game has a dominant strategy equilibrium with Pareto efficiency (i.e., the non-run equilibrium). For $c \in (c^{early}, 2y]$, we have $T_2 < T_1$ and $T_4 < T_3$. (E, E) is the dominant strategy for each depositor. For $c \in (c^{wait}, c^{early}]$, we have $T_2 \geq T_1$ and $T_4 < T_3$. The interval $(c^{wait}, c^{early}]$ is the region of c for which the post-deposit game is "chicken" type and the patient depositors' withdrawal decisions exhibit strategic substitutability (rather than strategic complementarity): A patient depositor withdraws late if and only if he expects that the other depositor – if patient – to withdraw early. The chicken behavior might seem a bit exotic in banking, but nonetheless this equilibrium is like a partial run. Thus, in contrast to the SC parameters for which the set of contracts with non-run as the unique BNE is a strict subset of the set of

BIC contracts, now the two sets are the same and both of them are $[0, c^{wait}]$.

For b and R satisfying inequality (28), from the proof of Lemma 1, we can see that there is not a run equilibrium for any contract $c \in [0, 2y]$ in the post-deposit game. Therefore any BIC contract is also a contract with non-run as the unique BNE.

Appendix C.2: The Optimal Contract for the Pre-Deposit Game

According to the Revelation Principle, to find $c^*(s)$ in the pre-deposit game, we need only focus on the BIC contracts. As we have seen, for the SC parameters, a BIC contract is also a contract with non-run as the unique BNE. Hence, bank runs are not relevant for the optimal contract c^*, and $c^*(s)$ maximizes the expected welfare of the depositors at the non-run equilibrium:

$$c^*(s) = \arg\max_c \widehat{W}(c) \text{ for } s \in [0,1] \tag{29}$$
$$s.t.\ c \text{ satisfies ICC (i.e. condition (5))}$$

For b and R satisfying inequality (27), we know that c satisfies (5) if and only if $c \leq c^{wait}$. Hence the solution to problem (29) is

$$c^* = \min\{\widehat{c}(A), c^{wait}\}.$$

For b and R satisfying inequality (28), c^{wait} is not well-defined. From the proof of Lemma 2, we know that the difference between the left-hand side and the right hand side of inequality (5) is no longer decreasing in c. Let us denote that difference by $Diff(c)$. $Diff(c)$ is strictly decreasing in c for

$c \in [0, \overline{c^{wait}}]$ and strictly increasing in c when $c \in [\overline{c^{wait}}, 2y]$, where

$$\overline{c^{wait}} = \frac{2y}{[\frac{1-p/2}{-p(R^{1-b}-1/2)}]^{-1/b} + 1}.$$

Furthermore, $Diff(0) = +\infty$ and $Diff(2y) = +\infty$. Therefore, if $Diff(\overline{c^{wait}}) \geq 0$, (5) holds for any $c \in [0, 2y]$. If $Diff(\overline{c^{wait}}) < 0$, (5) holds for

$$c \in [0, c^{wait1}] \cup [c^{wait2}, 2y], \tag{30}$$

where $c^{wait1} < c^{wait2}$ and c^{wait1} and c^{wait2} are the two solutions for $Diff(c) = 0$. Hence if $Diff(\overline{c^{wait}}) \geq 0$, or $Diff(\overline{c^{wait}}) < 0$ but at the same time $\widehat{c}(A)$ satisfies condition (**??**), the ICC does not bind and the solution to the problem (29) is

$$c^* = \widehat{c}(A).$$

If $Diff(\overline{c^{wait}}) < 0$ and at the same time $\widehat{c}(A)$ doesn't satisfy condition (**??**), the ICC binds and c^* is equal to c^{wait1} or c^{wait2} depending on which one delivers higher expected welfare at the non-run equilibrium $\widehat{W}(c)$.