Problem Set #2

Problem Set 2 has been posted. It is due on Monday, September 24, at the beginning of the lecture.
Next Lecture

Next lecture will include a review of deriving demand functions from utility functions. It will also introduce the basics of optimization theory. Preparation is not necessary. (For the first consult — if you like — your micro text. For the second — if you like — consult in a math text “the rule of Lagrange” or “mathematical programming” or “Kuhn-Tucker Theorem”.)
Consider a market with one good per period $\ell = 1$, and two periods $t = 1, 2$.

$$\max_{x^1_h, x^2_h} u_h(x^1_h, x^2_h)$$

s.t. $$p^1 x^1_h + p^2 x^2_h = p^1 \omega^1_h + p^2 \omega^2_h$$

The equilibrium is a price vector $(p^1, p^2)$ such that

$$\sum_h x^t_h = \sum_h \omega^t_h \text{ for } t = 1, 2$$
The interest factor $R$ and interest rate $r$ are defined as follow:

\[
p^1 = Rp^2
\]
\[
r = R - 1
\]

Normalizing $p^1 = 1$:

\[(p^1, p^2) = (1, p^2)\]

Then

\[p^2 = \frac{1}{R} = \frac{1}{1 + r}\]
Again, consider a market with one good per period \( \ell = 1 \), and two periods \( t = 1, 2 \).

\[
\max \quad u_h(x^1_h, x^2_h)
\]

s.t.
\[
p^1 x^1_h + p^{m1} m^1_h = p^1 \omega^1_h \\
p^2 x^2_h + p^{m2} m^2_h = p^2 \omega^2_h
\]

An equilibrium in the money market is \((p^1, p^2, p^{m1}, p^{m2})\) such that
\[
\sum_h x^t_h = \sum_h \omega^t_h \quad \text{and} \quad \sum_h m^t_h = 0 \quad \text{for} \quad t = 1, 2
\]
Money Market

I will show that in equilibrium $p^{m_1} = p^{m_2} = p^m$. This is the no arbitrage property result.

Consumer $h$ maximizes his financial wealth in order to maximize his inter-temporal utility:

$$\max \quad W_h = p^{m_1} m^1_h + p^{m_2} m^2_h$$
$$\text{s.t.} \quad m^1_h + m^2_h = 0$$

Rearranging the constraint and plugging into the wealth equation give:

$$m^2_h = -m^1_h$$
$$W_h = (p^{m_1} - p^{m_2}) m^1_h$$
Money Market

- If $p^m_1 > p^m_2$, the consumer can borrow (high) in period 1 and re-pay (low) in period 2. This means $m^1_h > 0$.
- If $p^m_1 < p^m_2$, the consumer can lend (high) in period 1 and receive repayment (low) in period 2. This means $m^1_h < 0$.
- If $p^m_1 \neq p^m_2$, $W_h$ can be arbitrarily large by choosing $|m^1_h|$ arbitrarily large. This allows $x^1_h$ and $x^2_h$ to be arbitrarily large.
- So,
  \[ \sum_h x^t_h > \sum_h \omega^t_h \text{ for } t = 1, 2 \]
- Therefore, $p^m_1 = p^m_2$ is required for competitive equilibrium.
- Let $p^m = p^m_1 = p^m_2$. 
Equivalence

I will next show that if \((x_h^1, x_h^2)\) solves the futures market problem, it also solved the money market problem.

The excess demand is

\[ z_t^h = x_t^h - \omega_t^h \]

Since \(p^1 = 1\),

\[ z_h^1 + p^2 z_h^2 = 0 \]
\[ z_h^1 = -p^2 z_h^2 \]

Define scalar \(k\) by

\[ k = z_h^1 - p^2 z_h^2 \]
Set $k = p^m m^1_h = -p^m m^2_h$. Then

\[
\begin{align*}
    z^1_h + p^m m^2_h &= 0 \\
p^2 z^2_h + p^m m^2_h &= 0 \\
m^1_h + m^2_h &= 0
\end{align*}
\]

Hence, if $(x^1_h, x^2_h)$ is an equilibrium allocation in the futures market, it is also an equilibrium in the money market.
Next, I will show that if \((x_h^1, x_h^2)\) for \(h = 1, \ldots, n\) solves the money market problem with \(p^m > 0\), then it also solves the futures market problem.
Equivalence

\[ m^1_h + m^2_h = 0 \]

\[ m^1_h = -m^2_h \]

Since \( p^m > 0 \),

\[ z^1_h = -p^m m^1_h \]
\[ \frac{z^1_h}{p^m} = -m^1_h \]
\[ p^2 z^2_h = p^m m^1_h \]
\[ \frac{p^2 z^2_h}{p^m} = m^1_h \]

\( m^1_h \) is a slack variable. Hence, we have

\[ z^1_h + p^2 z^2_h = 0 \]

Hence, if \((x^1_h, x^2_h)\) is an equilibrium allocation in the money market economy with \( p^m > 0 \), then \((x^1_h, x^2_h)\) is also an equilibrium allocation in the corresponding futures market economy.