DETERMINATE FORECASTING IN GENERAL OLG MODELS

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- In forward-looking economic models, agents must necessarily forecast future economically-relevant variables
- In general overlapping generations models, Kehoe and Levine (1985) showed that simply adopting the equilibrium law of motion for the model as agents' forecasts led to equilibrium being indeterminate
- The argument for this is straight-forward in the standard two-period-lived agents setting
  - If the equilibrium law of motion for prices is given by

    $$p_{t+1} = g\left(p_t, p_{t-1}\right)$$

    which is derived from the zero excess demand condition for young and old

    $$z\left(p_{t-1}, p_t\right) + y\left(p_t, p_{t+1}\right) = 0$$

    then using g as the young agent's forecast function yields

    $$z\left(p_{t-1}, p_t\right) + y\left(p_t, g\left[p_{t-1}, p_t\right]\right) \equiv 0$$

  - Hence, given p_{t-1}, any p_t will be an equilibrium price.

THE KEHOE-LEVINE FORECAST

- K&L showed that if there is a first-order forecast function

$$p_{t+1} = f(p_t)$$

which was consistent with the equilibrium dynamics on a neighborhood of the steady-state equilibrium price $\hat{p}$ in the sense that

$$f \circ f(p) = g(p, f[p])$$

then this forecast function would render the rational expectations equilibrium determinate

THE KEHOE-LEVINE FORECAST

- In addition to the consistency requirement, K&L also imposed the stability requirement that all eigenvalues of $Df(\hat{p})$ were inside the unit circle
- The stability requirement is consistent with Blanchard and Kahn's (1980) characterization of the equilibria of linear rational expectations models
  - To see this, let

$$q_{t+1} = \begin{bmatrix} p_{t+1} \\ p_t \end{bmatrix} = \begin{bmatrix} g(p_t, p_{t-1}) \\ p_t \end{bmatrix} = \hat{g}(q_t)$$

  - Then if the eigenvalues of $D_q\hat{g}(\hat{q})$ split, the steady-state equilibrium is determinate; if there are more stable eigenvalues than unstable, the steady-state is indeterminate; if there are more unstable eigenvalues that stable, the steady-state is explosive

THE KEHOE-LEVINE FORECAST

- K&L focus their analysis on an OLG model in which agents live for two periods, trading an arbitrary (but finite) number of commodities
  - For deterministic models, this assumption is without loss of generality due to the applicability of the Balasko-Cass-Shell transformation (see Balasko, Cass and Shell, "Existence of competitive equilibrium in a general overlapping generations model", *JET*, 1980)
    - The BCS transformation shows that by reinterpreting commodities available at different dates as different commodities at the same transformed date, arbitrary but finite-lived agents can be viewed as living two periods
  - The BCS transformation doesn't work for stochastic models
    - In stochastic OLG models, agents are born into a given state of the world, but are assumed to be able to trade contingent on all future states
    - Under the BCS transformation, young agents would necessarily lose the ability to make some future contingent trades after the reinterpretation of the model's timing; this necessarily has real effects

THE KEHOE-LEVINE FORECAST

- In other work, Eungsik Kim and I have shown that in general multi-period-lived stochstic OLG models, the law of motion for the lagged, endogenous state variables in the model is linear
  - for the recursive equilibrium, with endogenous state variables given by the wealth distribution (i.e. asset holdings of agents)
  - for the case of small shocks
- Deriving this fact requires the use of generalized forecast functions depending, at any give time, only on the pre-determined state variables
  - We focus in particular on the case where these forecast functions render the recursive equilibrium determinate

GENERALIZED FORECASTS

- The K&L forecast construction is based on differentiating the identity that determines the consistency of the first-order forecast with the general equilibrium dynamic system

$$\begin{bmatrix} f \circ f(p) \\ f(p) \end{bmatrix} = \begin{bmatrix} g(f(p), p) \\ f(p) \end{bmatrix}$$

- Differentiating with respect to p at $\hat{p}$ yields

$$\begin{bmatrix} Df \\ I \end{bmatrix} Df = \begin{bmatrix} G_1 & G_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} Df \\ I \end{bmatrix}$$

where G_i, $i = 1, 2$ are the derivative matrices of g with respect to p_{t+1-i}

- Putting this into canonical form yields

$$\begin{bmatrix} \Lambda \\ I \end{bmatrix} \Lambda = \begin{bmatrix} G_1 & G_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} \Lambda \\ I \end{bmatrix}$$

which immediately tells us that the eigenvalues of Df will be a subset of the (stable) eigenvalues of $D\hat{g}$

GENERALIZED FORECASTS

- I will illustrate construction of the generalized forecast function for an agent who lives $L > 2$ periods, but trades in only a single perishable good
- For this version of the model, the recursive equilibrium law of motion for prices takes the form

$$p_{t+L-1} = z\left(\hat{q}_{t+L-2}\right)$$

where$\hat{q}_{t+L-2} = \left(p_{t+L-2}, p_{t+L-3}, \ldots, p_{t-L+1}\right)$and $z : \mathbb{R}_{++}^{(2L-2)} \to \mathbb{R}_{++}$

- We can convert this high-order dynamic system into a first-order system in the usual way

$$\hat{q}_{t+L-1} = \hat{z}\left(\hat{q}_{t+L-2}\right) = \begin{bmatrix} z\left(\hat{q}_{t+L-2}\right) \\ p_{t+L-2} \\ \vdots \\ p_{t-L+2} \end{bmatrix}$$

GENERALIZED FORECASTS

- Let $q_{t-1} = (p_{t-1}, p_{t-2}, \ldots, p_{t-L+1})$ be the vector of predetermined variables at time t
- We define the generalized forecast function

$$p_t = f(q_{t-1})$$

and, as with $\hat{z}$ we define the related first-order forecast system as

$$q_t = \hat{f}(q_{t-1}) = \begin{bmatrix} f(q_{t-1}) \\ p_{t-1} \\ \vdots \\ p_{t-L+2} \end{bmatrix}$$

where $\hat{f} : \mathbb{R}^{(L-1)}_{++} \rightarrow \mathbb{R}^{(L-1)}_{++}$

GENERALIZED FORECASTS

- The consistency requirement on the forecast functions with the full system price dynamics requires that at time t, every agent is using f (or compositions of f) to forecast forward prices
- In a three-period-lived agents model, we can illustrate this as follows
  - Equilibrium law of motion: $p_{t+1} = z(p_t, p_{t-1}, p_{t-2})$
  - Generalized forecast function: $p_{t+1} = f(p_t, p_{t-1}) = f[f(p_{t-1}, p_{t-2}), p_{t-1}]$
  - Consistency requires

$$f[f(p_{t-1}, p_{t-2}), p_{t-1}] = z[f(p_{t-1}, p_{t-2}), p_{t-1}, p_{t-2}]$$

CONSISTENCY REQUIREMENTS

- To analyze the general consistency requirements, we define

$$Z = \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_{2L-3} & Z_{2L-2} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = \begin{bmatrix} Dz \\ J_{(2L-2)} \end{bmatrix}$$

where Z_i is the i^{th} partial derivative of z and $J_{(2L-2)} = \begin{bmatrix} I_{(2L-3)} & 0_{(2L-3)} \end{bmatrix}$ where $I_{(2L-3)}$is a $2L-3$ square identity matrix, and $0_{(2L-3)}$ is a $2L-3$ dimension vector of zeros

CONSISTENCY REQUIREMENTS

- Also define

$$F = \begin{bmatrix} F_1 & F_2 & \cdots & F_{L-2} & F_{L-1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = \begin{bmatrix} Df \\ J_{(L-1)} \end{bmatrix}$$

where F_i is the i^{th}partial derivative of the generalized forecast f

- Finally, define

$$K = \begin{bmatrix} DfF^{L-2} \\ \vdots \\ DfF \\ Df \\ I_{(L-1)} \end{bmatrix}$$

- K is the matrix of derivatives of the various compositions of the forecast functions that enter into the consistency condition
- K is given recursively in terms of the gradient Df and powers of the matrix F

CONSISTENCY REQUIREMENTS

- **Proposition:** For a consistent forecast function f, we will have

$$ZK = KF$$

- The proof of this is a tedious recursion; it is transparent for $L = 3$, but less so the larger L becomes, so I omit it here.
- This structural relationship between the derivative matrices of the two dynamic systems implies the existence of an invariant subspace of Z which is also invariant under the actions of F and which can be associated with the stable eigenvalues of Z
  - For the remainder of the talk, we will assume that the eigenvalues of Z split, i.e. there $L-1$ eigenvalues inside the unit circle and $L-1$ outside the unit circle

CONSISTENCY REQUIREMENTS

- To show the invariance result, note that the consistency relationship has the following implication
  - Let
    $$K = [k_1, k_2, \ldots, k_{L-1}]$$
    so that k_i is the i^{th} column of K
  - From the definitions of Z and F, we then have (using the companion matrix structure of F)
    $$\begin{aligned} Z^{L-1}k_1 &= F_1 Z^{L-2}k_1 + Z^{L-2}k_2 \\ Z^{L-2}k_2 &= F_2 Z^{L-3}k_1 + Z^{L-3}k_3 \\ &\vdots \\ Z^2 k_{L-2} &= F_{L-2}Zk_1 + Zk_{L-1} \\ Zk_{L-1} &= F_{L-1}k_1 \end{aligned}$$
  - Recursive substitution yields:
    $$\left[Z^{L-1} - F_1 Z^{L-2} - \cdots - F_{L-2}Z - F_{L-1}I\right] k_1 = 0$$

- This equation implies

$$ch_F(Z)\,k_1 = 0$$

- Since the characteristic function of F is not the characteristic function of Z, it cannot be that $ch_F(Z) = 0$, so it must be that $ch_F(Z)$annihilates the vector k_1
- This, in turn, implies that $k_1$is a cyclic vector for Z, and, via the cyclic decomposition theorem, that the powers of Z operating on k_1 generate in invariant (cyclic) subspace of Z

CONSISTENCY REQUIREMENTS

- With ch_F, we can associate the stable eigenvalues of Z in Λ_f with the elements in Df such that:

$$ch_F(c_i) = c_i^{L-1} - F_1 c_i^{L-2} - \cdots - F_{L-2} c_i - F_{L-1} = 0$$

for all of the $(L-1)$ stable eigenvalues, $\{c_i\}_{i=1}^{L-1}$

- One can find the general forecast functions by transforming this equation into the following linear system

$$\begin{bmatrix} c_1^{L-2} & \cdots & c_1 & 1 \\ c_2^{L-2} & \cdots & c_2 & 1 \\ \vdots & \iddots & \vdots & \vdots \\ c_{L-1}^{L-2} & \cdots & c_{L-1} & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{L-1} \end{bmatrix} = \begin{bmatrix} c_1^{L-1} \\ c_2^{L-1} \\ \vdots \\ c_{L-1}^{L-1} \end{bmatrix}$$

and solving for Df

CONCLUSIONS

- We have shown how to construct a general forecast function depending only on pre-determined variables for any forward-looking dynamic price function
  - We have constructed it for the rational expectations equilibria for overlapping generations models, but the technique can be applied more generally to any forward-looking economic system that generates the same kind of equilibrium pricing dynamics
- Via the Blanchard-Kahn results, we can also chacterize the equilibria for the class of models we examine as explosive (more eigenvalues of Z outside the unit circle than inside), determinate (eigenvalues of Z split), or indeterminate (more eigenvalues of Z inside the unit circle than outside)
  - For the indeterminate case, there will exist multiple forecasts depending only on pre-determined variables, and one can construct sunspot equilibrium forecast functions by randomizing over the choice of forecast functions