Multiple equilibria in “Expectations and the neutrality of money”

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Lucas (JET '72) is a counter-example to the view that

- an observed positive correlation between the growth rate of the stock of money and real output is exploitable (is invariant to the rule generating the stock of money)

Is it an *interesting* counter-example?

- is the correlation an implication of optimizing behavior under rational expectations in a setting that is not too bizarre?

We provide two new reasons to doubt that it is an interesting counter-example

The Lucas model

It has a physical environment and a solution concept or equilibrium notion

We are not going to quarrel with the physical environment, which includes who knows what and when they know it

We will, however, follow Wallace 1992 and use a version which is simpler in an innocuous way

The Lucas equilibrium concept is competitive-equilibrium (CE), or, given the uncertainty in the model, what is sometimes called rational-expectations equilibrium (REE)

We point out a CE-multiplicity and also use a different solution concept, the *market game*

The physical environment

Two-period-lived overlapping generations and one good per discrete date

The integer size of a generation is denoted N where $N \in \{N_l, N_h\}$ and where $N_h > N_l \geq 2$. (When we replicate, we replace N by $kN \in \{kN_l, kN_h\}$ and let $k \to \infty$.)

Proportional transfers to each old person: if a person when young acquired m amount of money, each offers when old $(1+\gamma)m$ amount of money where $\gamma \in \{\gamma_l, \gamma_h\}$, and where

$$(1+\gamma_l)/N_l = (1+\gamma_h)/N_h > 0. \tag{1}$$

Both N and γ are distributed uniformly and independently of each other and over time

Preferences and endowments

Each young agent chooses real saving $x \in [0, w]$ and has information I

The payoff to a young agent who chooses x is

$$P(x) = u(w - x) + E_I v(Rx), \tag{2}$$

where R, an endogenous random variable, is the real return on x, and E_I denotes expectation conditional on I

As in Lucas, the functions u and v are strictly increasing, strictly concave, twice differentiable with $u'(0) = v'(0) = \infty$, and v satisfies (a slight strengthening of gross substitutes)

$$\frac{xv''(x)}{v'(x)} \in (-1, -a) \text{ for some } a > 0. \tag{3}$$

real GDP at a date is Nx

First reason

No matter what the young know about current realizations of the two shocks, there are 3 stationary CE allocations with *trade*:

- one implies a positive correlation between real GDP and $1 + \gamma$ (L-allocation)
- one has *constant* per capita output (C-allocation);
- one has per capita output dependent only on the realization of N (N-allocation)

Second reason

Comes from implications of a strategic formulation: the *market-game* or *trading-post* model of Shapley and Shubik

Why use a strategic formulation?

CE solution concept has two well-known shortcomings:

- people face prices which come from nowhere
- payoffs are defined only for equilibrium actions

The first is especially significant because prices may *convey* information in this model

In addition, given the CE multiplicity, a strategic formulation has the potential to select an allocation

Why use the market-game

Simplicity

- one post per date: money trades for some amount of the single good
- the old offer all their money

Competition is defined to be a limit under replication of the number of agents and that limit, as is well known, gives a CE allocation

However, in the market-game, the limit is determined by what the young know when they choose an amount of output to save

Information in the market game

The young know the model and the pre-transfer total quantity of money, but do not know γ and do not know real saving at the previous date

Three alternatives

(i) the young know nothing (about the current realizations)

(ii) the young know $(1 + \gamma)/N$

(iii) the young know N

Results

Active-trade, stationary, equilibrium allocations			
	what the young know		
	(i) nothing	(ii) $(1+\gamma)/N$	(iii) N
market-game limit	C	L	N
CE	{C,L,N}	{C,L,N}	{C,L,N}

Second reason: L is not a market-game limit under (i), the Lucas assumption

The market-game formulation

A simultaneous-move quantity game

The person who saves x acquires money

$$m(x, x_-) = \frac{M}{(N-1)x_- + x}x, \tag{4}$$

where x_- is the saving of each other member of the current cohort, N is the size of the current cohort, and M is the post-transfer total stock of money

Consumption when old, the argument of v, is

$$x' = \frac{N'x_+}{(1+\gamma')M}(1+\gamma')m(x, x_-) = \frac{N'x_+}{(N-1)x_- + x}x, \tag{5}$$

where γ' is the next-date transfer, N' is the size of the next cohort, and x_+ is their per-person saving

Because the real return R in $P(x)$ is defined to be x'/x, we get

$$R = \frac{N'x_+}{(N-1)x_- + x}. \tag{6}$$

or with k-replication

$$R_k = \frac{kN'x_+}{(kN-1)x_- + x} = \frac{N'x_+}{(N-1/k)x_- + x/k}. \tag{7}$$

Then we have the following definition of an equilibrium.

Definition. For a given k and a given specification of the information known by the young, $x = x_- = x_+ = \hat{x}$ is a symmetric, stationary, active-trade Nash equilibrium if

$$\hat{x} = \arg\max_x P(x) = \arg\max_x [u(w-x) + E_I v(Rx)]$$

when R is given by (7).

Information and the domain of x

In this definition, x, x_-, and x_+ are each functions, whose domain varies across the different versions of the model.

Version (i) (the young know nothing about current realizations): x is a scalar and the equilibrium condition is a single equation, the first-order condition, $\partial P(x)/\partial x = 0$, evaluated at $x = x_- = x_+$.

Version (ii) (the young know the ratio of the current shocks): the domain is the three-element set of such ratios,

$$\{(1+\gamma_l)/N_h \equiv \rho_l, (1+\gamma_l)/N_l = (1+\gamma_l)/N_l \equiv \rho_m, (1+\gamma_h)/N_l \equiv \rho_h\},$$

and the equilibrium condition is three simultaneous first-order conditions at equality in $\hat{x} = (\hat{x}_l, \hat{x}_m, \hat{x}_h)$

Version (iii) (the young know N): the domain is the two-element set $\{N_l, N_h\}$ and the equilibrium condition is two simultaneous first-order conditions in $\hat{x} = (\hat{y}_l, \hat{y}_h)$

Information and the limiting (uniform) $(1/k = 0)$ distributions of R

Version (i): The support is $\{\frac{N_l}{N_h}, \frac{N_l}{N_l}, \frac{N_h}{N_h}, \frac{N_h}{N_l}\}$

Version (ii): If ρ_l, then support is $\{\frac{N_h\hat{x}_l}{N_h\hat{x}_l}, \frac{N_l\hat{x}_m}{N_h\hat{x}_l}, \frac{N_h\hat{x}_m}{N_h\hat{x}_l}, \frac{N_l\hat{x}_h}{N_h\hat{x}_l}\}$; if ρ_m, then support is

$\{\frac{N_h\hat{x}_l}{N_h\hat{x}_m}, \frac{N_l\hat{x}_m}{N_h\hat{x}_m}, \frac{N_h\hat{x}_m}{N_h\hat{x}_m}, \frac{N_l\hat{x}_h}{N_h\hat{x}_m}, \frac{N_h\hat{x}_l}{N_l\hat{x}_m}, \frac{N_l\hat{x}_m}{N_l\hat{x}_m}, \frac{N_h\hat{x}_m}{N_l\hat{x}_m}, \frac{N_l\hat{x}_h}{N_l\hat{x}_m}\}$;

if ρ_h, then support is $\{\frac{N_h\hat{x}_l}{N_l\hat{x}_h}, \frac{N_l\hat{x}_m}{N_l\hat{x}_h}, \frac{N_h\hat{x}_m}{N_l\hat{x}_h}, \frac{N_l\hat{x}_h}{N_l\hat{x}_h}\}$

Version (iii): If N_l, then the support is $\{\frac{N_l\hat{y}_l}{N_l\hat{y}_l}, \frac{N_h\hat{y}_h}{N_l\hat{y}_l}\}$; if N_h, then the support is $\{\frac{N_l\hat{y}_l}{N_h\hat{y}_h}, \frac{N_h\hat{y}_h}{N_h\hat{y}_h}\}$

One result

Proposition. For any k and for each informational version, there exists a symmetric, stationary, active-trade Nash equilibrium $\hat{x}$ which is unique. Moreover, the limit of $\hat{x}$ as $k \to \infty$ is a stationary, active trade CE.

Remarks about the proof: We follow Lucas and use (3), gross substitutes, to show that the mapping from future saving to present saving implied by the first-order conditions is a uniform contraction for all sufficently large k.

An alternative proof would not rely on gross substitutes. It would follow Wallace 92 which relies on ideas in Rody Manuelli's Ph.D. dissertation: bound saving away from zero at $1/k = 0$ and use a max mapping from future saving to current saving which allows Brouwer to be applied. To get convergence: show that generically any such fixed point is *regular* which allows application of the implicit function in a neighborhood of $1/k = 0$.

The correlation between γ and Nx in the L-allocation

Let $\rho_l = \frac{1+\gamma_l}{N_h}$, $\rho_m = \frac{1+\gamma_l}{N_l} = \frac{1+\gamma_h}{N_h}$, $\rho_h = \frac{1+\gamma_h}{N_l}$ and let $\hat{x}_l$, $\hat{x}_m$, $\hat{x}_h$ be the respective $\hat{x}'$s in the L- allocation, where it follows from the gross substitutes assumption that $x_l < x_m < x_h$. In the L-allocation, we have

$$\text{if } \gamma = \gamma_l \text{, then } Nx = \begin{cases} N_l\hat{x}_m \text{ with prob } 1/2 \\ N_h\hat{x}_l \text{ with prob } 1/2 \end{cases},$$

and

$$\text{if } \gamma = \gamma_h \text{, then } Nx = \begin{cases} N_l\hat{x}_h \text{ with prob } 1/2 \\ N_h\hat{x}_m \text{ with prob } 1/2 \end{cases}.$$

It follows from $\hat{x}_l < \hat{x}_m < \hat{x}_h$ that the correlation between γ and Nx is positive

Concluding remarks

Next task: extend the analysis to games with richer strategies, strategies that look a bit like supply functions

Conjecture: if the young see nothing and if we allow contingent saving choices, where the contingencies are the 4-point support of (N, γ), then only the N-allocation is a limiting allocation of the resulting game