Economics 4905
Financial Fragility and the Macroeconomy
Cornell University
Fall 2017
Solutions to the Practice Questions for Prelim 2

1. The Overlapping Generations Model

The model is set up as follow:

- 2 period lives
- 1 commodity per period, $\ell = 1$
- Stationary environment
- 1 person per generation

The utility functions are given as:

$$u_0(x_0^1) = x_0^1$$
$$u_t(x_t^t, x_t^{t+1}) = (x_t^t)^\alpha (x_t^{t+1})^\beta \text{ for } t = 1, 2, \ldots$$

The endowments are 1 unit each person each period she is alive:

$$\omega_0^1 = \omega_t^t = \omega_t^{t+1} = 1 \text{ for } t = 1, 2, \ldots$$

Define:

$$z^t = \omega_t^t - x_t^t$$
$$z^{t+1} = x_t^{t+1} - \omega_t^{t+1}$$

Case 1: $\alpha = 1, \beta = 10, m_0^1 = 1, m_s^t = 0$ otherwise
Case 2: $\alpha = 5, \beta = 1, m_0^1 = 1, m_s^t = 0$ otherwise

For each of the above cases, solve for the following:

a) The equilibrium demand (x_t^t, x_t^{t+1})

Solution:
The consumer problem is:

$$\max_{x_t^t, x_t^{t+1}} \quad (x_t^t)^\alpha (x_t^{t+1})^\beta$$
$$\text{subject to} \quad p^t x_t^t + p^{t+1} x_t^{t+1} = p^t \omega_t^t + p^{t+1} \omega_t^{t+1}$$

The solution to the consumer problem is

$$x_t^t = \frac{\alpha}{\alpha+\beta}\frac{p^t\omega_t^t + p^{t+1}\omega_t^{t+1}}{p^t}$$
$$x_t^{t+1} = \frac{\beta}{\alpha+\beta}\frac{p^t\omega_t^t + p^{t+1}\omega_t^{t+1}}{p^{t+1}}$$

Case 1:

$$x_t^t = \frac{1}{11}\left(1 + \frac{p^{t+1}}{p^t}\right)$$
$$x_t^{t+1} = \frac{10}{11}\left(\frac{p^t}{p^{t+1}} + 1\right)$$

Case 2:

$$x_t^t = \frac{5}{6}\left(1 + \frac{p^{t+1}}{p^t}\right)$$
$$x_t^{t+1} = \frac{1}{6}\left(\frac{p^t}{p^{t+1}} + 1\right)$$

b) The offer curve (OC)

Solution:

From Problem Set 4, the offer curve is:

$$z^{t+1} = \frac{\alpha\omega_t^{t+1}z^t}{\beta\omega_t^t - (\alpha+\beta)z^t}$$

Case 1:

$$z^{t+1} = \frac{z^t}{10 - 11z^t}$$

Case 2:

$$z^{t+1} = \frac{5z^t}{1 - 6z^t}$$

c) The steady states

Solution:
Case 1:
Setting $\bar{z} = z^t = z^{t+1}$:

$$\bar{z} = \frac{\bar{z}}{10 + 11\bar{z}}$$

Solving for $\bar{z}$ gives:

$$\bar{z} = \frac{9}{11} \text{ or } 0$$

There are two steady states.

Case 2:
There is only one steady state $z = 0$.

d) The set of equilibrium money prices, $\mathcal{P}^m$

Solution:
Case 1:
The set of equilibrium money prices is

$$\mathcal{P}^m = [0, 9/11]$$

Case 2:
The set of equilibrium money prices is

$$\mathcal{P}^m = [\{0\}$$

e) The full dynamic analysis, including the stability of steady states

Solution:
Case 1:
If $0 < P^m < \frac{9}{11}$, then z^t is declining, and the bubble fades away through inflation. $z = 0$ is a stable steady state, in which money is worthless $P^m = 0$. $z = \frac{9}{11}$ is an unstable steady state. If $z > \frac{9}{11}$, hyperinflation ensues and the bubble bursts in finite time. We may note that this is the Samuelson case.

Case 2:
The non-monetary steady state where $P^m = 0$ is unstable, unique and Pareto optimal. Trajectories originating away from it will be deflationary. This is the Ricardo case.

2. Bank Runs

The probability of being impatient is $\lambda = 0.4$. The type (patient or impatient) is realized in period 1 and it is private information. The utility function is:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma},$$

where $\gamma = \frac{3}{2} > 0$. Each individual has one unit of endowment in period 0. There is costless storage. If the endowment $\omega = 1$ is invested in period 0 and is harvested in period 1, the return is 0. If harvested late, the return is 4. Assume that the banking industry is free-entry.

(a) What is the depositor's *ex-ante* expected utility W as a function of c_1 consumption in period 1, and c_2, consumption in period 2?

Solution:
$W = \lambda u(c_1) + (1 - \lambda)u(c_2)$
$W = \frac{\lambda c_1^{1-\gamma}}{1-\gamma} + \frac{(1-\lambda)c_2^{1-\gamma}}{1-\gamma}$

(b) Show that the depositor prefers consumption smoothing.

Solution:
$u'(c) = \frac{(1-\gamma)c_1^{-\gamma}}{1-\gamma} = c^{-\gamma} > 0$
$u''(c) = -\gamma c^{-\gamma-1} < 0$

So $u(c)$ is strictly concave (And the consumer is risk-averse. Concave functions lie above their chords (Jensen's inequality):
$u(\lambda c_1 + (1-\lambda)c_2) > \lambda u(c_1) + (1-\lambda)u(c_2)$ when $c1 \neq c2$,

So $W(\bar{c}, \bar{c}) > W(c_1, c_2)$ where $\bar{c} = \lambda c_1 + (1-\lambda)c_2$.

(c) What is the bank's resource constraint RC? Write this down precisely and explain in words.

Solution:

$(1-\lambda)d_2 \leq (1-\lambda d_1)R$
Where d_t is the withdrawal allowed in period $t = 1, 2$.

The LHS of the inequality is the funds to be withdrawn in period 2. The RHS is the resources available at the bank in period 2. If the inequality is violated, the bank is insolvent.

(d) What is the incentive problem? Write down the incentive constraint IC precisely, and explain it in words.

Solution:
$d_1 \leq d_2$ is the ICC.

If the inequality does not hold, every depositor will attempt to withdraw early: the depositors would not self-select correctly.

(e) Solve for the optimal deposit contract for the post-deposit bank assuming that there is no run. (That is: Write down the optimal first-period payment d_1^* as a function of λ, R and γ)

Solution:
The bank will act to maximize $W(d_1, d_2)$ while constrained by the resources such that $(1-\lambda)d_2 = (1-\lambda d_1)R$.

$$\arg\max_{d_1,d_2}\{W(d_1, d_2)\} \Rightarrow \arg\max_{d_1,d_2}\{\lambda u(c_1) + (1-\lambda)u(c_2)\}$$

$$\text{Subject to } (1-\lambda)d_2 - (1-\lambda d_1)R = 0$$

We may therefore write the Lagrangian

$$\mathscr{L}(d_1, d_2) = [\lambda u(d_1) + (1-\lambda)u(d_2)] - \delta[(1-\lambda)d_2 - (1-\lambda d_1)R]$$

So then the first order conditions are:

$$\frac{\partial \mathscr{L}}{\partial d_1} = \lambda u'(d_1) - \delta\lambda R = 0 \quad \Rightarrow \quad \lambda u'(d_1) = \delta\lambda R \;\Rightarrow\; u'(d_1) = \delta R$$

$$\frac{\partial \mathscr{L}}{\partial d_2} = (1-\lambda)u'(d_2) - \delta(1-\lambda) = 0 \quad \Rightarrow \quad (1-\lambda)u'(d_2) = \delta(1-\lambda) \;\Rightarrow\; u'(d_2) = \delta$$

Thus, it follows that, as in Problem 1:

$$\frac{u'(d_1)}{u'(d_2)} = \frac{\delta R}{\delta} = R$$

(If you're studying for Prelim 1, I recommend remembering that the optimization will yield this equation, often known as a kind of Euler equation. It's also equivalent to setting the MRS equal to the marginal utilities, in this two-dimensional case.)

Recall that $u'(c) = c^{-\gamma}$,

$$\frac{d_1^{-\gamma}}{d_2^{-\gamma}} = \left(\frac{d_1}{d_2}\right)^{-\gamma} = \left(\frac{d_2}{d_1}\right)^{\gamma} = R$$

So then $\frac{d_2}{d_1} = R^{\frac{1}{\gamma}}$. We may then may recall the resource constraint to state that, if $(1-\lambda)d_2 = (1-\lambda d_1)R$ then $d_2 = \frac{(1-\lambda d_1)R}{(1-\lambda)}$. Therefore:

$$\frac{(1-\lambda d_1)R}{d_1(1-\lambda)} = R^{\frac{1}{\gamma}} \;\Rightarrow\; (1-\lambda d_1) = R^{\frac{1}{\gamma}-1}d_1(1-\lambda)$$

Dividing through by d_1,

$$\frac{1}{d_1} - \lambda = R^{\frac{1}{\gamma}-1}(1-\lambda) \;\Rightarrow\; \frac{1}{d_1} = R^{\frac{1}{\gamma}-1}(1-\lambda) + \lambda$$

Finally, it becomes clear that

$$d_1^* = \frac{1}{R^{\frac{1}{\gamma}-1}(1-\lambda) + \lambda}$$

There was some confusion as to what was meant by "the rate of return is equal to 4". Some students took this as the return factor was $R = 4$, as in the previous problem. Others took it as $R = (1+r) = (1+4) = 5$. Calculations using both values have been given credit, although $R = 4$ is the more correct, literal interpretation of the problem statement.

Suppose we set $R = 4$.

$$d_1^* = \frac{1}{\lambda + (1-\lambda)R^{1/\gamma-1}} = \frac{1}{0.4 + 0.6 \cdot 4^{(1/\frac{3}{2}-1)}} = 1.2854$$

$$d_2^* = d_1^* \cdot R^{1/\gamma} = \frac{R^{1/\gamma}}{\lambda + (1-\lambda)R^{1/\gamma-1}} = 1.2854 \cdot 4^{\frac{2}{3}} = 3.2390$$

(f) Is there a run equilibrium to this “optimal contract”?

Solution:

Since $\gamma > 1$, we have $d_1^* > 1$, so a run would be possible.

However, if $\gamma < 1$, it holds that $d_1^* < \omega = 1$. The contract is dominant strategy incentive-compatible (DSIC). If patient depositors run, the bank will still have sufficient funds to pay off all depositors. Thus, there will not be a run equilibrium in this case.

3. The 2-Consumer, Pre-Deposit Bank

The utility function of the impatient agent is

$$u(x) = \frac{Ax^{1-\gamma}}{1-\gamma}$$

and the utility function of the patient agent is

$$v(x) = \frac{x^{1-\gamma}}{1-\gamma},$$

where $\gamma > 1$. The probability λ of being impatient is 50%. The parameter γ is 1.01. The endowment is y = 3. The rate of return on the asset if harvested late is 50%, i.e., R = 1.5. The probability of being first in line if 2 agents withdraw early is 50%.

(a) Solve for the numerical values of c^{early} and c^{wait}. Show that they are independent of the impulse parameter A.

Solution:

Refer to the lecture slides on the pre-deposit game and the related paper. From slide 17, we have:

$c^{wait} = 4.280878 > 4.155955 = c^{early}$ Hence, the parameters are "usual".

(b) Write down the expression for the *ex-ante* expected utility of the depositor, W.

Solution:

$c^*(s) = \arg \max_{c \in [0, c^{wait}]} W(c, s)$

where

$$W(c; s) = \begin{cases} \hat{W}(c) & c \leq c^{early} \\ (1-s)\hat{W}(c) + sW^{run}(c) & c \in (c^{early}, c^{wait}) \end{cases}$$

Where $\hat{W}(c)$ is welfare in the post-deposit game non-run equilibrium $W^{run}(c)$ is the welfare in the run equilibrium.

(c) Solve for $\hat{c}$, the value of c that maximizes W in the post-deposit game, as a function of A.
Solution:
$\hat{c} = \frac{2y}{(\frac{p}{2-p} + \frac{2(1-p)}{(2-p)AR\gamma-1})^{1/\gamma}+1} = \frac{6}{(\frac{1}{3}+\frac{1}{1.5^{1.01}A})^{1/1.01}+1}$

(d) Calculate the critical values A^{early} and A^{wait}.
Solution:
$A^{early} = 6.217686 < 10.27799 = A^{wait}$

(e) Let A = 7. Describe the optimal contract for the pre-deposit game, $c^*(s)$, as a function of s, the exogenous run probability.
Solution:
$\hat{c}(A = 7) = 4.19054$.

We are in Case 2. $c^*(s = 0) = \hat{c} = 4.19054$. c is strictly declining in s until $s = s_0 = .0003468$. For $s > s_0$, c is independent of s and equal to $c^{early} = 4.155955$.

4. Bank Runs

Consider the Diamond-Dybvig bank run model. The probability λ of being impatient is 50%. The utility function is:

$$u(c) = 10 - \frac{1}{(0.5)\sqrt{c}}.$$

The rate of return to the asset harvested late is 400%, i.e.,

$$R = 5.$$

(a) What is the depositor's *ex-ante* expected utility W as a function of c_1, consumption in period 1, and c_2, consumption in period 2?

Solution:
$W = \lambda u(c_1) + (1 - \lambda)u(c_2)$
$W = \lambda(10 - \frac{1}{0.5\sqrt{c_1}}) + (1 - \lambda)(10 - \frac{1}{0.5\sqrt{c_1}}) = \frac{1}{2}\left(10 - \frac{1}{0.5\sqrt{c_1}}\right) + \left(1 - \frac{1}{2}\right)\left(10 - \frac{1}{0.5\sqrt{c_2}}\right)$ So

$$W = 5 - \frac{1}{\sqrt{c_1}} + 5 - \frac{1}{\sqrt{c_2}} = 10 - \frac{1}{\sqrt{c_1}} - \frac{1}{\sqrt{c_2}}$$

(b) Show that the depositor prefers consumption smoothing.

Solution:

$u''(c) < 0$. Hence, $u(c)$ is a strictly concave. A concave functions lies above its chords (Jensen's inequality):
$u(\lambda c_1 + (1 - \lambda)c_2) > \lambda u(c_1) + (1 - \lambda)u(c_2)$ when $c_1 \neq c_2$,
That is she prefers $(\bar{c}, \bar{c})$ to (c_1, c_2) where $\bar{c} = \lambda c_1 + (1 - \lambda)c_2$.

(c) Why can't she insure on the market against liquidity shocks?

Solution:
Her type is purely her own private information. The insurance company would not trust her to report her type truthfully. She would say that she is impatient even if she is not.

Assume that her endowment is 100 and that she deposits her entire endowment in the bank.

(d) What is her utility W in autarky?
Solution:

$W_{autarky} = \lambda u(100) + (1-\lambda)u(500) = 9.855$

(e) What is her utility W under perfect smoothing, i.e. when $c_1 = c_2$?
Solution:

$W_{perfect-smoothing} = u(\lambda \cdot 100 + (1-\lambda) \cdot 500) = 9.88$

(f) What is the bank's resource constraint RC? Write this down precisely. Explain this in words.
Solution:

$$(0.50)(d_2) \leq (\omega - (0.50)d_1)R$$

Or

$$(0.50)(d_2) \leq (100 - (0.50)d_1) \cdot 5$$

Period-2 withdrawals cannot exceed period-2 bank resources. This may be re-written as

$$d_2 \leq 2(100 - 0.50 \cdot d_1) \cdot 5$$
$$d_2 \leq 1000 - 5 \cdot d_1$$

(g) What is the incentive problem? Write this down precisely and explain in words the incentive constraint IC.
Solution:

$d_1 \leq d_2$ (ICC)

If ICC does not hold, everyone will seek to withdraw in period 1.

(h) Find the optimal deposit contract for this bank. What is W if there is no run?
Solution:

$$\arg\max_{d_1, d_2} \lambda u(d_1) + (1-\lambda)u(d_2)$$

subject to RC and ICC.
Using Lagrangian Optimization (see the lecture notes, or Problem 2), we may find that

$$\frac{u'(d_1^*)}{u'(d_2^*)} = R$$

Since $u(c) = 10 - \frac{1}{(0.5)\sqrt{c}}$, it follows that $u'(c) = \frac{d}{dc}\left(10 - 2c^{-1/2}\right) = -2 \cdot \left(-\frac{1}{2}\right) c^{-3/2} = c^{-3/2}$,

$$\frac{(d_1^*)^{-3/2}}{(d_2^*)^{-3/2}} = \left(\frac{d_1^*}{d_2^*}\right)^{-3/2} = R \quad \Rightarrow \quad \left(\frac{d_2^*}{d_1^*}\right)^{3/2} = R$$

And therefore

$$\frac{d_2^*}{d_1^*} = R^{2/3} \quad \Rightarrow \quad d_2^* = d_1^* R^{2/3}$$

Recalling the resource constraint, $d_2^* = 1000 - 5 \cdot d_1^*$. Thus,

$$1000 - 5 \cdot d_1^* = d_1^* \cdot 5^{2/3} \quad \Rightarrow d_1^*(5 + 5^{2/3}) = 1000$$

So

$$d_1^* = \frac{1000}{(5 + 5^{2/3})} = 126.2$$

$$d_2^* = 1000 - 5 \cdot 126.2 = 369.0$$

As such, $d_1^* = 126.2, d_2^* = 369.0$
$W_{no-run} = 0.5u(d_1^*) + 0.5u(d_2^*) = 9.859$

(i) Why is there a run equilibrium for this bank?
Solution:

$d_1^* = 126.20 > 100$

If every depositor attempted to withdraw at once (not just the impatient ones, but the impatient ones, too), then the bank will not be able to pay everyone at once

(j) Calculate the following numerical values of *ex-ante* utility W and and rank them in numerical ascending order: $W_{autarky}$, $W_{perfect\ smoothing}$, $W_{no\ run}$, W_{run}.
Solution:
$W_{run} = (100/d_1^*)u(d_1^*) = 7.783$

$W_{run} < W_{autarky} < W_{no-run} < W_{perfectsmoothing}$

(k) Assume that the run probability s is 2%. Will individuals deposit in this bank? That is, will they accept this banking contract? Explain.
Solution:

$(.02)W_{run} + (.98)W_{no-run} = (0.02)(7.783) + (0.98)(9.86) = 9.812 < 9.855 = W_{autarky}$

Consumers will *not* deposit at the bank if there is a 2% run probability.