1 Question 1, Banking (15 minutes)

The probability of being impatient is $\lambda = 0.4$. The type (patient or impatient) is realized in period 1 and it is private information. The utility function is:

$$u(c) = \log(c),$$

where $c$ is consumption. Each individual has one unit of endowment in period 0. There is costless storage. If the endowment $\omega = 1$ is invested in period 0 and is harvested in period 1, the output is 1. If harvested late, the output is $R = 4$. Assume that the banking industry is free-entry. The bank contract is defined by $(d_1, d_2)$ where $d_t$ is the payment to the withdrawal in period $t$, $t = 1, 2$.

a) What is the depositor’s ex-ante expected utility $W$ as a function of $c_1$, consumption in period 1, and $c_2$, consumption in period 2?

**Solutions:**

$$W(c_1, c_2) = 0.6 \log(c_1) + 0.4 \log(c_2)$$

b) Show that the depositor prefers consumption smoothing.

**Solutions:**

Take the second derivative of the utility function:

$$u'(c) = \frac{1}{c} \quad u''(c) = -\frac{1}{c^2} < 0$$

Hence, $u(c)$ is a strictly concave. A concave function lies above its chords (Jensen’s inequality):

$$u(\lambda c_1 + (1 - \lambda) c_2) > \lambda u(c_1) + (1 - \lambda) u(c_2)$$

for $c_1 \neq c_2$. That is she prefers $(\bar{c}, \bar{c})$ to $(c_1, c_2)$ where $\bar{c} = \lambda c_1 + (1 - \lambda) c_2$. 

c) What is the bank’s resource constraint RC? Write this down precisely and explain it in words.

**Solutions:**

\[
\lambda d_2 \leq (1 - \lambda d_1)R \\
0.4d_2 \leq 4(1 - 0.6d_1)
\]

d) What is the incentive problem? Write down the incentive constraint IC precisely, and explain it in words.

**Solutions:**

\[
d_1 \leq d_2
\]

e) Suppose that the optimal contract is given by \( d_1^* = 1 \). Is there a run equilibrium to this “optimal contract”? If yes, what is the value of the ex-ante utility in the run equilibrium, \( W_{\text{run}} \)?

**Solutions:**

There will not be a run.

## 2 Question 2, Overlapping Generations (45 minutes)

- 2 period lives
- 1 commodity per period, \( \ell = 1 \)
- Stationary environment
- 1 person per generation

The utility functions are:

\[
\begin{align*}
u_0(x_0^1) &= 7 \log x_0^1 \\
u_t(x_t^t, x_{t+1}^t) &= \log x_t^t + 7 \log x_{t+1}^t \text{ for } t = 1, 2, \ldots
\end{align*}
\]

where \( x_s^t \) is the demand of generation \( s \) at period \( t \). The endowments are 5 units for each period of a person’s life:

\[
\omega_0^1 = \omega_t^t = \omega_{t+1}^t = 5 \text{ for } t = 1, 2, \ldots
\]

Define the excess demand/supply:

\[
\begin{align*}
z^t &= \omega_t^t - x_t^t \\
z_{t+1} &= x_{t+1}^t - \omega_{t+1}^t
\end{align*}
\]

Let \( m_s^t \) be the money transfer to generation \( s \) at period \( t \). The values are:

\[
\begin{align*}
m_0^1 &= 1 \\
m_s^t &= 0 \text{ otherwise}
\end{align*}
\]
a) For Mr $t$ ($t = 1, 2, \ldots$), write the equation for the indifference curve (IC) through the endowment and sketch the IC curve.

**Solutions:**

\[ \log x_t^t + 7 \log x_t^{t+1} = \log 5 + 7 \log 5 = 8 \log 5 \]

b) What is the slope of the IC at the endowment?

**Solutions:**

Taking the derivative of the utility function with respect to $x_t^t$ and $x_t^{t+1}$:

\[
\frac{\partial u(x_t^t, x_t^{t+1})}{\partial x_t^t} = \frac{1}{x_t^t} \quad \quad \frac{\partial u(x_t^t, x_t^{t+1})}{\partial x_t^{t+1}} = \frac{7}{x_t^{t+1}}
\]

The slope of the IC at the endowment is:

\[
\frac{\partial u(5, 5)}{\partial x_t^t} = \frac{1/5}{7/5} = \frac{1}{7}
\]

c) What is the interest rate $r$ at the endowment?

**Solutions:**

The interest rate is:

\[
1 + r = \frac{1}{7} \quad \quad r = -\frac{6}{7}
\]

d) Is the economy Samuelson or Ricardo? Why?

**Solutions:**

The economy is Samuelson because $r < 0$.

e) Derive the equation of the offer curve (OC).

**Solutions:**

Using the equation from Problem Set 4:

\[
z_t^{t+1} = \frac{\alpha \omega_t^{t+1} z^t}{\beta \omega_t^t - (\alpha + \beta) z^t}
\]

we have

\[
z_t^{t+1} = \frac{(1)(5)z^t}{(7)(5) - (1 + 7)z^t} = \frac{5z^t}{35 - 8z^t}
\]

f) Draw and label the reflected, translated OC in the first quadrant.

**Solutions:**

Refer to lecture for diagram.
g) What are the stationary solutions?

**Solutions:**
Set \( z^t = z^{t+1} = \bar{z} \):

\[
\bar{z} = \frac{5\bar{z}}{35 - 8\bar{z}}
\]

Solving for \( \bar{z} \) gives

\( \bar{z} = 0 \) or \( 15/4 \)

h) What is the maximum equilibrium price of money \( \bar{P}^m \)?

**Solutions:**
Since \( m_0^1 = 1 \), the maximum equilibrium price of money is

\( \bar{P}^m = \frac{15}{4} \)

i) Do the full dynamic analysis.

**Solutions:**
If \( 0 < \bar{P}^m < \frac{15}{4} \), then \( z^t \) is declining, and the bubble fades away through inflation. \( z = 0 \) is a stable steady state, in which money is worthless \( \bar{P}^m = 0 \). \( z = 15/4 \) is an unstable steady state. If \( z > 15/4 \), hyperdeflation ensues and the bubble bursts in finite time.

j) Assume that \( \bar{P}^m = 3 \). What are the values of \( z_0^1, z_1^1 \) and \( z_2^1 \)?

**Solutions:**
Since \( m_0^1 = 1 \), the initial old’s excess demand is

\( z_0^1 = 3 \)

Using market clearing condition, the initial old’s excess demand has to be equal to Mr 1’s excess supply at \( t = 1 \).

\( z_1^1 = z_0^1 = 3 \)

Using the OC, Mr 1’s excess demand at \( t = 2 \) is

\[
z_2^1 = \frac{5z_1^1}{35 - 8z_1^1} = \frac{5(3)}{35 - 8(3)} = \frac{15}{11}
\]

k) Do the full welfare analysis.

**Solutions:**
When \( \bar{P}^m = 15/4 \), the allocation is Pareto optimal. Otherwise, it is not.