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The Overlapping-Generations Model. II. The Case of Pure Exchange with Money*

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1. INTRODUCTION

Government debt instruments (e.g., money and government bonds) serve many functions in the private sector. They can be stores of value, vehicles for the payment of taxes, media for transactions, and so forth. Their roles vary from economy to economy depending upon institutions and conventions. One of the roles, however, is basic to all others: if a government debt instrument does not serve as a value store, then it cannot serve any other useful function.

In the present article, the paper assets created by the government are assumed to have two functions: (1) they are of course, potential stores of value; and (2) they are used by households in paying their taxes and by the government in making transfers. In addition, we make the extreme assumptions that markets are complete and transactions are costless. There is then no essential economic difference between the various forms of government debt; hence, we shall refer to all forms of "government paper" as simply "money."

Money does not in general serve as a proper store of value—i.e., money cannot have a positive price—in the finite-horizon economy in which the

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terminal date is known with certainty. The reason is obvious. Money is worthless at the end of the final period. Consequently, in the next-to-last period, individuals desire to dispose of money holdings in order to avoid capital losses. This drives the price of money to zero at the end of the next-to-last period. And so on. Individuals with foresight drive the price of money to zero in each period, i.e., the "general price level" in equilibrium must be infinite.

The natural way to permit money to be a proper store of value is to go beyond the finite-horizon model. Our analysis is cast, therefore, in terms of the overlapping-generations economy, first analyzed by Samuelson [17]. Samuelson's model is especially attractive for macroeconomic analysis. It is genuinely dynamic, reflecting the vitality and mortality of consumers, and the unbounded, unidirectional flow of time. Furthermore, the model is basically disaggregative—clearly distinguishing between the tastes and opportunities of the individual consumers.

In the present paper, we treat a pure-exchange economy. We go well beyond the analysis of [9, 11, 13, 15, 17, 18], allowing for many commodities in each period, for tastes and endowments which vary from individual to individual, and for *arbitrary* government monetary policies. In Section 2, the basic model is introduced and the concept of *monetary competitive equilibrium* is defined. The consequences of equilibrium for the money markets are examined in Section 3. We then define the closely related, but mathematically simpler, concept of *Walrasian equilibrium*. We focus on equilibrium in the markets for real commodities in Section 4.

A *monetary policy* is a sequence of money taxes and money transfers. For every monetary policy, there is at least one competitive equilibrium. Money, however, might be irrelevant in the sense that the equilibrium price of money is identically zero. In Section 5, we study *bonafide monetary policies*, those policies which permit the existence of a nontrivial price of money. A sequence of commodity prices and a sequence of individual incomes is said to be a *price-income equilibrium* if aggregate demands for commodities at those prices and incomes is equal to the given aggregate resources of the economy. We provide a characterization of the monetary policies consistent with a given price-income equilibrium. We proceed to establish that if long-run interest rates exceed long-run growth rates, then bonafide monetary policies entail long-run money stocks of zero. On the other hand, if asymptotically the money supply is growing faster than the real growth rates, then any proper monetary equilibrium is not Pareto-optimal.

In Section 6, we show that the set of bonafide monetary policies is connected. It is also shown that any Pareto-optimal allocation can be achieved as a competitive equilibrium if the government has control over monetary policy and the price of money, even if endowments of real commodities cannot be reassigned. A monetary policy is said to be *potent* if for some price of money the resulting equilibrium allocation is Pareto-optimal. We show that the set of potent monetary policies is connected.

The set of equilibrium money prices, given fixed endowments and a fixed monetary policy, is investigated in Section 7. A natural bijection relates this set to a subset of the set of bonafide monetary policies. If the monetary policy is neither trivial nor pathological, the set of equilibrium money prices is bounded. Typically, there is a vast multiplicity of such monetary equilibria; this has important consequences for monetary theory and policy.

In the Appendix, we establish the connectedness of the set of weakly-Pareto-optimal allocations and the arc-connectedness of the set of Pareto-optimal allocations in the overlapping-generations model. A list of some basic notation appears as a Glossary, which follows the Appendix.

2. The Monetary Economy

The pure-exchange, overlapping-generations model of Balasko and Shell [5] is extended to allow for consumer money holdings; the basic assumptions about the "real" economy along with most of the corresponding notation are taken from [5]. Our extension to the monetary case allows the government to pursue any arbitrary (active or inactive) monetary policy; no restriction is placed on the sequence of nominal lump-sum money transfers and money taxes.

We summarize the basic features of the formal model. For analytic convenience, we adopt the simple demographic pattern analysed in [5], but recall that this is done without serious loss in generality (see [4, Sect. IV]). Consumers, who are indexed by their order of birth $h = 0, 1, \ldots$ are either present at the inception of the economy (in which case they live out the balance of their lives during period 1) or are born at the beginning of some period t ($t = 1, 2, \ldots$) and live out the whole of their lives in periods t and $t+1$. Each generation consists of a single consumer indexed uniquely by its birthdate t ($t = 0, 1, \ldots$).

In each period t ($t = 1, 2, \ldots$) there are l completely perishable commodities. There is also a completely imperishable fiat money. There is no production. Let $x_t^{s,i}$ be consumption of commodity i ($i = 1, \ldots, l$) by consumer t in period s. Consumer t derives utility from consuming goods during his lifetime; thus utility functions can be represented as

$$u_t(x_t) \qquad \text{for} \quad t = 0, 1, \ldots,$$

where

$$x_0 = x_0^1 = (x_0^{1,1}, \ldots, x_0^{1,l}) \in \mathbb{R}_{++}^l \qquad \text{for} \quad t = 0$$

and

$$x_t = (x_t^t, x_t^{t+1}) = (x_t^{t,1},..., x_t^{t,l}, x_t^{t+1,1},..., x_t^{t+1,l}) \in \mathbb{R}^{2l}_{++} \quad \text{for} \quad t = 1, 2,....$$

When convenient, we also denote by x_0 and x_t the respective sequences

$$x_0 = (x_0^1, 0,..., 0,...)$$

and

$$x_t = (0,..., 0, x_t^t, x_t^{t+1}, 0,...) \qquad \text{for} \quad t \geq 1.$$

Let $x = (x_0, x_1,...)$ be the sequence of commodity allocations and X be the set of all such allocation sequences, i.e., $X = \mathbb{R}^l_{++} \times \mathbb{R}^{2l}_{++} \times \mathbb{R}^{2l}_{++} \times \cdots$.

The utility functions $u_0(\cdot)$ and $u_t(\cdot)$ are assumed to have strictly positive first-order partial derivatives (i.e., to be smoothly monotonic) and to be strictly quasi-concave (i.e., to exhibit diminishing marginal rates of commodity substitution). Furthermore, in order to rule out complicated boundary behavior, the closure of every indifference surface in $\mathbb{R}^l$ (resp. $\mathbb{R}^{2l}$) is assumed to be contained in the corresponding strictly positive orthant $\mathbb{R}^l_{++}$ (resp. $\mathbb{R}^{2l}_{++}$). Each consumer has strictly positive endowments of the commodities available during his lifetime,

$$\omega_0 = \omega_0^1 = (\omega_0^{1,1},..., \omega_0^{1,l}) \in \mathbb{R}^l_{++}$$

and

$$\omega_t = (\omega_t^t, \omega_t^{t+1}) = (\omega_t^{t,1},..., \omega_t^{t,l}, \omega_t^{t+1,1},..., \omega_t^{t+1,l}) \in \mathbb{R}^{2l}_{++} \qquad \text{for } t \geq 1.$$

When convenient, we also denote by ω_0 and ω_t the respective sequences

$$\omega_0 = (\omega_0^1, 0,..., 0,...)$$

and

$$\omega_t = (0,..., 0, \omega_t^t, \omega_t^{t+1}, 0,...) \qquad \text{for} \quad t \geq 1.$$

Let $\omega = (\omega_0, \omega_1,...)$ denote the positive sequence of commodity endowments in X.

We turn to the monetary aspects of the model. The government cannot redistribute commodity endowments but can create nominal money and distribute it to individuals. It can also levy nominal money taxes on individuals. We assume that transfers and taxes are made in a lump-sum fashion during the lifetimes of individual consumers and that they are perfectly foreseen. As a result of the government's policy, therefore, consumers can be considered to be "endowed" with nominal money (in possibly positive, zero, or negative amounts) for each period of their lives. Monetary transfers (alternatively, endowments of money) are

$$m_0^1 \in \mathbb{R} \qquad \text{for} \quad t = 0$$

and

$$(m_t^t, m_t^{t+1}) \in \mathbb{R}^2 \qquad \text{for} \quad t = 1, 2, \ldots,$$

where m_t^s is the lump-sum money transfer to consumer t in period s. Since only the living can receive transfers or pay taxes, we have that for $t \geqslant 1$, $m_t^s = 0$ if $s < t$ or $s > t+1$, while $m_0^s = 0$ for $s \neq 1$. The nominal supply of money extant at time t is

$$m^t = \sum_{s=1}^{t} \sum_{\tau} m_\tau^s \in \mathbb{R},$$

where $\sum_\tau m_\tau^s$ is the aggregate amount of money created by the government in period s.

The government's control of the economy is exercised through its monetary transfers, which can be summarized by the sequence $m = (m_0^1, m_1^1, m_1^2, \ldots, m_t^t, m_t^{t+1}, \ldots)$ in the space of feasible monetary transfers, $M = \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \cdots$. With only one type of government debt instrument, no useful distinction can be drawn between monetary and fiscal policy. We adopt the convention of the overlapping-generations literature by referring to the sequence $m \in M$ as *monetary policy*. The reader should be warned, however, that in the classical macroeconomics literature the sequence $m \in M$ is frequently referred to as fiscal policy, leaving for "monetary policy" the composition of the public debt between monetary and nonmonetary instruments.

Just as an individual chooses his lifetime consumption profile, he must also choose a lifetime profile of money holdings. Let $x_t^{s,m} \in \mathbb{R}$ be the gross addition to his holdings of money committed by consumer t in period s. If $x_t^{s,m}$ is positive, he is increasing his inventory of money; if negative, he is decreasing his inventory. If $(x_t^{s,m} - m_t^s)$ is positive, consumer t is a net purchaser of money in period s, since he is committing to inventory more money than he receives in transfers. If $(x_t^{s,m} - m_t^s)$ is negative, he is a net seller of money. If $x_t^{s,m} = m_t^s$, he is neither a net purchaser nor a net seller of money: he salts away his entire transfer, or, alternatively, he purchases only enough money to cover his taxes.

Only the government is permitted to create outside money, i.e., money which is not redeemed. Hence, all individual money holdings are constrained to be nonnegative at death, yielding constraints which take the form

$$x_0^{1,m} \geqslant 0$$

and (2.1)

$$x_t^{t,m} + x_t^{t+1,m} \geqslant 0 \qquad \text{for} \quad t \geqslant 1,$$

under the realistic convention that $x_t^{s,m} = 0$ for $s > t + 1$.

Define the sequence of gross additions to money holdings by

$$x^m = (x_0^{1,m}, x_1^{1,m}, x_1^{2,m}, \ldots, x_t^{t+1,m}, \ldots).$$

Each consumer can buy and sell on the spot market and the forward market at perfectly foreseen prices. Let $p^{t,i}$ denote the price of commodity i in period t. Denote by p^t the vector $(p^{t,1}, \ldots, p^{t,l}) \in \mathbb{R}^l_{++}$ and by p the price sequence $(p^1, p^2, \ldots)$. We choose the normalization $p^{1,1} = 1$ and thus restrict attention to the set of sequences of *present* prices,

$$\mathscr{S} = \{p \mid p^{1,1} = 1\}.$$

Denote by $p^{t,m} \in \mathbb{R}_+$ the *present* price (in terms of the *numéraire* commodity) of money delivered in period t.

Each consumer's lifetime consumption profile and profile of money holdings is then the solution of a budget-constrained utility-maximization problem

$$\begin{aligned}
&\text{maximize } u_0(x_0^1)\\
&\text{subject to}\\
&\qquad p_1 \cdot x_0^1 + p^{1,m}x_0^{1,m} \leqslant p^1 \cdot \omega_0^1 + p^{1,m}m_0^1 = w_0,\\
&\qquad x_0^{1,m} \geqslant 0, \qquad x_0^1 \geqslant 0, \qquad\qquad\qquad \text{for } t = 0,\\
&\text{maximize } u_t(x_t^t, x_t^{t+1}) \qquad\qquad\qquad\qquad\qquad\qquad (2.2)\\
&\text{subject to}\\
&\qquad p^t \cdot x_t^t + p^{t+1} \cdot x_t^{t+1} + p^{t,m}x_t^{t,m} + p^{t+1,m}x_t^{t+1,m}\\
&\qquad \leqslant p^t \cdot \omega_t^t + p^{t+1} \cdot \omega_t^{t+1} + p^{t,m}m_t^t + p^{t+1,m}m_t^{t+1}\\
&\qquad = w_t,\\
&\qquad x_t^{t,m} + x_t^{t+1,m} \geqslant 0, \qquad (x_t^t, x_t^{t+1}) \geqslant 0, \qquad \text{for } t \geqslant 1,
\end{aligned}$$

where w_t then denotes the present value of consumer t's wealth.

2.3 DEFINITION. Let $\omega \in X$ be a sequence of strictly positive commodity endowments and $m \in M$ be a sequence of monetary transfers. A *monetary competitive equilibrium* associated with ω and m is a sequence of commodity prices $p \in \mathscr{S}$ and a sequence of nonnegative money prices $(p^{1,m}, p^{2,m},...)$ (along with the associated consumption profiles $x \in X$ and individual gross monetary additions x^m) satisfying the market-clearing conditions

$$\sum_t x_t = \sum_t \omega_t$$

and

$$x_{t-1}^{t-1,m} + x_{t-1}^{t,m} + x_t^{t,m} \leqslant m^t \qquad \text{for} \quad t \geqslant 1,$$

where $x_0^{0,m} = 0$, and $(x_{t-1}^{t-1,m} + x_{t-1}^{t,m} + x_t^{t,m})$ and m^t are, respectively, the aggregate demand for and supply of money in period t, and where x and x^m are solutions to the system (2.2) for given ω and m.

3. EQUILIBRIUM IN THE MONEY MARKETS

We derive some basic consequences of equilibrium in the money markets. These results are of interest in their own right, and will be used in Section 4 to develop a simplified representation of (Walrasian) equilibrium, which can replace the monetary-competitive-equilibrium concept defined in Secion 2.

3.1 PROPOSITION. *Let the price sequences $p \in \mathscr{S}$ and $(p^{1,m}, p^{2,m},...) \geqslant 0$ be a monetary competitive equilibrium for the endowment sequence $\omega \in X$ and the monetary transfer sequence $m \in M$. Then, the present price of money is a nonnegative constant, i.e., $p^{t,m} = p^m \geqslant 0$ for $t = 1, 2,...$.*

Proof. Assume that for some t, $p^{t,m} \neq p^{t+1,m}$. We must show that this is inconsistent with monetary competitive equilibrium. The budget constraints of consumer t $(t \geqslant 1)$ can be rewritten (from system (2.2)) as

$$\begin{aligned} p^t \cdot x_t^t + p^{t+1} \cdot x_t^{t+1} \leqslant p^t \cdot \omega_t^t &+ p^{t+1} \cdot \omega_t^{t+1} + p^{t,m}(m_t^t - x_t^{t,m}) \\ &+ p^{t+1,m}(m_t^{t+1} - x_t^{t+1,m}) \\ x_t^{t,m} + x_t^{t+1,m} \geqslant 0. \end{aligned}$$

Assume, for example, that $p^{t+1,m} > p^{t,m}$. By setting $x_t^{t+1,m} = -x_t^{t,m}$ and choosing an arbitrarily large purchase of money in period t, so that $x_t^{t,m}$ is arbitrarily large, along with a correspondingly large sale of money in period $t+1$, so that $(-x_t^{t+1,m})$ is arbitarily large, consumer t is able to afford any consumption profile $(x_t^t, x_{t+1}^t) \in \mathbb{R}_{++}^{2l}$. Since the utility function $u_t(\cdot)$ is strictly increasing this entails a contradiction to the equilibrium requirement that materials balance in the commodity market, i.e., $\sum_t x_t = \sum_t \omega_t$. ∎

3.2 *Remark*. Proposition (3.1) is a zero-profit condition, excluding the possibility of arbitrage from intertemporal trading in money.

Since no separate sign restrictions are imposed on $x_t^{t,m}$ or $x_t^{t+1,m}$, the model can be interpreted as having perfect borrowing and lending markets, or perfect markets for outside money *and* inside money. Indeed, the constancy of the *present* price of money p^m has a familiar interpretation equivalent to the standard conditions for perfect "capital" markets. In our model, money serves only as a store of value. Also, the (nominal) own rate of interest on money is assumed to be zero. Hence money's *current* price in terms of the lth commodity, $(p^{t,m}/p^{t,l})$, must in equilibrium increase at the commodity rate of interest $\rho^{t,l}$, where by definition we have

$$\rho^{t,l} = \frac{p^{t,l} - p^{t+1,l}}{p^{t+1,l}} = \frac{p^{t,l}}{p^{t+1,l}} - 1.$$

The current commodity price of money, $(p^{t,m}/p^{t,l})$, is increasing at the commodity interest rate if and only if the present price of money, $p^{t,m}$, is constant $(=p^m \geqslant 0)$.

Proposition (3.2) allows us to employ a more condensed notation. We say that $q \in \mathcal{Q} = \{(p, p^m) \mid p \in \mathcal{P} \text{ and } p^m \in \mathbb{R}_+\}$ defines a monetary competitive equilibrium for $\omega \in X$ and $m \in M$ if the sequences p and $(p^m, p^m, \ldots)$ satisfy Definition (2.3).

3.3. Proposition. *Let the sequence $q = (p, p^m) \in \mathcal{Q}$ define a monetary competitive equilibrium for $\omega \in X$ and $m \in M$. Then, it follows that*

$$p^m x_0^{1,m} = 0$$

and

$$p^m(x_t^{t,m} + x_t^{t+1,m}) = 0 \qquad \textit{for} \quad t \geqslant 1.$$

Proof. Because utility functions are strictly increasing, a solution to the budget-constrained utility maximization system (2.2) must have $x_0^{1,m} = 0$ and $x_t^{t,m} + x_t^{t+1,m} = 0$ whenever $p^m > 0$. ∎

There is no individual bequest motive in our model. Proposition (3.3) states that when money has positive value, it is not optimal for any individual to die holding positive money balances.

3.4. PROPOSITION. *Let $q=(p, p^m) \in \mathscr{Q}$ define a monetary competitive equilibrium for given sequences $\omega \in X$ and $m \in M$. Then the associated equilibrium allocations, $x \in X$ and $x^m \in \mathbb{R}^\infty$, must satisfy the following constraints*:

$$p^1 \cdot x_0^1 = -p^m(x_0^{1,m} - m_0^1) + p^1 \cdot \omega_0^1 \qquad \textit{for} \quad t=0$$

and

$$\begin{aligned} &p^t \cdot x_t^t + p^m(x_t^{t,m} - m_t^t) = p^t \cdot \omega_t^t, \\ &p^{t+1} \cdot x_t^{t+1} = -p^m(x_t^{t+1,m} - m_t^{t+1}) + p^{t+1} \cdot \omega_t^{t+1} \qquad \textit{for} \quad t \geqslant 1. \end{aligned} \tag{3.4.1}$$

Proof. Equation (3.4.1) for $t=0$ is derived from the System (2.2) and Proposition (3.1), using the facts that $u_0(\cdot)$ is strictly increasing and p^1 is positive. We next establish that Eqs. (3.4.1) hold for $t=1$. Because $u_1(\cdot)$ is strictly increasing and p^2 is positive, the system (2.2) yields

$$p^1 x_1^1 + p^m(x_1^{1,m} - m_1^1) + p^2 \cdot x_1^2 = p^1 \cdot \omega_1^1 - p^m(x_1^{2,m} - m_1^2) + p^2 \cdot \omega_1^2$$

after applying Propositon (3.1). Adding the above equation to Eqs. (3.4.1) for $t=0$ yields (3.4.1) for $t=1$ after substituting the equalities

$$p^1 \cdot (x_0^1 + x_1^1) = p^1 \cdot (\omega_0^1 + \omega_1^1)$$

and

$$p^m(x_0^{1,m} + x_1^{1,m}) = p^m(m_0^1 + m_1^1),$$

which are consequences of equilibrium (Definition 2.3). The remainder of the proof is by induction on t. Assume that Eqs. (3.4.1) hold for $t=s-1$. Then, by using the equilibrium conditions

$$\begin{aligned} p^s \cdot (x_{s-1}^s + x_s^s) &= p^s \cdot (\omega_{s-1}^s + \omega_s^s), \\ p^m(x_{s-1}^{s,m} + x_s^{s,m}) &= p^m(m_{s-1}^s + m_s^s), \end{aligned}$$

Eqs. (3.4.1) are established for $t=s>1$. ∎

3.5. *Remark*. Notice that at equilibrium two separate "budget constraints" are satisfied for consumer $t \geqslant 1$, cf. (3.4.1). Thus, at equilibrium prices, consumer t can be thought of as purchasing and selling commodities solely on the current spot market, while providing for his future by savings held in the form of nonperishable money.

3.6. PROPOSITION. *If x^m is a sequence of individual money additions with a constant present price of money $p^{t,m} = p^m > 0$ for $t = 1, 2, \ldots$, then*

$$\begin{aligned} x_0^{1,m} &= 0, && \\ x_t^{t,m} &= -x_t^{t+1,m} = m^t && \textit{for} \quad t \geqslant 1, \\ x_t^{s,m} &= 0 && \textit{otherwise.} \end{aligned} \tag{3.6.1}$$

Proof. Proposition (3.1) allows us to set $p^{t,m} = p^m$. From Proposition (3.3), $x_0^{1,m} = 0$ and $x_t^{t,m} = -x_t^{t+1,m}$ for $t \geqslant 1$. From the equilibrium condition for clearing in the money market, we have that $x_t^{t,m} = m^t$ for $t \geqslant 1$. ∎

3.7. *Remark.* Proposition (3.6) reflects our simple demographic assumptions. In each period, there are only two consumers: An "old" consumer in his last period of life, and a "young" consumer with only one more period of life remaining. In equilibrium, the old consumer exchanges all of his outside money for goods of the young consumer. With this simple generational structure, there is in equilibrium no role for inside money; the money holdings of the young equal the stock of outside money, or government debt. Proposition (3.6) also provides a "dichotomy" between the money markets and the commodity markets. One should not however, make too much of the seeming separation of the economy into monetary and real parts. The demand for money is a correspondence depending on the sequence $p^{1,m}, p^{2,m}, \ldots$. If for some s and t $(s \neq t)$, $p^{s,m} \neq p^{t,m}$, then the demand for money in some periods is arbitrarily large and the supply of money in some other periods is arbitrarily large. Furthermore, if $p^{s,m} \neq p^{t,m}$, then the demand for physical commodities is unbounded. If $p^{t,m} = p^m$ for $t = 1, 2, \ldots$, then individuals are indifferent as to how much money they should hold in their first periods of life. The constancy of the present price of money ensures that consumers are indifferent at the margin between a "dollar" of increased money holdings and a "dollar" of current consumption.

4. THE WALRASIAN EQUILIBRIUM EQUATIONS

We can derive from the system (2.2) individual demands for the physical commodities as functions of commodity prices $p \in \mathscr{P}$ and individual incomes $w \in W$. Thus, if f_t is the demand function for consumer t, then

$$\begin{aligned} f_0 &: S \times \mathbb{R}_{++} \to \mathbb{R}^l_{++}, \\ f_t &: S \times \mathbb{R}_{++} \to \mathbb{R}^{2l}_{++} \qquad \text{for} \quad t \geqslant 1, \end{aligned} \tag{4.1}$$

where

$$w_0 = p \cdot \omega_0 + p^m m_0^1,$$
$$w_t = p \cdot \omega_t + p^m(m_t^t + m_t^{t+1}) \qquad \text{for} \quad t \geq 1, \tag{4.2}$$

when attention is restricted to the case where $p^{t,m} = p^m$. In this case a monetary policy affects individual wealth through (4.2) and thus affects individual demands, f_0 and f_t.

Notice from (4.1) and (4.2) that the effect of a monetary transfer $m \in M$ on individual demands for commodities depends only on the sums $m_0^1, m_1^1 + m_1^2,..., m_t^t + m_t^{t+1},...$. This is an immediate consequence of perfect "capital" markets. A consumer is indifferent as to the timing of taxes and transfers as long as the total *present* value of his net tax bill is constant. We can thus rewrite (4.2) as

$$w_t = p \cdot \omega_t + p^m \mu_t \qquad \text{for} \quad t \geq 0, \tag{4.3}$$

where $\mu = (\mu_0, \mu_1,..., \mu_t,...) = (m_0^1, m_1^1 + m_1^2,..., m_t^t + m_t^{t+1},...)$ is a *monetary policy*. Let $\mathscr{M}$ be the set of feasible government monetary policies μ. Since no restrictions are imposed on these policies, $\mathscr{M}$ is $\mathbb{R} \times \mathbb{R} \times \cdots = \mathbb{R}^\infty$, the case of full monetary control.

We next define Walrasian equilibrium and then establish the relationship between this equilibrium concept and the concept of monetary competitive equilibrium given in Definition (2.3).

4.4. Definition. For given commodity endowments $\omega \in X$ and a given monetary policy $\mu \in \mathscr{M}$, a *Walrasian equilibrium* is a price system $q = (p, p^m) \in \mathscr{Q}$ which solves the following equations:

$$\sum_t f_t(p, w_t) = \sum_t \omega_t$$

and

$$w_t = p \cdot \omega_t + p^m \mu_t \qquad \text{for} \quad t \geq 0.$$

4.5. Proposition. *Let $q = (p, p^m)$ be a Walrasian equilibrium for $\omega \in X$ and $\mu \in \mathscr{M}$. Then $p \in \mathscr{S}$ and the sequence $p^{1,m}, p^{2,m},...$, where $p^{t,m} = p^m$ for $t = 1, 2,...$, is a monetary competitive equilibrium (Definition 2.1) for ω and $m = m_0^1, m_1^1, m_1^2,..., m_t^t, m_t^{t+1},...$ if $\mu = (m_0^1, m_1^1 + m_1^2,..., m_t^t + m_t^{t+1},...)$. Furthermore, the Walrasian allocation $f = (f_0, f_1,...)$ is also the monetary competitive allocation, $f = x = (x_0, x_1,...)$. If $p^m > 0$, competitive equilibrium money holdings are given by Proposition (3.9); if $p^m = 0$, any pattern of money holdings consistent with* *materials balance in the money market is also a competitive allocation of money.*

Proof. Follows directly (3.1), (3.3), (3.4), and (4.4). ∎

Proposition (4.5) allows us to loosely refer to "Walrasian equilibria" in what follows as "monetary competitive equilibria." Similarly, we refer to $\mu \in \mathscr{M}$ as a "monetary policy"; there should be no confusion with $m \in M$, also referred to as a "monetary policy."

4.6. Definition. Let $\mathscr{Q}(\omega, \mu)$ denote the *set of Walrasian equilibria* (alternatively, the *set of monetary competitive equilibria*) for $\omega \in X$ and $\mu \in \mathscr{M}$. Thus $\mathscr{Q}(\omega, \mu) = \{q \in \mathscr{Q} \mid \text{Definition (4.4) is satisfied for } \omega \in X \text{ and } \mu \in \mathscr{M}\}$. The price system $q = (p, p^m)$ is said to define a *proper monetary competitive equilibrium* for $\omega \in X$ and $\mu \in \mathscr{M}$ if $q \in \mathscr{Q}(\omega, \mu)$ *and* $p^m \neq 0$.

4.7. Proposition (The "Neutrality" of Money). *For each positive scalar* λ,

$$\mathscr{Q}(\omega, \lambda\mu) = \{p, (p^m/\lambda) \mid (p, p^m) \in \mathscr{Q}(\omega, \mu)\}.$$

That is, two economies which differ only in the units used for measuring money will have the same set *of equilibrium commodity prices and the same* set *of equilibrium* real *monetary transfers* $p^m\mu$.

Proof. From (4.3), $w_t = p \cdot \omega_t + p^m\mu_t$. Since the only effect of μ or p^m on f_t is through w_t, the result follows immediately. ∎

4.8. *Remark.* Proposition (4.7) is weaker than the usual "Quantity Theory of Money," which necessarily equates "doubling the nominal money supply" with "doubling the general price level." Here, if for *each period* the perfectly foreseen money transfer is doubled, then halving the present price of money is *consistent* with equilibrium, but is hardly necessary for equilibrium.

4.9. *Remark.* The constancy of the equilibrium price of money (Proposition (3.1)) depends on the assumption that the nominal rate of interest (the own rate of interest on money) is zero. To extend our analysis to the more general setting is quite simple. For example, if the nominal rate of interest is θ, the zero-profit condition becomes

$$\frac{p^{t+1,m}}{p^{t,m}} = \frac{1}{1+\theta},$$

so that a change in the nominal interest rate is exactly offset by an induced change in the inflation rate. In the more general setting, given ω the equilibrium allocation $f \in X$ is affected only by the sequence of real money transfers $(p^{1,m}m_0^1, \ldots, p^{t,m}m_t^t + p^{t+1,m}m_t^{t+1}, \ldots)$. As long as government policy does not affect this sequence, it does not affect the equilibrium allocation $f \in X$. In particular, if the government were to increase the nominal interest rate and the growth rate of nominal monetary transfers, all in the same proportion, then the set of equilibrium allocations would be unaffected. This simple property of the model is known as the *superneutrality of money*.

5. Bonafide Monetary Policies

From Definition (4.4), a Walrasian equilibrium $q = (p, p^m)$ is determined (not necessarily uniquely) by the demand functions, the endowments, and the government monetary policy. Fixing the demand functions f, if the government selects monetary policy μ when the endowments of the economy are ω, then we say that μ is an *ω-monetary policy*.

The purpose of monetary policy is to affect the equilibrium allocation $x = f$. If, for some nontrivial monetary policy μ, all Walrasian equilibria associated with the endowments ω have the property that $p^m = 0$, then the monetary policy μ cannot possibly have any affect on the Walrasian allocation, $x = f$. In this case, the government could have no "*good-faith*" expectation that its monetary policy would be effective. The monetary policy μ is then not *bonafide* for the economy with endowments ω.

5.1. Definition. The ω-monetary policy $\mu \in \mathscr{M}$ is said to be *ω-bonafide* if there is a *proper* monetary equilibrium $q = (p, p^m) \in \mathscr{P} \times \mathbb{R}_{++}$ associated with (ω, μ), i.e., there is a $q = (p, p^m) \in \mathscr{Q}(\omega, \mu)$ such that $p^m \neq 0$. The monetary policy $\mu \in \mathscr{M}$ is said to be *bonafide* if it is ω-bonafide for some $\omega \in X$.

Clearly, the existence of ω-bonafide and bonafide monetary policies is critical for the integration of money within the general-equilibrium model of overlapping generations.[1] We investigate in some detail the properties of these policies.

5.2. Proposition. *The set of ω-bonafide monetary policies and the set of bonafide monetary policies are cones.*

[1] The finite-horizon model is unsatisfactory for monetary analysis because only very special monetary policies permit a nonzero price of money. In the finite-horizon, n-consumer model, only policies in which the algebraic sum of taxes is zero, $\sum_1^n \mu_i = 0$, are bonafide. See, e.g., Balasko and Shell [7].

Proof. If μ is ω-bonafide, then by Definitions (4.4) and (5.1), there is $q = (p, p^m) \in \mathscr{Q}$ with $p^m \neq 0$ such that $q \in \mathscr{Q}(\omega, \mu)$. Then by Proposition (4.7), $(p, p^m/\lambda) \in \mathscr{Q}(\omega, \lambda\mu)$, where $\lambda > 0$. ∎

Thus, the neutrality of money guarantees that the set of ω-bonafide monetary policies and hence the set of bonafide monetary policies are cones. It suffices, therefore, to analyze convenient cross sections of these sets; such normalizations are provided in the following definition.

5.3. DEFINITION. The ω-bonafide (resp., bonafide) monetary policy $\mu \in \mathscr{M}$ is said to be *normalized* if $(p, 1)$ is a (proper) monetary competitive equilibrium associated with (ω, μ) (resp., with (ω, μ) for some ω). We denote by $\mathscr{M}_B(\omega) \subset \mathscr{M}$ (resp., $\mathscr{M}_B \subset \mathscr{M}$) the *set of normalized ω-bonafide* (resp., *normalized bonafide*) *monetary policies*. The set of ω-bonafide monetary policies is the positive cone in $\mathscr{M}$ generated by $\mathscr{M}_B(\omega)$ (resp., $\mathscr{M}_B$).

In what follows, we characterize the set of normalized ω-bonafide monetary policies in terms of the price sequences $q = (p, 1) \in \mathscr{Q}$ and income sequences $w = w_0, w_1, \ldots$ consistent with the existence of (proper) monetary competitive equilibria. To do this, we extend to the overlapping-generations context, the price-income equilibrium concept employed in standard equilibrium theory (see, e.g., Balasko [3]).

The resource (or aggregate endowment) sequence $r = \sum_t \omega_t$ is taken as fixed. We are interested in the set of prices and incomes (p, w) that solve $\sum_t f_t(p, w_t) = r$. Let μ be an ω-bonafide normalized monetary policy where $(p, 1)$ is the associated monetary competitive equilibrium, so that $w_t = p \cdot \omega_t + \mu_t$ for $t \geqslant 0$. Given the normalization $p^m = 1$, we also ask whether or not μ is consistent with a given price-income equilibrium (p, w). These ideas are formalized in the following definition.

5.4. DEFINITION. The sequence (p, w) is a *price-income equilibrium* associated with the total resources $r = (r^1, \ldots, r^t, \ldots)$ if and only if $\sum_t f_t(p, w_t) = r$. The ω-monetary (resp., monetary) policy μ is said to be *consistent with the price-income equilibrium* (p, w) if and only if $w_t = p \cdot \omega_t + \mu_t$ for $t \geqslant 0$ (resp., there is an ω such that $\sum_t \omega_t = r$ and $w_t = p \cdot \omega_t + \mu_t$ for $t \geqslant 0$).

Note that if μ is consistent with (p, w), then clearly $(p, 1)$ is a (proper) monetary equilibrium associated with (ω, μ) for some ω; in other words, μ is appropriately normalized.

The next proposition provides a characterization of the monetary policies consistent with a *given* price-income equilibrium.

5.5. PROPOSITION. *The (normalized) monetary policy μ is consistent with the price-income equilibrium (p, w) if and only if the following system of inequalities is satisfied*

$$-p^{t+1} \cdot f_{t+1}^{t+1}(p, w_{t+1}) < \sum_{l=0}^{t} \mu_l < p^{t+1} \cdot f_t^{t+1}(p, w_t) \qquad \text{for} \quad t \geqslant 0. \tag{5.5.1}$$

Proof. *Step* 1. Consider the equilibrium equations

$$\begin{aligned} f_0^1(p, w_0) + f_1^1(p, w_1) &= r^1, \\ \cdots \qquad \cdots \quad &= \cdots \\ f_t^{t+1}(p, w_t) + f_{t+1}^{t+1}(p, w_{t+1}) &= r^{t+1}. \end{aligned}$$

Note that we consider only the first $(t+1)$ equilibrium equations. Multiply equation i by p^i, and add up. Applying Walras' law to each individual demand function, we obtain

$$w_0 + w_1 + \cdots + w_t + p^{t+1} \cdot f_{t+1}^{t+1} = p^1 \cdot r^1 + \cdots + p^{t+1} \cdot r^{t+1}.$$

Therefore, we have

$$\sum_{l=0}^{t} w_l - \sum_{l=1}^{t+1} p^l \cdot r^l = -p^{t+1} \cdot f_{t+1}^{t+1}$$

and

$$\sum_{l=0}^{t} w_l - \sum_{l=1}^{t} p^l \cdot r^l = p^{t+1} \cdot (r^{t+1} - f_{t+1}^{t+1}) = p^{t+1} \cdot f_t^{t+1}.$$

We can replace the left-hand side and the right-hand side of (5.5.1) by the above expressions, yielding

$$\sum_{l=0}^{t} w_l - \sum_{l=1}^{t+1} p^l \cdot r^l < \sum_{l=0}^{t} \mu_l < \sum_{l=0}^{t} w_l - \sum_{l=1}^{t} p^l \cdot r^l \qquad \text{for} \quad t \geqslant 0. \tag{5.5.2}$$

Step 2. We now determine explicitly the sequence of endowments $\omega = (\omega_0, \ldots, \omega_t, \ldots)$ such that $(p, 1)$ is a monetary equilibrium associated with (ω, μ).

Start with $\omega_0 = \omega_0^1$. Clearly, we must have $p^1 \cdot \omega_0^1 = w_0 - \mu_0$. The inequality (5.5.2) reduces, in the case $t = 0$, to $w_0 - p^1 \cdot r^1 < \mu_0 < w_0$. We want an endowment ω_0^1 such that $0 < \omega_0^1 < r^1$. Such a solution exists if $0 < p^1 \cdot \omega_0^1 < p^1 \cdot r^1$, i.e., $0 < w_0 - \mu_0 < p^1 \cdot r^1$ or, equivalently,

$w_0 - p^1 \cdot r^1 < \mu_0 < w_0$. Therefore, (5.5.2) for $t = 0$ provides us with ω_0^1 such that

$$p^1 \cdot \omega_0^1 = w_0 - \mu_0$$
$$0 < \omega_0^1 < r^1.$$

Let $\omega_1^1 = r^1 - \omega_0^1$. We clearly have $0 < \omega_1^1 < r^1$. Let us now determine ω_1^2 such that

$$p^1 \cdot \omega_1^1 + p^2 \cdot \omega_1^2 = w_1 - \mu_1,$$
$$0 < \omega_1^2 < r^2.$$

We have

$$\begin{aligned} p^2 \cdot \omega_1^2 &= w_1 - \mu_1 - p^1 \cdot \omega_1^1 \\ &= w_1 - \mu_1 - p^1 \cdot (r^1 - \omega_0^1) \\ &= w_1 - \mu_1 - p^1 \cdot r^1 + w_0 - \mu_0 \\ &= w_0 + w_1 - p^1 \cdot r^1 - \mu_0 - \mu_1. \end{aligned}$$

There is an ω_1^2 satisfying

$$p^2 \cdot \omega_1^2 = w_0 + w_1 - p^1 \cdot r^1 - \mu_0 - \mu_1,$$
$$0 < \omega_1^2 < r^2,$$

if and only if $0 < p^2 \cdot \omega_1^2 < p^2 \cdot r^2$, i.e.,

$$0 < w_0 + w_1 - p^1 \cdot r^1 - \mu_0 - \mu_1 < p^2 \cdot r^2,$$

or, equivalently,

$$w_0 + w_1 - p^1 \cdot r^1 - p^2 \cdot r^2 < \mu_0 + \mu_1 < w_0 + w_1 - p^1 \cdot r^1,$$

which corresponds to (5.5.2) for $t = 1$.

The rest of the proof proceeds by induction. Therefore, we can assume that ω_t^t is determined if and only if the inequality (5.5.2) for $(t-1)$ holds. We also have

$$p_t \cdot \omega_t^t = -\sum_{l=0}^{t-1} w_l + \sum_{l=1}^{t} p^l \cdot r^l + \sum_{l=0}^{t-1} \mu_l.$$

Therefore, the problem is to find ω_t^{t+1} such that

$$w_t = p^t \cdot \omega_t^t + p^{t+1} \cdot \omega_t^{t+1} + \mu_t,$$
$$0 < \omega_t^{t+1} < r^{t+1}.$$

The first equality becomes

$$p^{t+1} \cdot \omega_t^{t+1} = \sum_{l=0}^{t} w_l - \sum_{l=1}^{t} p^l \cdot r^l - \sum_{l=0}^{t} \mu_l.$$

Therefore, there is an ω_t^{t+1} satisfying the above constraint if and only if

$$0 < p^{t+1} \cdot \omega_t^{t+1} < p^{t+1} \cdot r^{t+1},$$

which, as before, is easily transformed into

$$0 < \sum_{l=0}^{t} w_l - \sum_{l=1}^{t} p^l \cdot r^l - \sum_{l=0}^{t} \mu_l < p^{t+1} \cdot r^{t+1},$$

which corresponds to (5.5.2) for $t > 1$. ∎

The "only if" part of the proof of Proposition (5.5) is very easy. An alternative "only if" proof follows directly from Propositions (3.4) and (3.6). Given p, w, and μ, there is a positive ω_t^{t+1} for $t \geqslant 0$ only if

$$p^{t+1} \cdot f_t^{t+1}(p, w_t) > m^t + m_t^{t+1} = \sum_{l=0}^{t} \mu_l.$$

Also from Propositions (3.4) and (3.6), there is a positive ω_{t+1}^{t+1} for $t \geqslant 0$ consistent with p, w, and μ only if

$$\sum_{l=0}^{t} \mu_l > -p^{t+1} \cdot f_{t+1}^{t+1}(p, w_{t+1}).$$

Combining the inequalities yields the system (5.5.1).

5.6. *Remark.* The trivial monetary policy $\mu = 0$ obviously satisfies the inequality system (5.5.1). This is not surprising since the price-income equilibrium (p, w) can be associated with $(\omega, 0)$, where $\omega_t = f_t(p, w_t)$ for $t \geqslant 0$.

5.7. PROPOSITION. *Let $(p, 1)$ be a monetary competitive equilibrium associated with (ω, μ), i.e., $(p, 1) \in \mathscr{Q}(\omega, \mu)$. If the total resources $r = \sum_t \omega_t$ are bounded from above (i.e., there is $0 < K < +\infty$ such that $\|r^t\| < K$ for $t \geqslant 1$), then*

$$\|p^t\| > \frac{1}{K} \left\| \sum_{l=0}^{t-1} \mu_l \right\| \qquad \textit{for} \quad t \geqslant 1,$$

where for convenience we use the Euclidean norm.

Proof. From (5.5.1), we deduce that

$$\left\| \sum_{i=0}^{t-1} \mu_i \right\| < \max(\| p^t \cdot f_{t-1}^t \|, \| p^t \cdot f^t \|) \leqslant \| p^t \cdot r^t \| \leqslant \| p^t \| \, \| r^t \|,$$

so that

$$\| p^t \| > \frac{1}{\| r^t \|} \left\| \sum_{i=0}^{t-1} \mu_i \right\| > \frac{1}{K} \left\| \sum_{i=0}^{t-1} \mu_i \right\|. \quad ∎$$

Proposition (5.7) is important since it relates the dynamic behavior of the "money supply," $\{\sum_{i=0}^{t} \mu_i\}$, with properties of the sequence of commodity prices, $\{p^t\}$, which properties are known to be critical in determining whether or not the competitive allocation $\{f_t(p, w_t)\}$ is Pareto-optimal. (See Balasko and Shell [5, Sect. 5].)

First, we apply Proposition (5.7) to an important class of equilibria satisfying a long-run "positive interest-rate" condition: $\| p^t \| \to 0$ as $t \to \infty$. We know that such equilibria are Pareto-optimal if the allocation sequence $\{x_t\}$ is bounded from above; cf. [5, Proposition (5.3)].

5.8. PROPOSITION. *Assume*

> *Property* B: *The allocation sequence* $\{x_t\}$ *is bounded from above, so that there is* $0 < K < +\infty$ *such that* $\| r^t \| < K$ *for* $t \geqslant 1$.

Let $\mu = (\mu_0, \mu_1, ..., \mu_t, ...) \in \mathscr{M}$ *be a bonafide monetary policy admitting the proper monetary competitive equilibrium* (p, p^m) *(so that* $p^m \neq 0$*). If* $\| p^t \|$ *tends to zero as* t *tends to infinity, then* μ *necessarily satisfies the equality* $\sum_{t=0}^{\infty} \mu_t = 0$.

Proof. If $\mu \in \mathscr{M}_B(\omega)$ admits the monetary equilibrium $(p, 1)$, then from Proposition (5.7), we have that

$$\left\| \sum_{i=0}^{t-1} \mu_i \right\| < K \| p^t \| \qquad \text{for} \quad t \geqslant 1.$$

Since $\| p^t \| \to 0$ as $t \to \infty$ the right-hand side of this inequality approaches zero as t approaches infinity. Hence, the series $\{\sum_{0}^{t-1} \mu_i\}$ converges and $\sum_{0}^{\infty} \mu_t = 0$. Proposition (5.2) allows this result to be extended to (nonnormalized) bonafide monetary policies. ∎

Proposition (5.8) has a simple economic interpretation. If the present prices of commodities are falling to zero, the *current* price of money in terms of commodities is becoming arbitrarily large. Unless the endowment of commodities is growing, this situation is inconsistent with monetary equilibrium: ultimately the demand for commodities must exceed their supply.

From Proposition (5.8), we conclude that a "positive-interest-rate" economy with a nonpositive long-run real growth rate allows only a limited set of bonafide monetary policies, i.e., those satisfying the restriction that the money supply tends to zero as t becomes large. Such economies are like finite-horizon economies in that for a monetary policy to be bonafide, the algebraic sum of transfers must be zero; cf. [7]. Proposition (5.8) generalizes to the case of real economic growth (i.e., growth in resources, r^t): If the long-run interest rates exceed the long-run real growth rates, then the long-run money supply must be zero for any ω-bonafide monetary policy. On the other hand, if the rate of monetary growth is greater than the real rate of growth of the economy, then Pareto efficiency is ruled out, as the next proposition shows for an economy in which consumers satisfy some mild uniformity properties.

5.9. PROPOSITION. *Consider the monetary policy $\mu \in \mathscr{M}$ satisfying the growth condition*

$$\sum_{s=0}^{t} \mu_s \geqslant \mu_0(1+\gamma)^t \qquad \textit{for} \quad t \geqslant 0,$$

where γ is a positive scalar. Let $q = (p, p^m)$ be a proper monetary competitive equilibrium associated with μ and some $\omega \in X$, i.e., $(p, p^m) \in \mathscr{Q}(\omega, \mu)$ and $p^m \neq 0$ and let $x = \{x_t\} = \{f_t(p, p \cdot \omega_t + p^m \mu_t)\} \in X$ be the corresponding equilibrium allocation. Assume, furthermore, that consumers satisfy uniformity properties (a–e) *of Proposition* (5.6) *in Balasko and Shell* [5].[2] *Then, the allocation x is not Pareto-optimal.*

Proof. Since we include in our uniformity conditions the boundedness of the allocation sequence,we know from Proposition (5.7) that there is a positive scalar K such that

$$\|p^t\| > \frac{1}{K}\mu_0(1+\gamma)^{t-1} \qquad \text{for} \quad t \geqslant 1,$$

in the normalized case, where $p^m = 1$. It follows from Balasko and Shell [4], Proposition (5.6), that the resulting normalized monetary competitive equilibrium allocation, $x = f(p, w)$ is not Pareto-optimal. From

[2] A close reading of the proof of Proposition (5.6) in [5] shows that not all of the uniformity properties (a–e) are needed for Proposition (5.9). In particular, while the upper bound on the Gaussian curvature of indifference surfaces is crucial, we do not require here that the curvature be bounded above zero.

Proposition (5.2), we immediately deduce that the corollary applies to all proper monetary competitive equilibria, i.e., for any positive p^m. ∎

Our result can be taken as a generalization of a single-commodity steady-state result in Wallace [19] (cf. especially his Proposition 6), a provocative paper full of insights and results. Our Propositions (5.8) and (5.9) along with Proposition (5.6) in [5] strongly suggest that attention be focused on monetary policies for which the asymptotic growth rate of the money supply is equal to "the" asymptotic growth rate of the real economy. The equation of these rates of growth is consistent with some of the prescriptions of the Chicago School.[3] Nothing in our analysis, however, supports a restriction to a fixed short-run growth rate for money.

6. The Set of ω-Bonafide Monetary Policies

Proposition (5.5), the central result of the preceding section, provides a characterization of those bonafide monetary policies which are consistent with a given price-income equilibrium (p, w). This characterization aids us in recognizing whether or not a monetary policy $\mu \in \mathscr{M}$ is bonafide and consistent with (p, w) for some suitably assigned sequence of endowments of commodities $\omega \in X$.

These results are not, however, sufficient for the analysis of money as a policy instrument. For example, the government might consider choosing a monetary policy that would lead to allocations which are Pareto-superior to those attainable without money. Or, more generally, the government might attempt to promote through monetary policy an allocation which maximizes some interpersonal welfare or distributional criterion. In pursuing these goals, the government may well have no power to alter the endowments of physical commodities, $\omega \in X$, but is free to employ monetary policy, $\mu \in \mathscr{M}$, as a control variable.

In this section, we analyze properties of the set of ω-bonafide monetary policies. The first problem we consider is whether each Pareto-optimal (PO) allocation, $x \in X$, can be achieved as a monetary competitive allocation associated with the given endowments, $\omega \in X$ ($\sum_t x_t = \sum_t \omega_t$), and some $\mu \in \mathscr{M}_B(\omega)$, the set of normalized bonafide monetary policies.

6.1. Proposition. *There is one and only one monetary policy $\mu \in \mathscr{M}_B(\omega) \subset \mathscr{M}$ such that a given Pareto-optimal allocation $x = \{x_t\}$ is a*

[3] A precise citation to the oral tradition of the Chicago School is often difficult or impossible. For some clues, see Friedman [12, especially pp. 133–156]. The emphasis in [12] is on fixed rules. The desirability of equating real and monetary growth rates is, at best, implicit in [12].

competitive allocation associated with (ω, μ) *for some monetary competitive equilibrium* $(p, 1)$, *i.e., where* $x_t = f_t(p, w_t)$ *and* $w_t = p \cdot \omega_t + \mu_t$ *for* $t \geqslant 0$.

Proof. Since x is Pareto optimal (PO) and thus weakly Pareto optimal (WPO), there is a unique price sequence $p \in \mathscr{S}$ supporting x, i.e., $x = f_t(p, p \cdot x_t)$ for $t \geqslant 0$. (See [5, Lemma (4.3)].) Let $w_t = p \cdot x_t$ for $t \geqslant 0$. We have that $\sum_t f_t(p, w_t) = r$, and hence (p, w) is a price-income equilibrium (Definition (5.4)). To obtain x as the competitive allocation associated with (ω, μ) for the (proper) monetary competitive equilibrium $(p, 1)$, we merely require that $w_t = p \cdot \omega_t + \mu_t$, or $\mu_t = w_t - p \cdot \omega_t$, for $t \geqslant 0$. Given $x \in X$, $p \in \mathscr{S}$ is unique, hence $w \in W$ is unique, so that $\mu \in \mathscr{M}$ (as constructed above) exists and is unique. ∎

A result close to our Proposition (6.1) was established by Okuno and Zilcha in [15, Theorem 2].

To achieve the PO allocation x through monetary policy alone is always possible, but it is not assured. The Pareto-optimizing μ is unique in $\mathscr{M}_B(\omega)$, the set of *normalized* ω-bonafide monetary policies. The Pareto-optimizing ω-bonafide monetary policy, however, is in general of the form (μ/p^m) where the monetary competitive equilibrium is $(p, p^m) \in \mathscr{S} \times \mathbb{R}_{++}$. Thus the scale of the Pareto-optimizing ω-bonafide monetary policy is indeterminate. There is no assurance that the competitive economy will choose the present price of money (given μ and ω) to achieve the intended PO allocation. Proposition (6.1) then has the interpretation that to ensure a PO allocation x, the government must in general control both $\mu \in \mathscr{M}$ and $p^m \in \mathbb{R}_+$.

Let $\mathscr{B}$ (for "budget") denote the set of price-income equilibria, i.e.,

$$\mathscr{B} = \left\{ (p, w) \in \mathscr{S} \times W \mid w_t > 0 \text{ for } t \geqslant 0 \text{ and } \sum_t f_t(p, w_t) = r \right\},$$

where r, the sequence of total resources, is given and held constant. Let Φ_ω: $\mathscr{B} \to \mathscr{M}$ denote the mapping given by

$$(p, w) \mapsto (\mu_0 = w_0 - p \cdot \omega_0, \ldots, \mu_t = w_t - p \cdot \omega_t, \ldots).$$

6.2. Proposition. $\mathscr{M}_B(\omega) = \Phi_\omega(\mathscr{B}) = \operatorname{Im} \Phi_\omega$.

Proof. Obvious. ∎

Proposition (6.2) is a reflection of the simple structure of equilibrium in the money markets (cf. Sections 3 and 4). Proposition (6.2) simplifies the analysis of the set of ω-bonafide monetary policies, since variations of μ in $\mathscr{M}_B(\omega)$ can be related to variations of (p, w) in $\mathscr{B}$.

6.3. Proposition. *The monetary policy sequence* $0 \in \mathscr{M}$ *belongs to the set* $\mathscr{M}_B(\omega)$.

Proof. Let $p \in \mathscr{P}$ be an equilibrium price sequence associated with the *nonmonetary* economy with endowments $\omega \in X$ (cf. [5, Definition (2.3)]). Let $x = \{x_t\}$ be the corresponding competitive allocation. By definition, $w_t = p \cdot x_t = p \cdot \omega_t$ for $t \geqslant 0$ and hence $\Phi_\omega(p, w) = 0$ if $w = \{w_t\} = \{p \cdot \omega_t\}$. ∎

In general, $\mathscr{M}_B(\omega)$ is a proper subset of $\mathscr{M}$. There can be monetary policies $\mu \in \mathscr{M}$ which are not ω-bonafide; see, for example, the highly regular "classical-case" examples in Gale [13]. Regular examples can also be constructed in which the sequence 0 does not necessarily belong to the interior of $\mathscr{M}_B(\omega)$. (This last contrasts with our analysis of the regular *finite* economy, where in fact 0 belongs to the interior of the set of ω-bonafide monetary (or, alternatively, tax) policies; see Balasko and Shell [7].)

An interesting question which arises in the analysis of economic policy is whether the control instruments can be altered "continuously" within some set. We investigate here whether it is possible to move in a continuous way from one ω-bonafide monetary policy to another. The continuity properties of $\mathscr{M}_B(\omega)$ turn out to enable us to deduce properties of the set of monetary competitive equilibria; see Section 7. We also investigate the continuity of the set of monetary policies consistent with Pareto-optimality.

The natural concept of continuity in this context turns out to be that of connectedness.

6.4. Proposition. *The set $\mathscr{M}_B(\omega)$ is connected.*

Proof. The mapping Φ_ω: $\mathscr{P} \times W \to \mathscr{M}$ is continuous. The set of price-income equilibria $\mathscr{B}$ being connected (see Corollary (A.1.3) in the Appendix), its image $\Phi_\omega(\mathscr{B})$ is, therefore, connected (cf. Bourbaki [8, Chap. I, Sect. 11.3, Proposition 4]). The result follows after applying Proposition (6.2). ∎

The fact that PO allocations are potentially implementable through some bonafide monetary policies leads us to introduce the concept of a *potent* normalized monetary policy.

6.5. Definition. A monetary policy $\mu \in \mathscr{M}_B(\omega)$ is said to be *potent* if there exists a monetary competitive equilibrium $(p, 1)$ associated with (ω, μ) such that the allocation $x = \{x_t\} = \{f_t(p, p \cdot \omega_t + \mu_t)\}$ is Pareto-optimal.

Let $\mathscr{M}_P(\omega)$ denote the set of potent ω-monetary policies. Clearly, it follows that $\mathscr{M}_P(\omega) \subset \mathscr{M}_B(\omega) \subset \mathscr{M}$. Note that the sequence 0 does not necessarily belong to $\mathscr{M}_P(\omega)$; see, for example, the "Samuelson-case" example in Gale [13].

Let $\mathscr{B}_P$ denote the subset of the set of price-income equilibria which correspond to Pareto-optimal allocations, i.e., for which $\{f_t(p, w_t)\}$ is Pareto-optimal. It follows that $\mathscr{M}_P(\omega) = \Phi_\omega(\mathscr{B}_P)$.

6.6. PROPOSITION. *The set* $\mathscr{M}_p(\omega)$ *is connected.*

Proof. From Proposition (A.2.2) in the Appendix, it follows that $\mathscr{B}_p$ is arc-connected and hence connected. Therefore, $\Phi_\omega(\mathscr{B}_p) = \mathscr{M}_p(\omega)$ is connected. ∎

7. THE SET OF EQUILIBRIUM MONEY PRICES

Let $\mathscr{Q}(\omega, \mu)$ be the set of monetary competitive equilibria $q = (p, p^m)$ for given ω and μ. (Note that μ is neither necessarily ω-bonafide nor necessarily normalized.) The *set of equilibrium (present) money prices* associated with the monetary competitive equilibrium is denoted by

$$\mathscr{P}^m(\omega, \mu) = \{p^m \mid (p, p^m) \in \mathscr{Q}(\omega, \mu)\} \subset \mathbb{R}_+.$$

In this section, we establish some basic properties of the set $\mathscr{P}^m(\omega, \mu)$.

7.1. PROPOSITION. *The set* $\mathscr{P}^m(\omega, \mu)$ *is not empty.*

Proof. Take $p^m = 0$. Then $q = (p, 0)$ is a monetary competitive equilibrium if and only if p is an equilibrium of the nonmonetary economy defined by the endowment sequence ω. From [5], Proposition (3.10), we know that such an equilibrium sequence p exists. Then $(p, 0) \in \mathscr{Q}(\omega, \mu)$ and $0 \in \mathscr{P}^m(\omega, \mu)$. ∎

Proposition (7.1), although important, merely restates in the monetary framework the existence of equilibrium in the nonmonetary economy. It reminds us that fiat money's value is crucially dependent on the "faith" of consumers. No matter what monetary policy is pursued—bonafide or not—if consumers mistrust government debt it may well bear a zero value in equilibrium. Proposition (7.1) should be sharply contrasted with other monetary theories which claim a unique general price level which is positive and finite. It is not enough to assert that the community would benefit from a positive p^m if consumers refuse to accept that price as an equilibrium value.

We go on to investigate further the set $\mathscr{P}^m(\omega, \mu)$, now focusing on nontrivial equilibria.

7.2. PROPOSITION. $\mathscr{P}^m(\omega, \mu) \neq \{0\}$ *if and only if* μ *is an* ω*-bonafide monetary policy.*

Proof. Let $p^m \in \mathscr{P}^m(\omega, \mu)$ be such that $p^m \neq 0$. Clearly, the sequence $p^m\mu \in \mathscr{M}$ is an ω-bonafide normalized monetary policy. ∎

Fixing the monetary policy $\mu \in \mathscr{M}$, define

$$L(\mu) = \{\lambda\mu \mid \lambda \geqslant 0\} \subset \mathscr{M},$$

the ray generated by μ.

7.3. PROPOSITION. *Consider a nontrivial monetary policy $\mu \neq 0$ in $\mathscr{M}$. The set of equilibrium money prices $\mathscr{P}^m(\omega, \mu)$ is related to the set $\mathscr{M}_B(\omega) \cap L(\omega)$ by a one-to-one mapping.*

Proof. Map p^m from $\mathscr{P}^m(\omega, \mu)$ to $p^m\mu \in \mathscr{M}_B(\omega)$. This mapping is clearly a bijection, naturally arising because of the Neutrality of Money (Proposition (4.7)). ∎

The sets $\mathscr{M}_B(\omega)$ and $L(\mu)$ are depicted in Fig. 7.1. The plane of the page is intended to portray the infinite dimensional space $\mathscr{M}$ and the intersection of the axes is the point $\mu = 0$. The shaded set depicts $\mathscr{M}_B(\omega)$. $L(\mu)$ is the ray through the point μ. The set $\mathscr{M}_B(\omega) \cap L(\mu)$ is thus depicted by the two heavy line segments.

If the set $\mathscr{M}_B(\omega)$ were convex or even star-shaped about the origin, then the $\mathscr{M}_B(\omega) \cap L(\mu)$ would be an interval and thus $\mathscr{P}^m(\omega, \mu)$ would be an interval. In the general connected case, however, one can expect the set $\mathscr{M}_B(\omega) \cap L(\mu)$ to exhibit a rather complicated structure.

The set $\mathscr{P}^m(\omega, \mu)$ can be reduced to $\{0\}$ or to a proper interval; see [6, Proposition (3.2)]. We do not have a regular example where $\mathscr{P}^m(\omega, \mu)$ is a disconnected subset of $\mathbb{R}_+$, but it seems likely that one can construct such an example. Nonetheless, one cannot in general expect to have $\mathscr{P}^m(\omega, \mu) = \{0\} \cup \{p^{m^*}\}$, where p^{m^*} is the unique positive price of money. This has an important theoretical consequence: For a bonafide monetary policy, there is typically a (vast) indeterminacy of the general price level and a consequent indeterminacy of the resulting competitive allocation. This prevents the elaboration of a utility theory of money along the lines of Patinkin [15, Chaps V–VIII], where it is implied that the determination of p^m is unique at some positive level. In particular, Balasko and Shell [6] shows that for the log-linear case $\mathscr{P}^m(\omega, \mu)$ is an interval, so that for monetary models one cannot in general expect the equilibria to be isolated.

The set of equilibrium money prices may not be bounded from above. In particular, if the monetary policy is trivial, $\mu = 0$, then $\mathscr{P}^m(\omega, \mu) = \mathbb{R}_+$ for each $\omega \in X$. If the present price of money is arbitrarily large, then the current "price level" must be arbitrarily close to zero. This strange set of affairs is ruled out for usual monetary policies (those in which the first government action is to create a positive money stock). This is shown in the next proposition.

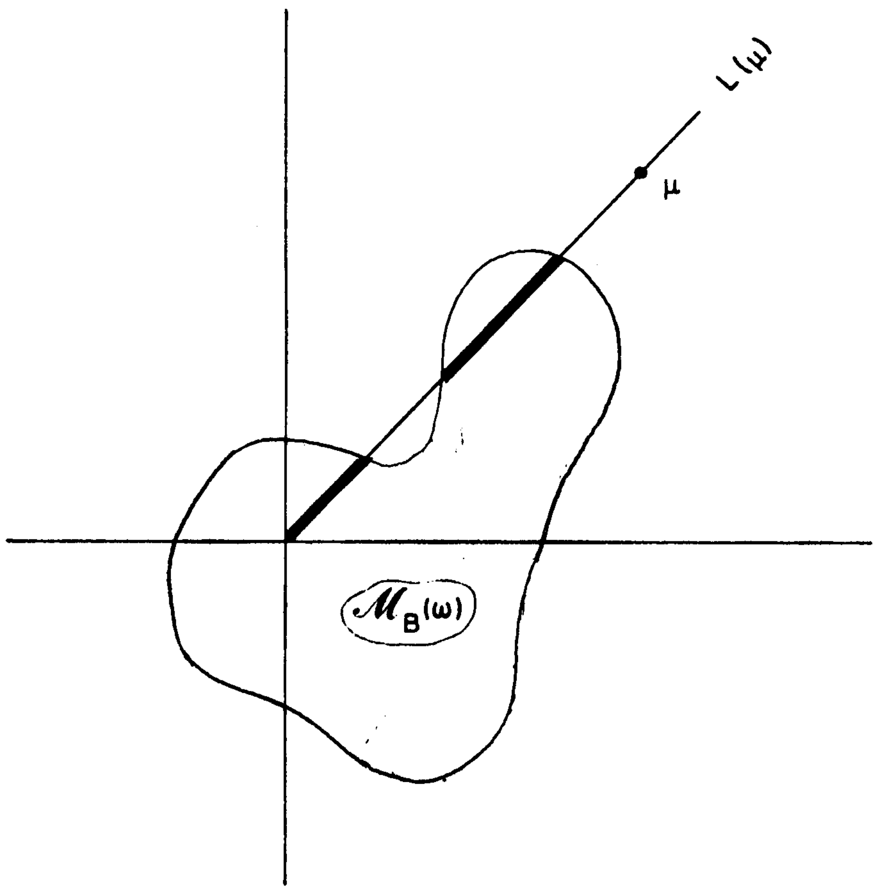

FIGURE 7.1

7.4. PROPOSITION. *Let* $\mu = (\mu_0, \mu_1, \ldots, \mu_t, \ldots) \in \mathscr{M}$. *Assume*:

(a) *For some* t, $\mu_t \neq 0$, *and*

(b) $\mu_{t_0} > 0$ *where* t_0 *is the first* t *with* $\mu_t \neq 0$,

Then, the set of equilibrium money prices $\mathscr{P}^m(\omega, \mu)$ *for the economy* (ω, μ) *is bounded if* ω *is bounded.*

The second hypothesis of Proposition (7.4) rules out pathological cases where the first nonzero monetary action of the government leads to the creation of a negative aggregate money supply.

Proof. The equilibrium allocation $x = (x_0, x_1, ..., x_{t_0}, x_{t_0+1})$ must satisfy the inequalities

$$\begin{aligned} &u_0(x_0) \geqslant u_0(\omega_0) \quad \text{and} \quad x_0 \leqslant r^1 \quad && \text{for} \quad t = 0, \\ &u_t(x_t) \geqslant u_t(\omega_t) \quad \text{and} \quad x_t \leqslant (r^t, r^{t+1}) \quad && \text{for} \quad t = 1, ..., t_0. \end{aligned} \tag{7.4.1}$$

Let (p, p^m) by a monetary competitive equilibrium. Because of the regularity of utility functions, we have from Inequalities (7.4.1) that prices $(p^1, ..., p^{t_0}, p^{t_0+1})$ are bounded above zero and bounded from above; cf. [5, Lemma (3.4)]. In particular there is $\beta^{t_0+1} \in \mathbb{R}^l_{++}$ such that $p^{t_0+1} \leqslant \beta^{t_0+1} < +\infty$. But, from Proposition (3.4), we have

$$p^{t_0+1} \cdot x^{t_0+1}_{t_0+1} + p^m(x^{t_0+1,m}_{t_0+1} - m^{t_0+1}_{t_0+1}) = p^{t_0+1} \cdot \omega^{t_0+1}_{t_0+1}.$$

From Proposition (3.6), we have $x^{t_0+1,m}_{t_0+1} = m^{t_0+1}$, so $x^{t_0+1,m}_{t_0+1} - m^{t_0+1}_{t_0+1} = \mu_{t_0}$. Hence we have

$$p^m \mu_{t_0} < p^{t_0+1} \cdot \omega^{t_0+1}_{t_0+1} \leqslant \beta^{t_0+1} \cdot \omega^{t_0+1}_{t_0+1} < +\infty,$$

since $\omega^{t_0+1}_{t_0+1}$ is bounded. The proposition then follows because μ_{t_0} is positive. ∎

Appendix

The purpose of this appendix is to establish connectedness (cf. Bourbaki [7, Chap. I, Sect. 11, Definition 1]) of the set of Pareto optima (PO) and the set of weak Pareto optima (WPO) in the pure-exchange, overlapping-generations model of Balasko and Shell [5]. These connectedness properties are interesting for their own sake and play a crucial role in the study of bonafide monetary policies and of potent monetary policies (cf. Section 6 of this paper, especially Propositions (6.4) and (6.6)).

A.1. *The Set of Weak Pareto Optima*

Let $\mathfrak{P}$ denote the set of weak Pareto optima (cf. Balasko and Shell [5, Definition 2.5]). The relationship between WPO allocations and competitive allocations has been clarified in [5, Proposition (4.4)]. We reformulate this relationship in the following form, where X, the space of feasible allocations, and $\mathscr{S} \times W$ are endowed with the product topology.

A.1.1. PROPOSITION. *The set of price-income equilibria* $\mathscr{B} = \{(p, w_0, ..., w_t, ...) \in \mathscr{S} \times W \mid \sum_t f_t(p, w_t) = r\}$ *is homeomorphic to the set of weak Pareto optima* $\mathfrak{P}$.

A.1.2. PROPOSITION. *The set of weak Pareto optimal $\mathfrak{P}$ is connected.*

A.1.3. COROLLARY. *The set of price-income equilibria $\mathscr{B}$ is connected.*

Proof of (A.1.3). Obvious from (A.1.1) and (A.1.2). ∎

Proof of (A.1.2). Several steps are required.

Step 1. Define the set $\mathfrak{P}_n(t)$ (where n and t are natural numbers) as the set of allocations $x = (x_0, x_1, \ldots, x_t, \ldots) \in X$ such that

(a) $u_0(x_0) \geqslant u_0(r^1/n)$,
$u_1(x_1) \geqslant u_1(r^1/n, r^2/n)$,
$\ldots$
$u_t(x_t) \geqslant u_t(r^t/n, r^{t+1}/n)$,
$\ldots$

(b) $\sum_t x_t^l \leqslant r^l, \qquad l = 1, 2, \ldots$, and

(c) the allocation $(x_0, x_1, \ldots, x_{t-1}, x_t^t)$ is Pareto optimal with respect to the utility functions $u_0(\cdot)$, $u_1(\cdot, \cdot), \ldots, u_{t-1}(\cdot, \cdot)$, $u_t(\cdot, x_t^{t+1})$.

We also need the set $\mathfrak{X}_n$ consisting of the elements $x \in X$ which satisfy (a) and (b). Clearly, $\mathfrak{X}_n$ is convex and compact.

Let $\mathfrak{U}_n(t-1)$ denote the subset of $\mathbb{R}^t$ which is defined as the image of $\mathfrak{X}_n$ by the mapping $x \mapsto u_0(x_0), \ldots, u_{t-1}(x_{t-1})$. Since $\mathfrak{X}_n$ is convex and hence arc-connected, $\mathfrak{U}_n(t-1)$ is arc-connected as the image of an arc-connected set by a continuous mapping.

Let $\mathfrak{P}_n(t, x_t^{t+1})$ denote the t-truncation of $\mathfrak{P}_n(t)$ associated with a *fixed* x_t^{t+1}, i.e., $\mathfrak{P}_n(t, x_t^{t+1}) = \{(x_0, x_1, \ldots, x_t^t)$, which are PO with respect to $u_0(\cdot)$, $u_1(\cdot), \ldots, u_{t-1}(\cdot)$, $u_t(\cdot, x_t^{t+1})\}$.

A.1.4. LEMMA. *The set $\mathfrak{P}_n(t, x_t^{t+1})$ is homeomorphic to $\mathfrak{U}_n(t-1)$, every element of $\mathfrak{P}_n(t, x_t^{t+1})$ being continuously parameterized by the utility levels $u_0, u_1, \ldots, u_{t-1}$.*

Proof of (A.1.4). The lemma is a simple extension of known results; (cf. Arrow and Hahn [1, Chap. 5, Sect. 2, pp. 111–114] and Balasko [3, Appendix 3]). In the overlapping-generations model, there is no commodity which is an argument of every utility function, but the nature of generational overlap provides sufficient "relatedness" among consumers to prove the lemma. ∎

A.1.5. COROLLARY. *The set $\mathfrak{P}_n(t, x_t^{t+1})$ is arc-connected.*

Proof of (A.1.5). Obvious from (A.1.4) and the arc-connectedness of $\mathfrak{U}_n(t-1)$. ∎

Step 2. We now establish

A.1.6. LEMMA. *The set $\mathfrak{P}_n(t)$ is compact and arc-connected.*

Proof of (A.1.6). It follows from the inequalities $u_t(x_t) \geqslant u_t(r^t/n, r^{t+1}/n)$ that $\mathfrak{P}_n(t)$ is compact as a closed subset of the compact set X_n.

We must show that given x and x' in $\mathfrak{P}_n(t)$ they can be linked by a continuous path in $\mathfrak{P}_n(t)$. Fix $x_t'^{t+1}, x_{t+1}, x_{t+2},\ldots$. From (A.1.1), there is $x'' \in \mathfrak{P}_n(t, x_t^{t+1})$ such that

$$u_0(x_0'') = u_0(x_0'), \qquad u_1(x_1'') = u_1(x_1'),\ldots, u_{t-1}(x_{t-1}'') = u_{t-1}(x_{t-1}').$$

From (A.1.5), x and x'' can be linked by a continuous path in $\mathfrak{P}_n(t, x_t^{t+1})$, hence in $\mathfrak{P}_n(t)$. Now take, for example, the segment linking $(x_t^{t+1}, x_{t+1},\ldots)$ to $(x_t'^{t+1}, x_{t+1}',\ldots)$. For any $\bar{x}_t^{t+1}$ belonging to the segment $[x_t^{t+1}, x_t'^{t+1}]$, there is by Lemma (A.1.4) a unique $x''(\bar{x}_t^{t+1}) \in \mathfrak{P}_n(t, \bar{x}_t^{t+1})$ such that $u_0(x_0''(\bar{x}_t^{t+1})) = u_0(x_0')$, $u_1(x_1''(\bar{x}_t^{t+1})) = u_1(x_1'),\ldots, u_{t-1}(x_{t-1}''(\bar{x}_t^{t+1})) = u_{t-1}(x_{t-1}')$. Furthermore, $x''(\bar{x}_t^{t+1})$ is a continuous function of $\bar{x}_t^{t+1}$. This construction defines a continuous path linking x'' to x' in $\mathfrak{P}_n(t)$. ∎

Step 3. Let $\mathfrak{P}(t)$ be the set of allocations $x \in X$ such that $(x_0, x_1,\ldots, x_t^t)$ is Pareto-optimal with respect to the utility functions $u_0(\cdot), u_1(\cdot),\ldots, u_{t-1}(\cdot), u_t(\cdot, x_t^t)$. Note that there are now no restrictions on the utility levels $u_t(x_t)$. Then, we have

A.1.7. LEMMA. $\mathfrak{P}(t) = \bigcup_n \mathfrak{P}_n(t)$.

Proof. Obvious. ∎

A.1.8. LEMMA. $\mathfrak{P} = \bigcap_t \mathfrak{P}(t)$.

Proof. Straightforward. (A similar construction is used in the Balasko and Shell [5] study of competitive equilibrium; cf. [5, Proposition (3.10) and Remark (3.12)].) ∎

Step 4. From (A.1.7) and (A.1.8), we have

$$\mathfrak{P} = \Big(\bigcap_t \bigcup_n \mathfrak{P}_n(t)\Big)$$

and hence

$$\mathfrak{P} = \Big(\bigcup_n \bigcap_t \mathfrak{P}_n(t)\Big).$$

Clearly, we have

$$\cdots \subset \mathfrak{P}_n(t+1) \subset \mathfrak{P}_n(t) \subset \cdots.$$

Therefore, $\bigcap \mathfrak{P}_n(t)$ is the intersection of a decreasing sequence of compact, connected, nonempty sets, hence is compact and connected. (Cf. e.g., Bourbaki [8, Chap. II, Sect. 4, Exercise 14, p. 212]. For a proof in a metric setting, cf., e.g., Lefschetz [14, Chap. I, Sect. 4, pp. 7–9].)

The sequence $\{\bigcap_t \mathfrak{P}_n(t)\}$ is clearly increasing in n, hence $\mathfrak{P} = \bigcup_n (\bigcap_t \mathfrak{P}_n(t))$ is connected; cf., e.g., Bourbaki [18, Chap. I, Sect 11, Proposition 2, p. 108]. ∎

A.1.9. *Remark.* We have shown only that $\mathfrak{P}$ is connected although we believe that one can probably establish the stronger result of arc-connectedness. (Arc-connectedness of $\mathfrak{P}$ becomes equivalent here to local arc-connectedness. It is not, however, in general true that the intersection of a nested sequence of compact, arc-connected, nonempty sets is arc-connected.)

A.2. *The Set of Pareto-Optima*

Let $\mathfrak{P}$ denote the set of Pareto-optimal allocations. Let $\mathfrak{U}_1$ be the image of X under the continuous mapping

$$x \mapsto (u_1(x_1),..., u_t(x_t),...).$$

Clearly, $\mathfrak{U}_1$ is arc-connected. Next we show that $\mathfrak{P}$ is homeomorphic to $\mathfrak{U}_1$ and hence $\mathfrak{P}$ is arc-connected.

A.2.1. LEMMA. *The set of Pareto-optimal allocations $\mathfrak{P}$ is homeomorphic to $\mathfrak{U}_1$.*

Proof of (A.2.1). Essentially, we repeat the proof used for analyzing the structure of the PO set for finite pure-exchange economies. Cf. (A.1.4) above and Balasko [2, Corollary 1, p. 564].

Let $(\bar{u}_1, \bar{u}_2,..., \bar{u}_t,...) \in \mathfrak{U}_1$ be fixed. Consider the set $\{x \in X \mid u_1(x_1) = \bar{u}_1,..., u_t(x_t) = \bar{u}_t,...\}$, which is nonempty because $(\bar{u}_1,..., \bar{u}_t,...)$ belongs to $\mathfrak{U}_1$. Furthermore, the set is clearly closed and every component x_t is bounded; therefore, the set is compact. The utility function u_0 can be viewed as a continuous function of the sequence x. Hence, it has a maximum on the compact set $\{x \in X \mid u_1(x_1) = \bar{u}_1,..., u_t(x_t) = \bar{u}_t,...\}$. By the strict quasiconcavity of every $u_t(\cdot)$, this maximum is unique and is a continuous function of $(\bar{u}_1,..., \bar{u}_t,...)$. We have thus defined a continuous mapping from $\mathfrak{U}_1$ into $\mathfrak{P}$. The inverse mapping $x \in \mathfrak{P} \mapsto (u_1(x_1),..., u_t(x_t),...)$ is obviously continuous.

A.2.2. PROPOSITION. *The set of Pareto-optimal allocations* $\mathfrak{P}$ *is arc-connected.*

Proof. Follows directly from (A.2.1) and the arc-connectedness of $\mathfrak{U}_1$. ∎

GLOSSARY OF BASIC NOTATION

$x = (x_0, x_1, \ldots)$ is the (intertemporal) commodity allocation sequence in the allocation space $X = \mathbb{R}^l_{++} \times \mathbb{R}^{2l}_{++} \times \mathbb{R}^{2l}_{++} \times \cdots$.

$\omega = (\omega_0, \omega_1, \ldots) \in X$ is the sequence of commodity endowments and $r = \sum_t \omega_t$ is the sequence of resources (or, aggregate endowments).

$p = (p^1, p^2, \ldots) \in \mathbb{R}^l_{++} \times \mathbb{R}^l_{++} \times \cdots$ is the (intertemporal) commodity price sequence. $\mathscr{P}$ is the set of normalized prices $\{p \mid p^{1,1} = 1\}$, $\mathscr{P} \subset \mathbb{R}^l_{++} \times \mathbb{R}^l_{++} \times \cdots$.

The monetary policy (or, monetary transfer) sequence is $m = (m_0^1, m_1^1, m_1^2, m_2^2, m_2^3, \ldots) \in M = \mathbb{R}^\infty$, the set of such monetary policies (or, monetary transfers). We also refer to $\mu = (\mu_0, \mu_1, \mu_2, \ldots) = (m_0^1, m_1^1 + m_1^2, m_2^2 + m_2^3, \ldots)$ as a monetary policy. The set of such μ is $\mathscr{M} = \mathbb{R}^\infty$.

$m^t \in \mathbb{R}$ is the aggregate money supply at time $t \geqslant 1$.

$x_t^{s,m} \in \mathbb{R}$ is the gross addition to money balances committed by consumer t in period s; x^m is the sequence $(x_0^{1,m}, x_1^{1,m}, x_1^{2,m}, \ldots)$.

$p^m \in \mathbb{R}_+$ is the present price of money.

$q = (p, p^m) \in \mathscr{Q} = \{(p, p^m) \mid p \in \mathscr{P}$ and $p^m \in \mathbb{R}_+\} \subset \mathbb{R}_+ \times \mathbb{R}^l_{++} \times \mathbb{R}^l_{++} \times \cdots$.

$\mathscr{Q}(\omega, \mu) \subset \mathscr{Q}$ is the set of monetary competitive equilibria $q = (p, p^m)$ associated with the endowments $\omega \in X$ and the monetary policy $\mu \in \mathscr{M}$.

$\mathscr{P}^m(\omega, \mu) = \{p^m \mid (p, p^m) \in \mathscr{Q}(\omega, \mu)\} \subset \mathbb{R}^+$ is the set of equilibrium money prices given the endowment $\omega \in X$ and the monetary policy $\mu \in \mathscr{M}$.

$w = (w_0, w_1, \ldots)$ is the sequence of "incomes" in the set $W = \mathbb{R}^\infty_{++}$.

$\mathscr{B} \subset \mathscr{P} \times W$ is the set of price-income equilibria (p, w) given fixed resources r.

$\mathscr{M}_B(\omega) \subset \mathscr{M}$ is the set of normalized ω-bonafide monetary policies. The set of ω-bonafide monetary policies is the cone in $\mathscr{M}$ generated by $\mathscr{M}_B(\omega)$. $\mathscr{M}_B \subset \mathscr{M}$ is the set of normalized bonafide monetary policies. The set of bonafide monetary policies is the cone in $\mathscr{M}$ generated by $\mathscr{M}_B$. $\mathscr{M}_P(\omega)$ is the set of potent ω-monetary policies.

$\bar{\mathfrak{P}} \subset X$ is the set of weakly-Pareto-optimal (WPO) allocations. $\mathfrak{P}$ is the set of Pareto-optimal (PO) allocations, thus $\mathfrak{P} \subset \bar{\mathfrak{P}}$.

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Erratum

Volume **24**, Number 1 (1981), in the article, "The Overlapping-Generations Model. II. The Case of Pure Exchange with Money" by Yves Balasko and Karl Shell, pp. 112–142:

Monetary economics is a mysterious subject. Cf., e.g., Fig. 7.1 on page 136 of our paper on overlapping generations with money. The set $\mathcal{M}_B(\omega)$ is not visible. We offer Fig. 7.1 once again. This time we have "pressed harder." What you should see is the set of normalized ω-bonafide monetary policies and the intersection of that set with the ray $\{\lambda\mu \mid \lambda \geqslant 0\}$.

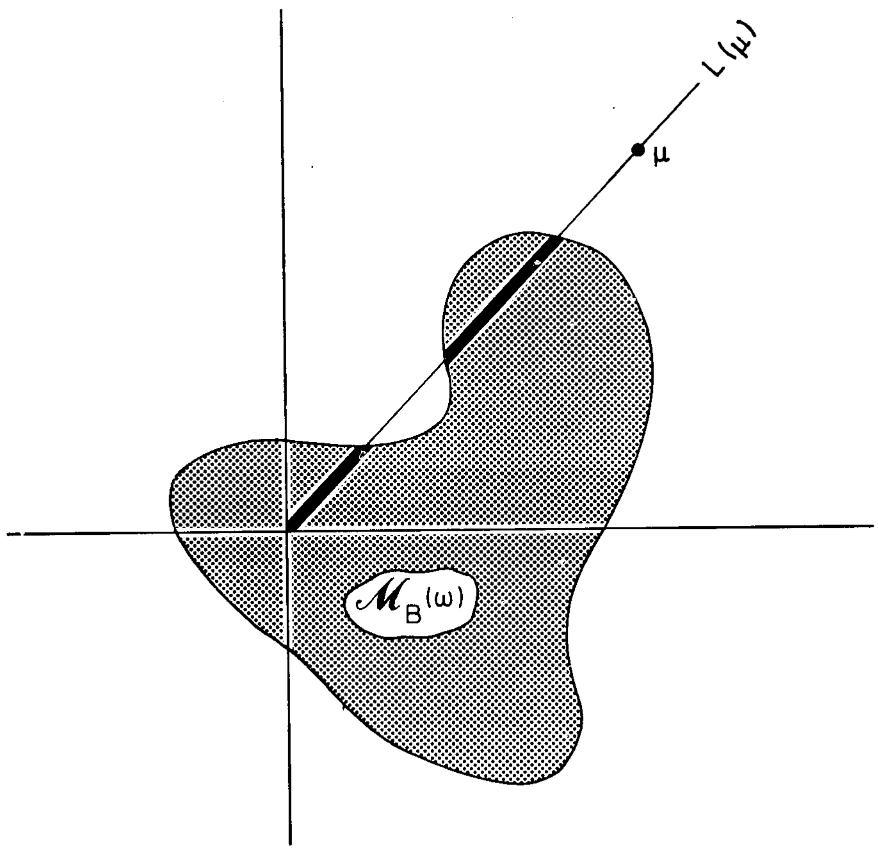

FIG. 7.1

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